EFFICIENT SOLUTIONS OF MULTICRITERIA LOCATION PROBLEMS WITH RECTILINEAR NORM IN $\boldsymbol{R}^3$

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Abstract. A multicriteria location problem with rectilinear norm in $\boldsymbol{R}^3$ is considered. We propose an algorithm to find all efficient solutions of the location problem.

1. Introduction. Given demand points in $\boldsymbol{R}^3$, a problem to locate a new facility in $\boldsymbol{R}^3$ is called a single facility location problem. The problem is usually formulated as a minimization problem with an objective function involving distances between the facility and demand points. It is assumed that m demand points $\boldsymbol{d}_i \equiv (d_i^1, d_i^2, d_i^3)^T \in \boldsymbol{R}^3$, $i \in M \equiv \{1, 2, \cdots, m\}$ and rectilinear norm $\|\cdot\|_1$ defined on $\boldsymbol{R}^3$ are given. Let $\boldsymbol{x} \equiv (x^1, x^2, x^3)^T \in \boldsymbol{R}^3$ be the variable location of the facility. We put $D \equiv \{\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_m\}$. Without loss of generality, it is assumed that $D \not\subset \{(x^1, x^2, x^3)^T \in \boldsymbol{R}^3\colon x^j = x_0\}$ for each $j \in J \equiv \{1, 2, 3\}$ and any $x_0 \in \boldsymbol{R}$. If its assumption is not satisfied, then our three-dimensional problems reduce to one or two-dimensional problems as in [2, 3]. Our main problem is a multicriteria location problem formulated as follows:

$$\text{(P)} \qquad \min_{\boldsymbol{x} \in \boldsymbol{R}^3} \boldsymbol{f}(\boldsymbol{x}) \equiv (\|\boldsymbol{x} - \boldsymbol{d}_1\|_1, \|\boldsymbol{x} - \boldsymbol{d}_2\|_1, \cdots, \|\boldsymbol{x} - \boldsymbol{d}_m\|_1)^T.$$

(P) is a problem to find an efficient solution. A point $\boldsymbol{x}_0 \in \boldsymbol{R}^3$ is called *an efficient solution of* (P) if there is no $\boldsymbol{x} \in \boldsymbol{R}^3$ such that $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x}_0)$ and $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{f}(\boldsymbol{x}_0)$. Let $E(D)$ be the set of all efficient solutions of (P). By the definition of the efficiency, each demand point is efficient in (P). We also consider a minisum location problem formulated as follows:

$$(\mathrm{P}_{\boldsymbol{\lambda}}) \qquad \min_{\boldsymbol{x} \in \boldsymbol{R}^3} g(\boldsymbol{x}) \equiv \sum_{i=1}^{m} \lambda^i \|\boldsymbol{x} - \boldsymbol{d}_i\|_1$$

where λ^i is a positive weight for each $\boldsymbol{d}_i$, $i \in M$. We put $\boldsymbol{\lambda} \equiv (\lambda^1, \lambda^2, \cdots, \lambda^m)^T$ and denote the set of all optimal solutions of $(\mathrm{P}_{\boldsymbol{\lambda}})$ as $S^*(\boldsymbol{\lambda})$.

In $\boldsymbol{R}^2$, the set of all efficient solutions of (P) can be determined by using an algorithm in [2]. $(\mathrm{P}_{\boldsymbol{\lambda}})$ can be solved by using an algorithm in [3]. (P) and $(\mathrm{P}_{\boldsymbol{\lambda}})$ with another norm or distance instead of rectilinear norm in $\boldsymbol{R}^2$ are considered in [3, 6, 10-12]. In this article, we consider (P) and $(\mathrm{P}_{\boldsymbol{\lambda}})$ with rectilinear norm in $\boldsymbol{R}^3$. First, we characterize efficient solutions of (P) by using optimal solutions of $(\mathrm{P}_{\boldsymbol{\lambda}})$. Next, we give another characterization of efficient solutions of (P). Then we propose the Frame Generating Algorithm to find $E(D)$, which requires $O(m^4)$ computational time.

In section 2, we give some properties of optimal solutions of $(\mathrm{P}_{\boldsymbol{\lambda}})$. In section 3, we give some properties of efficient solutions of (P). In section 4, we propose the Frame Generating Algorithm to find $E(D)$, which requires $O(m^4)$ computational time. Finally, in section 5, we give some conclusions.

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2. Optimality of ($\mathrm{P}_{\boldsymbol{\lambda}}$). In this section, we give some properties of optimal solutions of ($\mathrm{P}_{\boldsymbol{\lambda}}$).

The following theorem gives the relation between efficient solutions of (P) and optimal solutions of ($\mathrm{P}_{\boldsymbol{\lambda}}$).

