ON MS–SEQUENCES IN ORDERED TOPOLOGICAL VECTOR SPACES

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ABSTRACT. Various properties of ordered topological vector spaces related to the so-called *ms*-sequences are investigated. Main results present characterizations of ordered topological vector spaces such that ms-sequences converge to zero.

We shall consider *ordered topological vector spaces* (OTVS) $E = (E, \tau)$ in the sense of [6], i.e., a vector ordering $\leqslant$ in a real vector space E is introduced by a non-empty set $\mathcal{C} \subset E$ satisfying the following two properties: $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$ and $\lambda\mathcal{C} \subset \mathcal{C}$ for every $\lambda \in [0, \infty)$. The set $\mathcal{C}$ is called a *cone* (or a *wedge* — see [4]). The vector ordering $\leqslant$ in E is generated by $\mathcal{C}$ as follows: $x \leqslant y$ iff $y - x \in \mathcal{C}$. The vector ordering $\leqslant$ is reflexive, transitive and compatible with the algebraic structure of E. Moreover $\mathcal{C} = \{x \in E : 0 \leqslant x\}$, i.e., $\mathcal{C}$ is the set of positive elements. A cone $\mathcal{C}$ is said to be: *proper* when $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$ and *generating* if $E = \mathcal{C} - \mathcal{C}$. The vector ordering $\leqslant$ is antisymmetric iff $\mathcal{C}$ is proper and E is directed upwards iff $\mathcal{C}$ is generating. In general, the vector ordering and the linear topology τ need not have any connection.

We shall write $A \downarrow 0$ ($x_n \downarrow 0$) whenever the set $A \subset E$ (the sequence $(x_n) \subset E$) is directed downwards (decreasing), $0 \leqslant a$ for every $a \in A$ ($0 \leqslant x_n$ for every n) and $y \leqslant 0$ for an arbitrary lower bound y of A (of $\{x_n : n \in \mathbb{N}\}$). Every set $A \subset E$ directed downwards forms a net, and so the notion $A \xrightarrow{\tau} 0$ is clear.

The topological dual of E (= the space of linear τ-continuous functionals) will be denoted by E^*. The phrases 'complete', 'sequentially complete' will always have their topological meaning. The terminology concerning OTVS, ordered vector spaces (OVS) and locally solid Riesz spaces (TRS) not explained in the text is essentially that of [6], [4], and [1].

We shall investigate *majorized sums sequences* $(x_n) \subset E$, i.e.,

$$0 \leqslant x_n \text{ and } x_1 + \cdots + x_n \leqslant x \text{ for some } x \in E \text{ and all } n$$

(this notion was introduced by G.J.O. Jameson). Majorized sums sequences will be shortly called *ms-sequences.* We shall concentrate on consequences of the assumption

$(*)$ *every ms-sequence in E τ-converges to zero.*

Below we give some examples of OTVS enjoying the property $(*)$.

Examples. Let (E, τ) be a Hausdorff locally convex and locally order-convex OTVS. According to [6] (5.12) Corollary (see also [4] 3.2.2) every $x^* \in E^*$ is the difference of two positive continuous linear functionals, and so ms-sequences in E are weakly null. Thus $(E, \sigma(E, E^*))$ is an OTVS such that ms-sequences tend to zero.

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If we suppose additionally that $\mathcal{C}$ is τ-closed and (E,τ) is weakly sequentially complete, then (E,τ) satisfies $(*)$. Indeed, considering an ms-sequence $(x_n) \subset E$ we obtain that the sequence (s_k) of partial sums, $s_k = \sum_{n=1}^{k} x_n$, is increasing and weakly convergent. Therefore (s_k) is τ convergent by [6] (5.8) Proposition (see also [4] 3.2.10 Corollary). Hence $x_n \xrightarrow{\tau} 0$.

On the other hand there are locally order-convex OTVS containing ms-sequences with terms far from zero — consider the Banach lattices ℓ^∞, the space of bounded real sequences, and c the space of convergent real sequences, equipped with the sup norm.

The theorem below presents simple consequences of assumption $(*)$.

