ON PROPERTIES OF G-SPEC(R)

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Abstract. Let G be a group with identity e and $R = \oplus_{g\in G} R_g$ be a G-graded ring. We use some facts about the graded prime spectra to study more properties of graded rings and also give more properties of the topological space G-spec(R).

0 Introduction. Let G be a group with identity e. Then a ring R is said to be G-graded if there exist additive subgroups R_g of R such that $R = \oplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by (R, G), and we consider $supp(R, G) = \{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$ where x_g is the component of x in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. From now on we assume R is a G-graded ring unless otherwise indicated.

In this paper, we continue the work done in [3,5] and use the facts concerning graded prime spectra to study more properties of graded rings like: first strong, semiprime, prime and faithful. Then we use these facts to give more properties of the topological space G-space (R). In Section 1, we give some basic definitions and facts which are necessary in this paper. In Section 2, we give some applications of the graded prime spectra of a graded ring. In Section 3, we use the definitions and facts given in Section 2 to give more properties of the topological space G-spec(R). In paticular we give the relation between G-spec(R) and G-spec$(R/G - nil(R))$, and also the relation between the graded prime spectra of two homogeneously equivalent graded rings.

1 Preliminaries. In this section we give some basic definitions and facts which are necessary in this paper.

Definition 1.1 *Let R be a G-graded ring. Then*

1. *(R, G) is strong if $R_gR_h = R_{gh}$ for all $g, h \in G$. Also, (R, G) is strong if $1 \in R_gR_{g^{-1}}$ for all $g \in G$ (Proposition 1.6[1]).*
2. *(R, G) is first strong if $1 \in R_gR_{g^{-1}}$ for all $g \in supp(R, G)$.*

Definition 1.2 *Let I be an ideal of R. Then I is a graded ideal of (R, G) if $I = \oplus_{g\in G}(R_g \cap I)$.*

Remark 1.3 *Clearly $\oplus_{g\in G}(R_g \cap I) \subseteq I$ and hence I is a graded ideal of (R, G) if $I \subseteq \oplus_{g\in G}(R_g \cap I)$.*

Definition 1.4 *Let R be a G-graded ring. Then (R, G) is semiprime if R has no non-zero nilpotent graded ideals.*

Definition 1.5 *A G-graded ring R is faithful if for any $a_g \in R_g - 0$, $a_gR_h \neq 0$ and $R_ha_g \neq 0$ for all $g, h \in G$.*

For more details concerning graded rings one can look in [2,4,6].

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2 Applications of the graded prime spectra of a graded ring.

In this section we use some facts concerning graded prime spectra given in [3] to study more properties of graded rings like: first strong, semiprime, prime and faithful.

Definition 2.1 *Let I be a graded ideal of (R, G). Then*

1. *I is a graded prime ideal (in abbreviation "G-prime ideal") if $I \neq R$ and whenever $rs \in I$, we have $r \in I$ or $s \in I$.*
2. *I is a graded maximal ideal (in abbreviation "G-maximal ideal") If $I \neq R$ and there is no graded ideal J of (R, G) such that $I \underset{\neq}{\subset} J \underset{\neq}{\subset} R$.*
3. *The graded radical of I (in abbreviation "$Gr(I)$") is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with ${x_g}^{n_g} \in I$. Note that if r is a homogeneous element of (R, G) then $r \in Gr(I)$ iff $r^n \in I$ for some $n \in \mathbf{N}$.*
4. *The graded nilradical of (R, G)(in abbreviation "G-nil(R)") is the set of all $x \in R$ such that x_g is a nilpotent element of R for each $g \in G$.*

Notation Let R be a G-graded ring. If $M \subseteq R$ then let $V(M)$ denote the set of all G-prime ideals of (R, G) that contains M. Clearly, $V(M) = V(h(M))$ where $h(M) = \{x \in h(R) : r = x_g \text{ for some } x \in M, g \in G\}$. Also, let GX denote the set of all G-prime ideals of (R, G).

Proposition 2.2 ([3]) *Let $\tau = \{GX - V(M) : M \subseteq R\}$. Then τ is a topology on GX.*

Notation The topology in the previous proposition is called the graded prime spectrum of (R, G) (in abbreviation "graded spectra of (R, G)" and we write G-spec(R)).

For $t \in R$, we denote by GX_t to be the open member $GX - V(t)$ of G-spec(R).