Theorem 1.(See [7].) *A point $\boldsymbol{x}_0 \in \boldsymbol{R}^3$ is efficient in* (P) *if and only if $\boldsymbol{x}_0$ is optimal in* ($\mathrm{P}_{\boldsymbol{\lambda}}$) *for some $\boldsymbol{\lambda} > \boldsymbol{0}$.*

From Theorem 1, $E(D)$ can be expressed as

$$E(D) = \{\boldsymbol{x}^* \in \boldsymbol{R}^3 : \boldsymbol{x}^* \in S^*(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} > \boldsymbol{0}\}. \tag{1}$$

Thus, in the following, we investigate properties of optimal solutions of ($\mathrm{P}_{\boldsymbol{\lambda}}$).

Since the objective function of ($\mathrm{P}_{\boldsymbol{\lambda}}$), g, can be rewritten as

$$g(\boldsymbol{x}) = \sum_{i=1}^{m} \lambda^i \|\boldsymbol{x} - \boldsymbol{d}_i\|_1 = \sum_{i=1}^{m} \lambda^i \sum_{j=1}^{3} |x^j - d_i^j| = \sum_{j=1}^{3} \sum_{i=1}^{m} \lambda^i |x^j - d_i^j|,$$

($\mathrm{P}_{\boldsymbol{\lambda}}$) reduces to three independent one-dimensional problems. Namely, $\boldsymbol{x}^* \equiv (x^{1*}, x^{2*}, x^{3*})^T \in S^*(\boldsymbol{\lambda})$ if and only if each x^{j*}, $j \in J$ is an optimal solution of the following one-dimensional problem:

$$\min_{x \in \boldsymbol{R}} g_j(x) \equiv \sum_{i=1}^{m} \lambda^i |x - d_i^j|. \tag{P_j}$$

These one-dimensional problems can be solved by using an algorithm in [3]. For each $j \in J$, we denote all optimal solutions of (P_j) for $\boldsymbol{\lambda}$ as $S_j^*(\boldsymbol{\lambda})$. In the following, we concentrate on (P_1). In other (P_j), $j \in \{2, 3\}$, we have the same results as in (P_1).

Let $f : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be a convex function. We denote its left and right derivatives and subdifferential, respectively, as $\frac{df(x)}{dx^-}$, $\frac{df(x)}{dx^+}$ and $\partial f(x)$. Namely,

$$\frac{df(x)}{dx^-} = \lim_{\alpha \uparrow 0} \frac{f(x+\alpha) - f(x)}{\alpha}, \quad \frac{df(x)}{dx^+} = \lim_{\alpha \downarrow 0} \frac{f(x+\alpha) - f(x)}{\alpha}$$

and

$$\partial f(x) = \left[\frac{df(x)}{dx^-}, \frac{df(x)}{dx^+}\right] \equiv \left\{ y \in \boldsymbol{R} : \frac{df(x)}{dx^-} \le y \le \frac{df(x)}{dx^+} \right\}.$$

Note that if f is differentiable at x_0, then $\partial f(x_0) = \left\{\frac{df(x_0)}{dx}\right\}$, and that x_0 minimizes f over $\boldsymbol{R}$ if and only if $0 \in \partial f(x_0)$ (see, for example, [5]).

For $x \in \boldsymbol{R}$, we put $L(x) \equiv \{i \in M : d_i^1 < x\}$, $R(x) \equiv \{i \in M : d_i^1 > x\}$ and $I(x) \equiv \{i \in M : d_i^1 = x\}$. The objective function of (P_1), g_1, is a piecewise linear convex function. It is not differentiable only at each d_k^1, $k \in M$, and we have

$$\left.\frac{dg_1(x)}{dx^+}\right|_{x=d_k^1} = \sum_{i \in L(d_k^1) \cup I(d_k^1)} \lambda^i - \sum_{i \in R(d_k^1)} \lambda^i, \quad \left.\frac{dg_1(x)}{dx^-}\right|_{x=d_k^1} = \sum_{i \in L(d_k^1)} \lambda^i - \sum_{i \in R(d_k^1) \cup I(d_k^1)} \lambda^i \tag{2}$$

and

$$\partial g_1(d_k^1) = \left[\sum_{i \in L(d_k^1)} \lambda^i - \sum_{i \in R(d_k^1) \cup I(d_k^1)} \lambda^i, \sum_{i \in L(d_k^1) \cup I(d_k^1)} \lambda^i - \sum_{i \in R(d_k^1)} \lambda^i \right].$$

We put $d_{\min} \equiv \min\{d_i^1 \colon i \in M\}$ and $d_{\max} \equiv \max\{d_i^1 \colon i \in M\}$. Note that

$$(3) \qquad \left.\frac{dg_1(x)}{dx^-}\right|_{x=d_k^1} = \left.\frac{dg_1(x)}{dx^+}\right|_{x=d_\ell^1} = \frac{dg_1(x)}{dx},\ x \in (d_\ell^1, d_k^1) \equiv \{y \in \boldsymbol{R} : d_\ell^1 < y < d_k^1\}$$

for d_k^1 and d_ℓ^1, where $d_k^1 > d_{\min}$ and $d_\ell^1 = \max\{d_i^1 \colon d_i^1 < d_k^1,\ i \in M\}$. From (2), $\frac{dg_1(x)}{dx} = -\sum_{i=1}^m \lambda^i < 0$ for $x < d_{\min}$ and $\frac{dg_1(x)}{dx} = \sum_{i=1}^m \lambda^i > 0$ for $x > d_{\max}$. Thus, we have the following lemma.