Theorem 1. *Let (E,τ) be an* OTVS *such that every ms-sequence is τ-null. Then we have*

(a) *Order intervals are τ-bounded.*
(b) *If τ is Hausdorff, then $\mathcal{C}$ is proper.*
(c) *If $\mathcal{C}$ is τ-closed and τ is complete (sequentially complete), then $A \downarrow 0$ implies $A \xrightarrow{\tau} 0$ ($x_n \downarrow 0$ implies $x_n \xrightarrow{\tau} 0$), i.e., τ is a Lebsgue topology (τ is a σ-Lebesgue topology).*
(d) *If $\mathcal{C}$ is τ-closed and τ is complete and metrizable, then τ is locally order-convex.*

Proof. (a) Suppose that $[a,b]$ is not τ-bounded. Therefore there exists a sequence $(x_n) \subset [a,b]$ and a sequence of reals (t_n) such that $t_n \to 0$ but $t_n x_n \stackrel{\tau}{\nrightarrow} 0$. Hence $t_n(b-x_n) \stackrel{\tau}{\nrightarrow} 0$. Without loss of generality we can assume that $|t_n|(b-x_n) \notin V$ for some τ-neighborhood of zero V and $t = \sum_{n=1}^{\infty} |t_n| < \infty$. We have

$$0 \leqslant |t_n|(b-x_n) \quad \text{and} \quad \sum_{n=1}^{m} |t_n|(b-x_n) \leqslant t(b-a) \text{ for all } m \in \mathbb{N}.$$

Thus $(|t_n|(b-x_n))$ is an ms-sequence. It should be $|t_n|(b-x_n) \in V$ for large n, a contradiction.

(b) Assume $x \in \mathcal{C} \cap (-\mathcal{C})$. Consider a sequence (x_n) defined as follows: $x_n = x$ for odd n and $x_n = -x$ for even n. The sequence (x_n) is an ms-sequence, and so $x_n \xrightarrow{\tau} 0$. Hence $x = 0$ because τ is Hausdorff.

(c) Let $A \downarrow 0$. We shall show that the net A is τ-Cauchy. If we suppose that it is not then we are able to find a τ-neighborhood of zero U and a decreasing sequence $(a_n) \subset A$ satisfying the condition $x_n = a_n - a_{n+1} \notin U$. The sequence (x_n) is an ms-sequence, and so $x_n \in U$ for large n, a contradiction.

Finally, the completeness of τ and the closedness of $\mathcal{C}$ imply that A is convergent to zero (see [6] (2.1) Proposition or [4] 3.1.14).

(d) According to [4] 3.2.5 it is enough to show that $0 \leqslant y_n \leqslant x_n$ in E and $x_n \xrightarrow{\tau} 0$ imply $y_n \xrightarrow{\tau} 0$. Let $|\cdot|_\tau$ be an F-norm inducing τ. If we suppose that (y_n) is not τ-null, then there exist a number $\varepsilon > 0$, a subsequence (n_k) satisfying $|y_{n_k}|_\tau > \varepsilon$ and $|x_{n_k}|_\tau < 2^{-k}$. Since τ is complete and $\mathcal{C}$ is τ-closed , the series $\sum_{k=1}^{\infty} x_{n_k}$ is convergent to an element $x \in \mathcal{C}$. But then $\sum_{k=1}^{m} y_{n_k} \leqslant x$ for every m, i.e., (y_{n_k}) is an ms-sequence separated from zero, a contradiction.

Combining [6] (2.1) Proposition and Theorem 1 (b) we obtain the following corollary.

Corollary 2. *Let (E,τ) be an* OTVS *such that τ is Hausdorff and every ms-sequence is τ-null. If $\mathcal{C}$ is τ-closed, then the vector ordering $\leqslant$ generated by $\mathcal{C}$ is Archimedean and almost Archimedean.*

Remarks. 1. The assumption of completeness in the part (c) of Theorem 1 is essential. The weak topology in ℓ^∞ is not σ-Lebesgue while ms-sequences in ℓ^∞ converge weakly to zero. But ℓ^∞ is not weakly sequentially complete.