Proposition 2.3 ([3]) *Let I be a graded ideal of (R, G) different from R. Then there exists a G-maximal ideal M of (R, G) such that $I \subseteq M$.*

Proposition 2.4 *Let R be a G-graded ring. Then (R, G) is first strong iff for any $g \in supp(R, G)$, GX can be written as a finite union of basic open sets GX_r with $r \in R_g$.*

Proof Suppose (R, G) is first strong. Let $g \in supp(R, G)$. Then $1 \in R_g R_{g^{-1}}$ and hence there exist $r_1, r_2, \cdots, r_n \in R_g$ and $s_1, s_2, \cdots s_n \in R_{g^{-1}}$ such that $1 = \sum_{i=1}^n r_i s_i$. Let $P \in GX$. Then $1 \notin P$ and hence there exists $j \in \{1, 2, \cdots, n\}$ such that $r_j \notin P$. So, $P \in GX_{r_j} \subseteq \cup_{i=1}^n GX_{r_i}$, i.e., $GX = \cup_{i=1}^n GX_{r_i}$.

Conversely, let $g \in supp(R, G)$. Then there exist $r_1, r_2, \cdots r_n \in R_g$ such that $GX = \cup_{i=1}^n GX_{r_i}$. Let I be the ideal of R generated by $\{r_1, r_2, \cdots, r_n\}$. Then I is a graded ideal of (R, G).

Suppose $I \neq R$ then by Proposition 2.3 there exists a G-maximal ideal P of (R, G) with $I \subseteq P$. Now $r_i \in P$ for all $i \in \{1, 2, \cdots, n\}$ implies $P \notin \cup_{i=1}^n GX_{r_i} = GX$, a contradiction. Therefore, $I = R$ and then there exist $s_1, s_2, \cdots s_n \in R$ such that $1 = \sum_{i=1}^n r_i s_i$. But $(1)_e = 1$ implies $1 = r_1 (s_1)_{g^{-1}} + \cdots + r_n (s_n)_{g^{-1}} \in R_g R_{g^{-1}}$. Hence (R, G) is first strong.

Proposition 2.5 *Let R be a G-graded ring. Then (R, G) is semiprime iff for any graded ideal I of (R, G) with $I^2 = 0$, we have $I = 0$.*

Proof Direct.

Proposition 2.6 *Let R be a G-graded ring. Then (R, G) is semiprime iff G-nil$(R) = 0$.*

Proof Let R be semiprime and $x \in G$-nil(R). Let $g \in G$. Then there exists positive integer n such that $x_g^n = 0$. Choose $I = Rx_g$, then I is a graded ideal of (R, G) and $I^n = 0$. Since (R, G) is semiprime we have $I = 0$ and hence $x_g = 0$. Therefore, $x = 0$, i.e., G-nil$(R) = 0$.

Conversely, suppose G-nil$(R) = 0$ and let I be a graded ideal of (R, G) with $I^2 = 0$. Let $x \in I$ and $g \in G$. Then $x_g \in I$ and $x_g^2 = 0$, i.e., $x \in G$-nil$(R) = 0$. Therefore, $I = 0$ and hence by Proposition 2.5, (R, G) is semiprime.

Proposition 2.7 *Let R be a G-graded ring with supp$(R, G) = G$. If $0 \in GX$ then (R, G) is faithful.*

Proof: Suppose $0 \in GX$. Let $r_g \in R_g$ and $h \in G$ with $r_g R_h = 0$. Then by assumption $r_g = 0$ or $R_h = 0$. But $h \in G = supp(R, G)$ implies $r_g = 0$. Therefore, (R, G) is faithful.

However, the converse of Proposition 2.7 need not be true in general as we see in the following example.

Example 2.8 *Let $R = \mathbf{Z}_4[x]$ and $G = \mathbf{Z}_2$. Suppose R is G-graded as follows:*
$R_0 =$ the additive subgroup of R generated by $\{Kx^{2i} : k \in \mathbf{Z}_4, i = 0, 1, 2, 3, \cdots\}$
$R_1 =$ the additive subgroup of R generated by $\{Kx^{2i+1} : k \in \mathbf{Z}_4, i = 0, 1, 2, \cdots\}$
Clearly (R, G) is faithful, while $0 \notin GX$ because $2.2 = 0$.

3 Properties of G-spec(R).

In this section we use the definitions and facts given in Section 2 to give more properties of the topological space G-spec(R).

The following proposition shows the relation between G-spec(R) and G-spec$(R/G - \text{nil}(R))$.

Proposition 3.1 *Let R be a G-graded ring. Then G-spec(R) and G-spec$(R/G - nil(R))$ are homeomorphic spaces.*

Proof Let $X = G$-spec(R) and $Y = G$-spec(R/I) where $I = G$-nil(R). Consider the usual homomorphism $\varphi : R \to R/I$ given by $\varphi(r) = r + I$. Clearly Ker $\varphi = I$. Define $\psi : Y \to X$ by $\psi(p) = \varphi^{-1}(p)$. We show ψ is a homeomorphism.