Lemma 1. *For any fixed $\boldsymbol{\lambda} > \boldsymbol{0}$, $S_1^*(\boldsymbol{\lambda}) \subset [d_{\min},\, d_{\max}]$.*

From (2), (3) and Lemma 1, (i) $S_1^*(\boldsymbol{\lambda}) = \{d_k^1\}$ for some d_k^1, $k \in M$; or (ii) $S_1^*(\boldsymbol{\lambda}) = [d_k^1,\, d_\ell^1]$ for some d_k^1, d_ℓ^1, k, $\ell \in M$ such that $d_k^1 < d_\ell^1$ and that $d_i^1 \le d_k^1$ or $d_\ell^1 \le d_i^1$ for any d_i^1, $i \in M$.

3. Properties of efficient solutions. In this section, we give some properties of efficient solutions of (P) by using properties of optimal solutions of ($\mathrm{P}_{\boldsymbol{\lambda}}$).

For $\boldsymbol{x}_0 \equiv (x_0^1,\, x_0^2,\, x_0^3)^T \in \boldsymbol{R}^3$, $\boldsymbol{x}_0$ is called *an intersection point* if $x_0^j \in \{d_i^j \colon i \in M\}$, $j \in J$. We denote the set of all intersection points as I, and put

$$d_{\min}^j \equiv \min\{d_i^j : i \in M\},\ d_{\max}^j \equiv \max\{d_i^j : i \in M\},\ j \in J.$$

Then

$$B \equiv \{(x^1, x^2, x^3)^T \in \boldsymbol{R}^3 : d_{\min}^j \le x^j \le d_{\max}^j, j \in J\}$$

is called *the intersection box* (see Figure 1). From Theorem 1 and Lemma 1, $E(D) \subset B$.

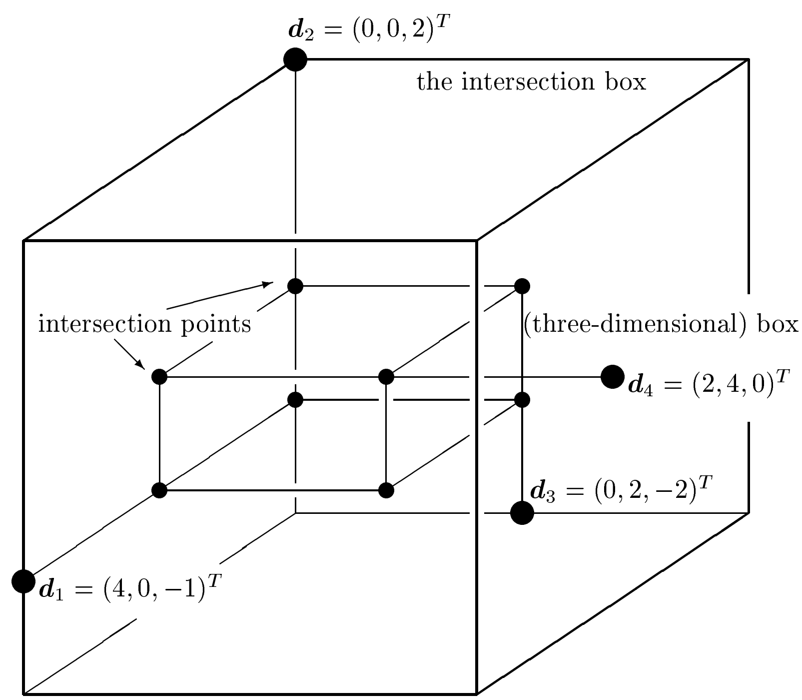


Figure 1. Intersection points, the intersection box, a box.

We put $\boldsymbol{e}_1 \equiv (1, 0, 0)^T$, $\boldsymbol{e}_2 \equiv (0, 1, 0)^T$, $\boldsymbol{e}_3 \equiv (0, 0, 1)^T \in \boldsymbol{R}^3$ and $\boldsymbol{x}_k \equiv (x_k^1, x_k^2, x_k^3)^T$, $\boldsymbol{x}_h \equiv (x_h^1, x_h^2, x_h^3)^T \in I$. For each $j \in J$, we call $\boldsymbol{x}_k$ *an* $\boldsymbol{e}_j$*-oriented intersection point* (resp. *a* $-\boldsymbol{e}_j$*-oriented intersection point*) *adjacent to* $\boldsymbol{x}_h$ if $x_k^j > x_h^j$ (resp. $x_k^j < x_h^j$), $x_k^{j'} = x_h^{j'}$, $j' \neq j$ and there is no $\boldsymbol{x}_s \equiv (x_s^1, x_s^2, x_s^3)^T \in I$ such that $x_h^j < x_s^j < x_k^j$ (resp. $x_k^j < x_s^j < x_h^j$).