2. Part (d) of Theorem 1 is also a consequence of [4] 3.5.9 Theorem and Theorem 1(b) (the proof of part (d) presented above is similar to the proof of [4] 3.5.9 Theorem).

Further results contain several characterizations of OTVS satisfying assumption $(*)$.

Theorem 3. *For an* OTVS (E,τ) *the following statements are equivalent:*

(a) *Every ms-sequence is τ-null.*
(b) *Every increasing order bounded sequence $(x_n) \subset \mathcal{C}$ is τ-Cauchy.*

Proof. (a) $\Rightarrow$ (b) Consider a sequence (x_n) satisfying $0 \leqslant x_n \uparrow \leqslant x$ and suppose it is not τ-Cauchy. Then there exists a subsequence (n_k) such that the elements $y_k = x_{n_{k+1}} - x_{n_k}$ form an ms-sequence which is not τ-null.

(b) $\Rightarrow$ (a) If (x_n) is an ms-sequence, then the sequence of partial sums $s_k = \sum_{n=1}^k x_n$ increases and it is order bounded. Therefore (s_k) is τ-Cauchy and so $x_n = s_n - s_{n-1} \xrightarrow{\tau} 0$.

Remark. The property described in the second statement of Theorem 3 is well-known in the theory of locally solid Riesz spaces. It is called 'pre-Lebesgue property' (see [1]). Moreover, if (E,τ) is a TRS where E is Archimedean, then the Lebesgue property for τ implies the pre-Lebesgue property and these two properties are equivalent whenever τ is Hausdorff and complete — see [1] Theorems 10.2 and 10.3. The proof of Theorem 10.2 shows that the implication Lebesgue property $\Rightarrow$ pre-Lebesgue property remains valid in the case when (E,τ) is an OTVS with E being an Archimedean vector lattice, τ Hausdorff, and $\mathcal{C}$ τ-closed. We do not know if the assumption that E is a vector lattice can be removed.

If (E,τ) is an OTVS such that $\mathcal{C}$ is generating and $\mathcal{N}_0$ is a τ-neighborhood - base at zero consisting of circeled sets, then the family $\{(U \cap \mathcal{C}) - (U \cap \mathcal{C}) : U \in \mathcal{N}_0\}$ determines a linear topology τ_D called the *topology with the open decomposition property associated with* τ (for more informations concerning τ_D see [6] Chapter 3).

Corollary 4. *For an* OTVS (E,τ) *the following statements are equivalent:*

(a) *Every ms-sequence in E is τ-null.*
(b) *Every ms-sequence in E is τ_D-null.*

Proof. (a) $\Rightarrow$ (b) Let $0 \leqslant x_n \uparrow \leqslant x$ and let U be a τ-neighborhood of zero. By Theorem 3 (x_n) is τ-Cauchy, and so there exists n_0 with $x_n - x_{n_0} \in U$ for every $n \geqslant n_0$. Thus $x_n - x_{n_0} \in U \cap \mathcal{C} \subset (U \cap \mathcal{C}) - (U \cap \mathcal{C})$ for $n \geqslant n_0$, i.e., (x_n) is τ_D-Cauchy. Using Theorem 3 again we conclude the assertion.

(b) $\Rightarrow$ (a) obvious because τ_D is finer than τ.

Theorem 5. *Let (E,τ) be a Hausdorff sequentially complete* OTVS *such that $\mathcal{C}$ is τ-closed and order intervals are τ-bounded. The following statements are equivalent:*

(a) *Every ms-sequence is τ-null.*
(b) *There does not exist a positive linear homeomorphism T from c into E, where c is the space of convergent real sequences equpped with the* sup-*norm.*

Proof. (a) $\Rightarrow$ (b) Clearly, if there is such a homeomorphism T, then for the unit vectors e_n, and $e = (1,1,1,\dots)$ for which we have $0 \leqslant e_1 + \dots + e_n \leqslant e$, there would follow that the sequence $(T(e_n))$ is an ms-sequence which is not τ-null.