1. Supose $\psi(P_1) = \psi(P_2)$. Then $\varphi^{-1}(P_1) = \varphi^{-1}(P_2)$ and hence $\varphi(\varphi^{-1}(P_1)) = \varphi(\varphi^{-1}(P_2))$. So, $P_1 = P_2$ because φ is surjective. Therefore, ψ is injective.

2. Let $P \in X$ then $I \subseteq P$ and hence $\varphi(P)$ is a graded ideal of R/I. Since $\varphi(P)$ is a G-prime of $(R/I, G)$ we have $\varphi(P) \in Y$. But clearly, $\varphi^{-1}(\varphi(P)) = P$ implies $\psi(\varphi(P)) = P$, i.e., ψ is surjective.

3. To show ψ is continuous, it is enough to show that $\psi^{-1}(X_r)$ is open in Y for all $r \in h(R)$. Let X_r be any basic set in Y. Then $P \in \psi^{-1}(X_r) \Leftrightarrow \psi(P) \in X_r \Leftrightarrow r \notin \psi(P) \Leftrightarrow r \notin \varphi^{-1}(P) \Leftrightarrow \varphi(r) \notin P \Leftrightarrow P \in Y_{\varphi(r)}$. Hence $\psi^{-1}(X_r) = Y_{\varphi(r)}$ which is open in Y.

4. To show ψ is open function, it is enough to show that $\psi(Y_r)$ is open in X for any basic open set Y_r of Y.
Let $r_g + I \in (R/I)_g$. Then $r_g + I = \varphi(r_g)$. By a similar argument to part 3, we have $\psi(Y_{r_g+I}) = X_{r_g}$ which is open in X. Therefore, ψ is a homeomorphism.

In [5] we have defined the “homogeneous equivalence” concept between graded rings, and we discussed some properties of G-graded rings and investigate which of these are preserved under homogeneous-equivalence maps. In Proposition 3.3 we give the relation between the graded prime spectra of two homogeneously equivalent graded rings.

Definition 3.2 **([4])** *Let G, H be groups, R be a G-graded ring and S be an H-graded ring. We say that R is homogeneously equivalent to S if there exists a ring isomorphism $f : R \to S$ sending $h(R)$ onto $h(S)$. We call such an f a homogeneous-equivalence of R with S.*

Proposition 3.3 *Suppose (R, G) and (R, H) are homogeneously equivalent graded rings. Then G-spec(R) and H-spec(R) are homeomorphic spaces.*

Proof Let $\varphi : R \to R$ be the homogeneous-equivalence map. Define $\psi : H\text{-spec}(R) \to G\text{-spec}(R)$ by $\psi(P) = \varphi^{-1}(P)$.

1. Suppose $\psi(P_1) = \psi(P_2)$. Then $\varphi^{-1}(P_1) = \varphi^{-1}(P_2)$ and hence $P_1 = P_2$ because φ^{-1} is bijective. Therefore, ψ is injective.

2. Let $P \in G\text{-spec}(R)$. Then $\varphi(P) \in H\text{-spec}(R)$. Now $\psi(\varphi(P)) = \varphi^{-1}(\varphi(P)) = P$ since φ is bijective. Thus, ψ is surjective.

3. To show ψ is continuous, let X_r be any basic open set of G-spec(R). Then $P \in \psi^{-1}(X_r) \Leftrightarrow \psi(P) \in X_r \Leftrightarrow r \notin \psi(P) = \varphi^{-1}(P) \Leftrightarrow \varphi(r) \notin P \Leftrightarrow P \in Y_{\varphi(r)}$. Thus $\psi^{-1}(X_r) = Y_{\varphi(r)}$ which is open in H-spec(R).

4. To show ψ is open, let Y_r be any basic open set in H-spec(R). Then there exists $t \in h(R, G)$ such that $\varphi(t) = r$. By similar argument to part 3, we have $\psi(Y_r) = X_t$ which is open in G-spec(R). Therefore, ψ is a homeomorphism between H-spec(R) and G-spec(R).

Remark 3.4 *For any commutative ring R with identity, Zariski defined a topology on the set of all prime ideals of R with a base $\beta = \{X - W(R) : r \in R\}$ where $W(r)$ is the set of all prime ideals of R that contains r, and X denotes the set of all prime ideals of R. Zariski called the resulting space the spectra of R and is denoted by spec(R).*

Now, we give an important result of the relation beween G-spec(R) and spec(R_e) in case the graduation is strong.

Proposition 3.5 *Suppose R is strongly G-graded ring. Then G-spec(R) and spec(R_e) are homeomorphic spaces.*

Proof Suppose R is a strongly G-graded ring. Define $\varphi : G\text{-spec}(R) \to \text{spec}(R_e)$ by $\varphi(P) = P \cap R_e$.