For each $j \in J$, let $d_{[1]}^j, d_{[2]}^j, \cdots, d_{[m_j]}^j$ be all distinct real numbers among $d_1^j, d_2^j, \cdots, d_m^j$ such that $d_{[1]}^j < d_{[2]}^j < \cdots < d_{[m_j]}^j$, and we put

$$F_{2k-1}^j \equiv \left\{ d_{[k]}^j \right\}, \ k = 1, 2, \cdots, m_j$$

and

$$F_{2k}^j \equiv \left[d_{[k]}^j, d_{[k+1]}^j \right], \ k = 1, 2, \cdots, m_j - 1.$$

For each $k_j \in \{1, 2, \cdots, 2m_j - 1\}$, $j \in J$, $F_{k_1}^1 \times F_{k_2}^2 \times F_{k_3}^3$ is called *a box*. Moreover, if k numbers are odd among k_1, k_2, k_3, then $F_{k_1}^1 \times F_{k_2}^2 \times F_{k_3}^3$ is called *k-dimensional box* (see Figure 1).

For any fixed $\boldsymbol{\lambda} > \boldsymbol{0}$, $S^*(\boldsymbol{\lambda}) = F_{k_1}^1 \times F_{k_2}^2 \times F_{k_3}^3$ for some $k_j \in \{1, 2, \cdots, 2m_j - 1\}$, $j \in J$. Therefore, $E(D)$ is the union of some boxes. The union of all one-dimensional boxes in $E(D)$ is called *the frame of* $E(D)$.

Theorem 2. ([13]) *Let* $h_1(w)$ *and* $h_2(w)$ *be convex functions defined on* $\boldsymbol{R}$, *where* h_i *is minimized at* w_i, $i = 1, 2$ *and* $w_1 < w_2$. *Then, given any* $\overline{w} \in [w_1, w_2]$, *there exists* $\theta \in [0, 1]$ *such that* $\overline{w}$ *minimizes* $\theta h_2(w) + (1 - \theta)h_1(w)$.

Corollary 1. *For* $\boldsymbol{x}_1 \equiv (x_1^1, x_1^2, x_1^3)^T$, $\boldsymbol{x}_2 \equiv (x_2^1, x_2^2, x_2^3)^T \in B$, *it is assumed that* $x_1^{j_0} \neq x_2^{j_0}$ *for some* $j_0 \in J$ *and* $x_1^j = x_2^j$, $j \neq j_0$. *We put* $\boldsymbol{x}^* \equiv \alpha \boldsymbol{x}_1 + (1 - \alpha)\boldsymbol{x}_2$, $\alpha \in (0, 1)$. *If* $\boldsymbol{x}_1, \boldsymbol{x}_2 \in E(D)$, *then* $\boldsymbol{x}^* \in E(D)$.

Proof. Without loss of generality, we assume that

$$x_1^1 < x_2^1, \ \ x_1^j = x_2^j, \ j \in \{2, 3\}.$$

Since $\boldsymbol{x}_1$, $\boldsymbol{x}_2 \in E(D)$, $\boldsymbol{x}_1 \in S^*(\boldsymbol{\lambda}_1)$ and $\boldsymbol{x}_2 \in S^*(\boldsymbol{\lambda}_2)$ for some $\boldsymbol{\lambda}_1 > \boldsymbol{0}$ and $\boldsymbol{\lambda}_2 > \boldsymbol{0}$ by Theorem 1. Thus, $x_1^1 \in S_1^*(\boldsymbol{\lambda}_1)$, $x_2^1 \in S_1^*(\boldsymbol{\lambda}_2)$ and x_1^j, $x_2^j \in S_j^*(\boldsymbol{\lambda}_1) \bigcap S_j^*(\boldsymbol{\lambda}_2)$, $j \in \{2, 3\}$. For each $j \in \{2, 3\}$, if we put $x^{j*} \equiv \alpha x_1^j + (1 - \alpha)x_2^j$, then $x^{j*} \in S_j^*(\theta\boldsymbol{\lambda}_1 + (1 - \theta)\boldsymbol{\lambda}_2)$ for any $\theta \in [0, 1]$. We put $x^{1*} \equiv \alpha\, x_1^1 + (1 - \alpha)x_2^1$. Since $x_1^1 < x^{1*} < x_2^1$, there exists $\theta_1 \in [0, 1]$ such that $x^{1*} \in S_1^*(\theta_1\boldsymbol{\lambda}_1 + (1 - \theta_1)\boldsymbol{\lambda}_2)$ by Theorem 2. Since $\boldsymbol{x}^* = (x^{1*}, x^{2*}, x^{3*})^T \in S^*(\theta_1\boldsymbol{\lambda}_1 + (1 - \theta_1)\boldsymbol{\lambda}_2)$, $\boldsymbol{x}^* \in E(D)$ by Theorem 1. □

Theorem 3. *Let* $h_1(w)$ and $h_2(w)$ *be convex functions defined on* $\boldsymbol{R}$, *where* h_i *is minimized at* w_i, $i = 1, 2$ *and* $w_1 \leq w_2$. *Then, given any* $\overline{w} \in [w_1, w_2]$,

$$H \equiv \{\theta \in [0, 1] : \overline{w} \ \textit{minimizes} \ \theta h_2(w) + (1 - \theta)h_1(w)\}$$

is a closed interval.