(b) $\Rightarrow$ (a) Suppose there are elements $x_n, x \in \mathcal{C}$, $n \in \mathbb{N}$ such that $x_1 + \dots + x_n \leqslant x$ for all n but $x_n \notin U$ for some τ-neighborhood of zero U. Note first that $\sum_{n=1}^\infty c_n x_n$ is τ-convergent whenever $(c_n) \in c_0$.

Indeed, fix a τ-neighborhood V of zero in E and $(c_n) \in c_0$. By the hypothesis we can also fix a $t > 0$ such that $t[-x, x] \subset V$. Choose $k_0 \in \mathbb{N}$ satisfying $\sup_{n¿k_0} |c_n| < t$. Thus for $j \geqslant m \geqslant k_0$ we have

$$\sum_{n=m}^{j} c_n x_n = t \sum_{n=m}^{j} t^{-1} c_n x_n \in t[-\sum_{n=m}^{j} x_n, \sum_{n=m}^{j} x_n] \subset t[-x, x] \subset V,$$

i.e., the sequence $(\sum_{n=1}^{j} c_n x_n)_{j=1}^{\infty}$ is τ-Cauchy. Since E is sequentially complete, we have the τ-convergence of $\sum_{n=1}^{\infty} c_n x_n$.

Define a linear operator $T_0 : c_0 \to E$ by $T_0((c_n)) = \sum_{n=1}^{\infty} c_n x_n$. The map T_0 is continuous. Indeed, let V be a τ-neighborhood of zero in E and $t > 0$ such that $t[-x, x] \subset V$. Then, for $(c_n) \in c_0$ satisfying $\|(c_n)\| = \sup_n |c_n| < t$ we have,

$$T_0((c_n)) = t \lim_{k\to\infty} \sum_{n=1}^{k} t^{-1} c_n x_n \in t[-x, x] \subset V;$$

the relation $\in$ holds since E_+ is τ-closed. Hence T_0 is continuous.

Moreover $T_0(e_n) = x_n \notin U$ for every $n \in \mathbb{N}$. Hence, by Drewnowski's theorem (see [2] Theorem 1) we are able to find an infinite subset $M \subset \mathbb{N}$ such that T_0 restricted to $c_0(M)$ is a homeomorphism. Without loss of generality we can assume $M = \mathbb{N}$. We can also assume $x \notin T_0(c_0)$. Indeed, suppose $x = \sum_{n=1}^{\infty} c_n x_n$ for some $(c_n) \in c_0$. Let $s > 0$ be such that $c_1 + s \neq 0$. we have $x + sx_1 \geqslant \sum_{k=1}^{n} x_k$ for all n and $x + sx_1 = T_0(c_1 + s, c_2, c_3, \dots)$. Thus $x + sx_1 \notin T_0(c_0(\{2, 3, \dots\}))$ because in the contrary case $x + sx_1 = T(0, b_1, b_2, \dots)$, and so the injectivity of T_0 implies $c_1 + s = 0$, a contradiction. Take $S : c_0 \to c_0$ given by $S((c_n)) = (0, c_1, c_2, \dots)$, i.e., $S((c_n)) = \sum_{n=1}^{\infty} c_n e_{n+1}$. The operator S is a positive linear isometry into. Hence for ${x_n}' = x_{n+1}$, $x' = x + sx_1$ we have ${x_n}' \geqslant 0$, $\sum_{k=1}^{n} {x_k}' \leqslant x'$. Moreover, $T_0 \circ S((c_n)) = \sum_{n=1}^{\infty} c_n {x_n}'$, $T_0 \circ S$ is a positive linear homeomorphism and $x' \notin T_0 \circ S(c_0)$.

Consider a map $T : c \to E$ defined by the equality

$$T((a_n)) = a_\infty x + \sum_{n=1}^{\infty} (a_n - a_\infty) x_n,$$

where $a_\infty = \lim_{n\to\infty} a_n$. The operator T is positive: if $a_n \geqslant 0$ for all n, then $a_\infty \geqslant 0$ and for an arbitrary $k \in \mathbb{N}$ there holds

$$a_\infty x + \sum_{n=1}^{k} (a_n - a_\infty) x_n = a_\infty (x - \sum_{n=1}^{k} x_n) + \sum_{n=1}^{k} a_n x_n \geqslant 0.$$

Since $\mathcal{C}$ is τ-closed we have $T((a_n)) \geqslant 0$. Since T maps linearly and homeomorphically each summand in $c = \mathbb{R}e \oplus c_0$, hence T is a linear homeomorphic embedding, contrary to (b).