1. Suppose $\varphi(P_1) = \varphi(P_2)$. Then $P_1 \cap R_e = P_2 \cap R_e$. Assume $P_1 \neq P_2$ then $P_1 \nsubseteq P_2$ or $P_2 \nsubseteq P_1$. If $P_1 \nsubseteq P_2$ then there exists $x \in P_1 - P_2$. Hence there exists $g \in G$ such that $x_g \in P_1 - P_2$. Since $R_{g^{-1}} x_g \subseteq P_1 \cap R_e = P_2 \cap R_e$, $R_{g^{-1}} x_g \subseteq P_2$ and hence $R_g R_{g^{-1}} x_g \subseteq P_2$. But R is strongly G-graded implies $R_g R_{g^{-1}} = R_e$ and then $x_g \in P_2$, a contradiction. Similarly if $P_2 \nsubseteq P_1$. Therefore, $P_1 = P_2$, i.e., φ is injective.

2. Let $P \in \text{spec}(R_e)$. Let J be the ideal of R generated by P. Then J is a graded ideal of (R, G). Assume $J = R$. Then there exist $r_1, r_2, \cdots, r_n \in R$ and $x_1, x_2, \cdots, x_n \in P \subseteq R_e$ such that $1 = \sum_{i=1}^{n} r_i x_i$. Since $(1)_e = 1$, we have $1 = (r_1)_e x_1 + \cdots + (r_n)_e x_n \in P$, a contradiction. Thus $J \neq R$.
   Claim $J \cap R_e = P$.
   Suppose $x \in J \cap R_e$. Then $x \in R_e$ and there exist $r_1, r_2, \cdots, r_n \in R$ and $s_1, s_2, \cdots, s_n \in P$ such that $x = \sum_{i=1}^{n} r_i s_i$. But $(x)_e = x$ implies $x = (r_1)_e s_1 + \cdots + (r_n)_e s_n \in P$. Clearly, $P \subseteq J \cap R_e$ and hence the claim is proved.
   Let us show $J \in G\text{-spec}(R)$. Suppose $r_g s_h \in J$ with $r_g \in R_g$ and $s_h \in R_h$. Then $R_{h^{-1}} R_{g^{-1}} r_g s_h \subseteq J \cap R_e = P$. Hence $(R_{h^{-1}} s_h)(R_{g^{-1}} r_g) \subseteq P$. Since $R_{h^{-1}} s_h$ and $R_{g^{-1}} r_g$ are ideals of R_e and P is prime ideal of R_e we have $R_{h^{-1}} s_h \subseteq P$ or $R_{g^{-1}} r_g \subseteq P$ and then $R_h R_{h^{-1}} s_h \subseteq J$ or $R_g R_{g^{-1}} r_g \subseteq J$. But $R_h R_{h^{-1}} = R_e = R_g R_{g^{-1}}$ implies $s_h \in J$ or $r_g \in J$. Thus $J \in G\text{-spec}(R)$ and $\varphi(J) = J \cap R_e = P$, i.e., φ is surjective.

3. To show φ is continuous, let Y_r be any basic open set of spec(R_e). Then $r \in R_e$. Now, $P \in \varphi^{-1}(Y_r) \Leftrightarrow \varphi(P) \in Y_r \Leftrightarrow r \notin \varphi(P) \Leftrightarrow r \notin P \cap R_e \Leftrightarrow r \notin P \Leftrightarrow P \in X_r$. Therefore, $\varphi^{-1}(Y_r) = X_r$ is open in G-spec(R).

4. To show φ is open, let X_r be any basic open set of G-spec(R), where $r \in R_g$. Let $V(r) =$ the set of all G-prime ideals of (R, G) that contains r and let $W(r) =$ the set of all prime ideals of R that contains r.

Claim $\varphi(V(r)) = W(R_{g^{-1}}r)$.
Let $P \in V(r)$ then $r \in P$ and hence $R_{g^{-1}}r \subseteq P \cap R_e = \varphi(P)$. Thus $\varphi(P) \in W(R_{g^{-1}}r)$. Conversely, let $P \in W(R_{g^{-1}}r)$ then $R_{g^{-1}}r \subseteq P$ and hence $R_{g^{-1}}r \subseteq \varphi^{-1}(P)$. So, $R_g R_{g^{-1}}r \subseteq \varphi^{-1}(P)$ and then $r \in \varphi^{-1}(P)$ because $R_g R_{g^{-1}} = R_e$. Therefore, $\varphi^{-1}(P) \in V(r)$ and then $p \in \varphi(V(r))$, i.e., $\varphi(V(r)) = W(R_{g^{-1}}r)$. Now, $\varphi(X_r) =$spec$(R_e) - W(R_{g^{-1}}r)$ which is open in spec(R_e).

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