Proof. Fix any $\overline{w} \in [w_1, w_2]$. If $w_1 = w_2$, then $H = [0, 1]$. Thus, we assume that $w_1 < w_2$. For $\theta \in [0, 1]$, $\overline{w}$ minimizes $\theta h_2(w) + (1 - \theta)h_1(w)$ if and only if

$$0 \in \partial(\theta h_2(\overline{w}) + (1 - \theta)h_1(\overline{w})) = \theta\partial h_2(\overline{w}) + (1 - \theta)\partial h_1(\overline{w}).$$

Holding the above equality is proven in [5]. If $y_1 \in \partial h_1(\overline{w})$ and $\overline{w} > w_1$, then $y_1 \geq 0$. If $y_2 \in \partial h_2(\overline{w})$ and $\overline{w} < w_2$, then $y_2 \leq 0$. These follow from the fact that $w' < w''$, $y' \in \partial h(w')$ and $y'' \in \partial h(w'')$ imply $y' \leq y''$ for any convex function h defined on $\boldsymbol{R}$ (see [5]).

(i) First, we suppose that $\overline{w} = w_1$. In this case, $\partial h_1(\overline{w}) = [a_1, a_2]$ and $\partial h_2(\overline{w}) = [-b_2, -b_1]$ for some $a_1, a_2, b_1, b_2 \in \boldsymbol{R}$ such that $a_1 \leq 0 \leq a_2$ and $0 \leq b_1 \leq b_2$. If $b_1 > 0$, then we have $H = [0, a_2/(b_1 + a_2)]$, otherwise $H = [0, 1]$.

(ii) Next, we suppose that $\overline{w} = w_2$. In this case, $\partial h_1(\overline{w}) = [a_1, a_2]$ and $\partial h_2(\overline{w}) = [b_1, b_2]$ for some $a_1, a_2, b_1, b_2 \in \boldsymbol{R}$ such that $0 \leq a_1 \leq a_2$ and $b_1 \leq 0 \leq b_2$. If $a_1 > 0$, then we have $H = [a_1/(a_1 - b_1), 1]$, otherwise $H = [0, 1]$.

(iii) Finally, we suppose that $w_1 < \overline{w} < w_2$. In this case, $\partial h_1(\overline{w}) = [a_1, a_2]$ and $\partial h_2(\overline{w}) = [-b_2, -b_1]$ for some $a_1, a_2, b_1, b_2 \in \boldsymbol{R}$ such that $0 \leq a_1 \leq a_2$ and $0 \leq b_1 \leq b_2$. If $a_1 + b_2 \neq 0$ and $b_1 + a_2 \neq 0$, then we have $H = [a_1/(a_1 + b_2), a_2/(b_1 + a_2)]$, otherwise $H = [0, 1]$. □

By Corollary 1, if all vertices of a box are efficient in (P), then any point in the box is efficient in (P).

Corollary 2. *For $\boldsymbol{x}_h \equiv (x_h^1, x_h^2, x_h^3)^T$, $\boldsymbol{x}_k \equiv (x_k^1, x_k^2, x_k^3)^T \in E(D) \bigcap I$, it is assumed that $J_0 \equiv \{j \in J : x_h^j \neq x_k^j\} \neq \emptyset$. For each $j \in J_0$, let $\boldsymbol{y}_j \equiv (y_j^1, y_j^2, y_j^3)^T$ be an $\boldsymbol{e}_j$-oriented intersection point (resp. a $-\boldsymbol{e}_j$-oriented intersection point) adjacent to $\boldsymbol{x}_k$* if $x_h^j > h_k^j$ *(resp. $x_h^j < x_k^j$). Then there exists $j_0 \in J_0$ such that $\boldsymbol{y}_{j_0} \in E(D)$.*

Proof. Without loss of generality, we assume that $x_h^j < x_k^j$, $j = 1, \cdots, s$ (≤ 3) and $x_h^j = x_k^j$, $j = s+1, \cdots, 3$. We shall show only the case $x_h^1 < x_k^1$, $x_h^2 < x_k^2$ and $x_h^3 = x_k^3$. In other cases, it can be shown similarly.