Remark. The above proof applies and generalizes ideas due to L. Drewnowski who showed in [3] a particular case of Theorem 5 concerning locally solid Riesz spaces.

The following fact holds true (see [5] Proposition 1.2; $C(K)$ is the space of real-valued continuous functions on the compact space K).

Theorem 6. *Let Y be a Hausdorff complete locally convex topological vector space. For a continuous linear operator $T : C(K) \to Y$ the following statements are equivalent:*

(i) *T is weakly compact.*
(ii) *$T(f_n) \to 0$ for every bounded sequence (f_n) of pairwise disjoint functions.*

Now we are able to deduce the following result.

Corollary 7. *Suppose that (E,τ) is a Hausdorff complete locally convex* OTVS *space and that every ms-sequence is τ-null. Then for an arbitrary compact space K every positive linear operator $T : C(K) \to E$ is weakly compact.*

Proof. If $T : C(K) \to E$ is positive and linear, then T is continuous by [4] 3.5.5 Theorem and Theorem 1(a). Consider a sequence $(f_n) \subset C(K)$ with terms satisfying the condition $|f_i| \wedge |f_j| = 0$ for $i \neq j$, and $\alpha = \sup_i \|f_i\| < \infty$. The sequences $(f_i^+), (f_i^-)$ are bounded by the constant function taking the value α and they form ms-sequences. Since T is positive, the sequences $(T(f_i^+))$, $(T(f_i^-))$ are ms-sequences and by hypothesis $T(f_n) \xrightarrow{\tau} 0$, i.e., T is weakly compact.

Now, combining Theorems 5 and Corollary 7 we get

Theorem 8. *Let (E,τ) be a Hausdorff complete locally convex* OTVS *such that $\mathcal{C}$ is τ-closed and order intervals are τ-bounded. The following statements are equivalent:*

(a) *Every ms-sequence is τ-null.*
(b) *For an arbitrary compact space K all positive linear operators $T : C(K) \to E$ are weakly compact.*
(c) *Every positive linear operator $T : c \to E$ is weakly compact.*

Let E be an OVS and let $(F,\mathfrak{t})$ be a topological vector space. A linear operator $T : E \to F$ is said to be *ms-null* if $T(x_n) \xrightarrow{\mathfrak{t}} 0$ for every ms-sequence $(x_n) \subset E$. The ms-null operators can be characterized as follows.

Theorem 9. *For a linear operator T mapping an* OVS *E into a Hausdorff sequentially complete topological vector space $(F,\mathfrak{t})$ the following statements are equivalent:*

(a) *T is an ms-null operator.*
(b) *The series $\sum_{n=1}^{\infty} Tx_n$ is unconditionally convergent for every ms-sequence $(x_n) \subset E$ (i.e., $\sum_{n=1}^{\infty} Tx_{\pi(n)}$ converges for every permutation π).*
(c) *The series $\sum_{n=1}^{\infty} Tx_n$ converges for every ms-sequence $(x_n) \subset E$.*
(d) *If $(x_n) \subset \mathcal{C} \subset E$ decreases, then (Tx_n) is $\mathfrak{t}$-convergent.*

Moreover, every ms-null operator maps order bounded sets into $\mathfrak{t}$-bounded sets.

Proof. (a) $\Rightarrow$ (b) Assume that $\sum_{n=1}^{\infty} Tx_n$ is not unconditionally convergent for an ms-sequence $(x_n) \subset E$. Then there exists a $\mathfrak{t}$-neighborhood of zero U and a sequence (Δ_k) of pairwise disjoint finite subsets of $\mathbb{N}$ satisfying $\sum_{n\in\Delta_k} Tx_n \notin U$ for every k. Thus $(\sum_{n\in\Delta_k} x_n)_{k=1}^{\infty}$ is an ms-sequence whose T-image is not $\mathfrak{t}$-null, a contradiction.