Since $\boldsymbol{x}_h, \boldsymbol{x}_k \in E(D)$, $\boldsymbol{x}_h \in S^*(\boldsymbol{\lambda}_h)$ and $\boldsymbol{x}_k \in S^*(\boldsymbol{\lambda}_k)$ for some $\boldsymbol{\lambda}_h \equiv (\lambda_h^1, \lambda_h^2, \cdots, \lambda_h^m)^T > \boldsymbol{0}$ and $\boldsymbol{\lambda}_k \equiv (\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^m)^T > \boldsymbol{0}$ by Theorem 1. Then, for any $j \in J$, $x_h^j \in S_j^*(\boldsymbol{\lambda}_h)$ and $x_k^j \in S_j^*(\boldsymbol{\lambda}_k)$. Since $x_h^1 \leq y_1^1 \leq x_k^1$, there exists $\theta_1 = \min\{\theta \in [0, 1] : y_1^1 \in S_1^*(\theta\boldsymbol{\lambda}_h + (1 - \theta)\boldsymbol{\lambda}_k)\}$ by Theorem 3. Then we shall show that $x_k^1 \in S_1^*(\delta\boldsymbol{\lambda}_h + (1 - \delta)\boldsymbol{\lambda}_k)$ for any $\delta \in [0, \theta_1]$. It is trivial when $\theta_1 = 0$. Thus, we assume that $\theta_1 > 0$. We put

$$\eta \equiv \frac{\delta}{\theta_1} \quad \text{and} \quad \boldsymbol{\lambda}_1 \equiv \theta_1\boldsymbol{\lambda}_h + (1 - \theta_1)\boldsymbol{\lambda}_k.$$

Then

$$\delta\boldsymbol{\lambda}_h + (1 - \delta)\boldsymbol{\lambda}_k,\ \delta \in [0, \theta_1]$$

can be expressed as

$$\eta\theta_1\boldsymbol{\lambda}_h + (1 - \eta\theta_1)\boldsymbol{\lambda}_k = \eta\boldsymbol{\lambda}_1 + (1 - \eta)\boldsymbol{\lambda}_k,\ \eta \in [0, 1].$$

We put

$$f_1(x) \equiv \sum_{i=1}^{m} \lambda_1^i |x - d_i^1| \text{ and } f_k(x) \equiv \sum_{i=1}^{m} \lambda_k^i |x - d_i^1|$$

where $\lambda_1^i \equiv \theta_1\lambda_h^i + (1 - \theta_1)\lambda_k^i$, $i \in M$. Since $y_1^1 \in S_1^*(\boldsymbol{\lambda}_1)$ and $x_k^1 \in S_1^*(\boldsymbol{\lambda}_k)$, $\partial f_1(y_1^1) = [a_1, a_2]$ and $\partial f_k(x_k^1) = [b_1, b_2]$ for some $a_1, a_2, b_1, b_2 \in \boldsymbol{R}$ such that $a_1 \leq 0 \leq a_2$ and $b_1 \leq 0 \leq b_2$. Since $y_1^1 < x_k^1$, $\partial f_1(x_k^1) = [a_1', a_2']$ and $\partial f_k(y_1^1) = [b_1', b_2']$ for some $a_1', a_2', b_1', b_2' \in \boldsymbol{R}$ such that $a_2 \leq a_1' \leq a_2'$ and $b_1' \leq b_2' \leq b_1$. For $\eta \in [0, 1]$, we put

$$G(\eta) \equiv \eta\partial f_1(x_k^1) + (1 - \eta)\partial f_k(x_k^1) = [b_1 + \eta(a_1' - b_1), b_2 + \eta(a_2' - b_2)]$$

and

$$H(\eta) \equiv \eta\partial f_1(y_1^1) + (1 - \eta)\partial f_k(y_1^1) = [b_1' + \eta(a_1 - b_1'), b_2' + \eta(a_2 - b_2')].$$

By definitions of θ_1 and η,

$$0 \in H(1), 0 \notin H(\eta), \eta \in [0,1] \setminus \{1\}. \tag{4}$$

Thus, it needs that $a_2 = 0$ and $b_2' < 0$. By Theorem 2, for $x_0 \in (y_1^1, x_k^1)$, there exists $\eta_0 \in [0, 1]$ such that $x_0 \in S_1^*(\eta_0\boldsymbol{\lambda}_1 + (1 - \eta_0)\boldsymbol{\lambda}_k)$. In this case, $[y_1^1, x_k^1] \subset S_1^*(\eta_0\boldsymbol{\lambda}_1 + (1 - \eta_0)\boldsymbol{\lambda}_k)$. Thus, we have

$$0 \in G(\eta_0) \bigcap H(\eta_0).$$

From (4), it needs that $\eta_0 = 1$. Since it needs that $a_1' = 0$, $0 \in G(\eta)$ for any $\eta \in [0, 1]$. By the definition of $G(\eta)$, $x_k^1 \in S_1^*(\eta\boldsymbol{\lambda}_1 + (1 - \eta)\boldsymbol{\lambda}_k)$ for any $\eta \in [0, 1]$. Namely, by definitions of η and $\boldsymbol{\lambda}_1$, $x_k^1 \in S_1^*(\delta\boldsymbol{\lambda}_h + (1 - \delta)\boldsymbol{\lambda}_k)$ for any $\delta \in [0, \theta_1]$. Similarly, there exists $\theta_2 = \min\{\theta \in [0, 1] : y_2^2 \in S_2^*(\theta\boldsymbol{\lambda}_h + (1 - \theta)\boldsymbol{\lambda}_k)\}$ by Theorem 3. Then $x_k^2 \in S_2^*(\delta\boldsymbol{\lambda}_h + (1 - \delta)\boldsymbol{\lambda}_k)$ for any $\delta \in [0, \theta_2]$.

On the other hand, since $x_k^3 \in S_3^*(\boldsymbol{\lambda}_h) \bigcap S_3^*(\boldsymbol{\lambda}_k)$, $x_k^3 \in S_3^*(\theta\boldsymbol{\lambda}_h + (1 - \theta)\boldsymbol{\lambda}_k)$ for any $\theta \in [0, 1]$. We put

$$\theta_{j_0} \equiv \min\{\theta_1, \theta_2\} \text{ and } \boldsymbol{\lambda}_{j_0} \equiv \theta_{j_0}\boldsymbol{\lambda}_h + (1 - \theta_{j_0})\boldsymbol{\lambda}_k.$$

Then $y_{j_0}^j \in S_j^*(\boldsymbol{\lambda}_{j_0})$, $j \neq j_0$ and $y_{j_0}^{j_0} \in S_{j_0}^*(\boldsymbol{\lambda}_{j_0})$ by the definition of θ_{j_0}. Since $\boldsymbol{y}_{j_0} \in S^*(\boldsymbol{\lambda}_{j_0})$, $\boldsymbol{y}_{j_0} \in E(D)$ by Theorem 1. □

From Corollary 1 and 2, there exists "zig-zag path" between any two efficient solutions of (P). Moreover, the frame of $E(D)$ is connected. If the frame of $E(D)$ is determined, then $E(D)$ can be constructed. Thus, we give an algorithm to find the frame of $E(D)$ in the next section.

4. Algorithm to Find All Efficient Solutions. In this section, we propose the Frame Generating Algorithm to find the frame of $E(D)$, which requires $O(m^4)$ computational time.

In the Frame Generating Algorithm, checking that an intersection point is efficient in (P) or not is needed. Thus, in the following, we state how to check it.

For $\boldsymbol{x} \in \boldsymbol{R}^3$, we put $B_{\boldsymbol{d}_i}(\boldsymbol{x}) = \{\boldsymbol{y} \in \boldsymbol{R}^3 \colon \|\boldsymbol{y} - \boldsymbol{d}_i\|_1 \leq \|\boldsymbol{x} - \boldsymbol{d}_i\|_1\}$, $i \in M$ and $B(\boldsymbol{x}) = \bigcap_{i=1}^m B_{\boldsymbol{d}_i}(\boldsymbol{x})$. By the definition of the efficiency, $\boldsymbol{x}_0 \in E(D)$ if and only if $B(\boldsymbol{x}_0)$ does not intersect the interior of any $B_{\boldsymbol{d}_i}(\boldsymbol{x}_0)$. For $\varepsilon > 0$ and $\boldsymbol{x} \in \boldsymbol{R}^3$, we put $D_\varepsilon(\boldsymbol{x}) \equiv N_\varepsilon(\boldsymbol{x}) \bigcap B(\boldsymbol{x})$, where $N_\varepsilon(\boldsymbol{x})$ is an ε-neighbourhood of $\boldsymbol{x}$. Then we have the following lemma.

Lemma 2. *A point $\boldsymbol{x}_0 \in \boldsymbol{R}^3$ is efficient in* (P) *if and only if $D_\varepsilon(\boldsymbol{x}_0)$ does not intersect the interior of any $B_{\boldsymbol{d}_i}(\boldsymbol{x}_0)$ for some $\varepsilon > 0$.*

Proof. If $\boldsymbol{x}_0 \in E(D)$, then $D_\varepsilon(\boldsymbol{x}_0)$ does not intersect the interior of any $B_{\boldsymbol{d}_i}(\boldsymbol{x}_0)$ for any $\varepsilon > 0$ by the definition of the efficiency.