(b) $\Rightarrow$ (c) obvious

(c) $\Rightarrow$ (d) If $(x_n) \subset \mathcal{C}$ is decreasing, then (Tx_n) is $\mathfrak{t}$-Cauchy because for every subsequence (n_k) the sequence of differences $(x_{n_k} - x_{n_{k+1}})$ forms an ms-sequence. Hence $T(x_n)$ converges.

(d) $\Rightarrow$ (a) Let $x_n \in \mathcal{C}$ and $x_1 + \cdots + x_n \leqslant x$ for every n. Since the sequence $u_k = x - \sum_{n=1}^{k} x_n$ is decreasing, then (Tu_k) is convergent, and so $Tx_n = Tu_{n-1} - Tu_n \xrightarrow{\mathfrak{t}} 0$.

Finally, suppose that T is an ms-null operator but $T[a,b]$ is not $\mathfrak{t}$-bounded. Therefore there exists a sequence $(x_n) \subset [a,b]$ and a sequence of reals (t_n), $t_n \to 0$ such that $t_n(Tb - Tx_n) \stackrel{\mathfrak{t}}{\nrightarrow} 0$. Repeating arguments used in the proof of Theorem 1(a) we shall construct an ms-sequence whose T-image is not $\mathfrak{t}$-null.

Remark. The last part of the proof shows that ms-null operators always map order bounded sets into topologically bounded sets — the assumption '$\mathfrak{t}$ is sequentially complete' is superfluous.

Applying the theorem above to the identity operator we obtain the next characterization of OTVS satisfying $(*)$.

Corollary 10. *For a Hausdorff sequentially complete* OTVS (E, τ) *the following statements are equivalent:*

(a) *Every ms-sequence is* τ*-null.*
(b) *The series* $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent for every ms-sequence* $(x_n) \subset E$.
(c) *The series* $\sum_{n=1}^{\infty} x_n$ *is convergent for every ms-sequence* $(x_n) \subset E$.
(d) *If* $(x_n) \subset \mathcal{C}$ *decreases, then* (x_n) *is* τ*-convergent.*

An immediate consequence of Theorem 9 and [4] 3.5.5 Theorem is the following result concerning continuity of ms-null operators.

Theorem 11. *Let* (E, τ) *be a complete metrizable* OTVS *such that the cone* $\mathcal{C}$ *is closed and generatring. Then every ms-null operator mapping* E *into a topological vector space is continuous.*

It is clear that the identity operator on c maps order bounded sets into topologically bounded sets but it is not ms-null. Such operators do not exists in the class of functionals.

Theorem 12. *Let* E *be an* OVTS*. A functional* $f : E \to \mathbb{R}$ *is ms-null iff* f *maps order bounded sets into bounded sets.*

Proof. Assume that f maps order bounded sets into bounded sets but $|f(x_n)| > \varepsilon > 0$ for some ms-sequence $(x_n) \subset E$ with $\sum_{k=1}^{n} x_k \leqslant x$ for every n. Consider two cases.

I. $f(x_n) > 0$ for infinitely many n's. Let $n_1 < n_2 < \dots$ be such that $f(x_{n_k}) > 0$ for all k. Putting $y_i = \sum_{k=1}^{i} x_{n_k}$ we obtain $0 \leqslant y_i \leqslant x$ and $f(y_i) > i\varepsilon$. Thus $\sup_{y \in [0,x]} |f(y)| = \infty$, a contradiction.

II. $f(x_n) > 0$ for at most finitely many n's. Therefore $f(x_n) < -\varepsilon$ for almost all n's. Let n_0 be so large that $f(x_n) < -\varepsilon$ for $n \geqslant n_0$. Put $y_i = \sum_{k=n_0}^{n_0+i} x_k$. We have $0 \leqslant y_i \leqslant x$ and $f(y_i) < -i\varepsilon$. Thus $\sup_{y \in [0,x]} |f(y)| = \infty$, a contradiction again.

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