Assume that $\boldsymbol{x}_0 \notin E(D)$. Then there exists $\boldsymbol{y} \in \boldsymbol{R}^3$ such that $\|\boldsymbol{y} - \boldsymbol{d}_i\|_1 \leq \|\boldsymbol{x}_0 - \boldsymbol{d}_i\|_1$, $i \in M$ and $\|\boldsymbol{y} - \boldsymbol{d}_k\|_1 < \|\boldsymbol{x}_0 - \boldsymbol{d}_k\|_1$ for some $k \in M$. For $\alpha \in (0, 1)$, $\|(1 - \alpha)\boldsymbol{x}_0 + \alpha\boldsymbol{y} - \boldsymbol{d}_i\|_1 \leq \|\boldsymbol{x}_0 - \boldsymbol{d}_i\|_1$, $i \in M$ by the convexity of $\|\cdot\|_1$. Since $\|\boldsymbol{y} - \boldsymbol{d}_k\|_1 < \|\boldsymbol{x}_0 - \boldsymbol{d}_k\|_1$, $\boldsymbol{y}$ is an interior point of $B_{\boldsymbol{d}_k}(\boldsymbol{x}_0)$. Thus, we have $\|(1 - \alpha)\boldsymbol{x}_0 + \alpha\boldsymbol{y} - \boldsymbol{d}_k\|_1 < \|\boldsymbol{x}_0 - \boldsymbol{d}_k\|_1$. For any $\varepsilon > 0$, if α is sufficiently small, then $(1 - \alpha)\boldsymbol{x}_0 + \alpha\boldsymbol{y} \in D_\varepsilon(\boldsymbol{x}_0)$ is an interior point of $B_{\boldsymbol{d}_k}(\boldsymbol{x}_0)$. □

Following [12], we introduce the concept of the summary diagram in order to check that an intersection point is efficient in (P) or not by applying Lemma 2. In [12], the summary diagram is introduced for multicriteria location problems with one-infinity norm in $\boldsymbol{R}^2$. We put

$$\begin{aligned}
O_1 &\equiv \{(x^1, x^2, x^3)^T \in \boldsymbol{R}^3 : x^1 \geq 0, x^2 \geq 0, x^3 \geq 0\},\\
O_2 &\equiv \{(x^1, x^2, x^3)^T \in \boldsymbol{R}^3 : x^1 \leq 0, x^2 \geq 0, x^3 \geq 0\},\\
O_3 &\equiv \{(x^1, x^2, x^3)^T \in \boldsymbol{R}^3 : x^1 \leq 0, x^2 \leq 0, x^3 \geq 0\},\\
O_4 &\equiv \{(x^1, x^2, x^3)^T \in \boldsymbol{R}^3 : x^1 \geq 0, x^2 \leq 0, x^3 \geq 0\}
\end{aligned}$$

and $O_{-\eta} \equiv -O_\eta$, $\eta = 1, 2, 3, 4$. For $\boldsymbol{x} \in \boldsymbol{R}^3$, *the summary diagram* of $\boldsymbol{x}$, $SD(\boldsymbol{x})$, is defined as follows:

$$SD(\boldsymbol{x}) \equiv \{\eta \in \{\pm 1, \pm 2, \pm, 3, \pm 4\} : \boldsymbol{d}_i \in O_\eta(\boldsymbol{x}) \text{ for some } i\}.$$

Conveniently, $SD(\boldsymbol{x})$ is represented in diagram form as follows: First, we draw the cube with vertices $\boldsymbol{v}_1 \equiv (1, 1, 1)^T$, $\boldsymbol{v}_2 \equiv (-1, 1, 1)^T$, $\boldsymbol{v}_3 \equiv (-1, -1, 1)^T$, $\boldsymbol{v}_4 \equiv (1, -1, 1)^T$ and $\boldsymbol{v}_{-\eta} \equiv -\boldsymbol{v}_\eta$, $\eta = 1, 2, 3, 4$; Next, for each $\eta \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$, dot $\boldsymbol{v}_\eta$ if $\eta \in SD(\boldsymbol{x})$. For example, $SD(\boldsymbol{x}) = \{2, 3, 4, -2, -3\}$ for $\boldsymbol{d}_1 = (3, 0, 4)^T$, $\boldsymbol{d}_2 = (4, 2, 0)^T$, $\boldsymbol{d}_3 = (2, 1, 3)^T$, $\boldsymbol{d}_4 = (0, 4, 5)^T$, $\boldsymbol{d}_5 = (1, 5, 2)^T$ and $\boldsymbol{x} = (3, 2, 1)^T$. Figure 2 shows its summary diagram in diagram form.

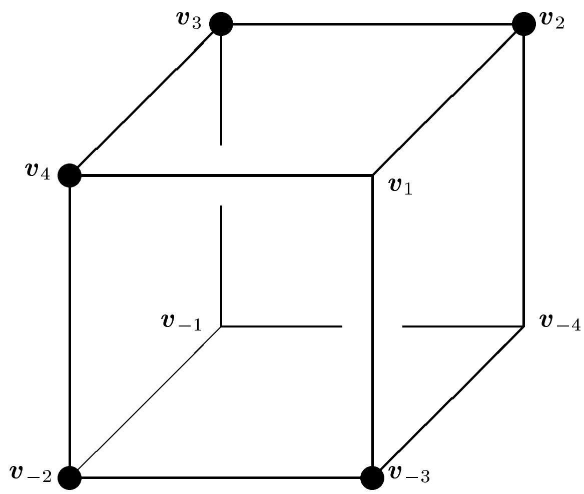
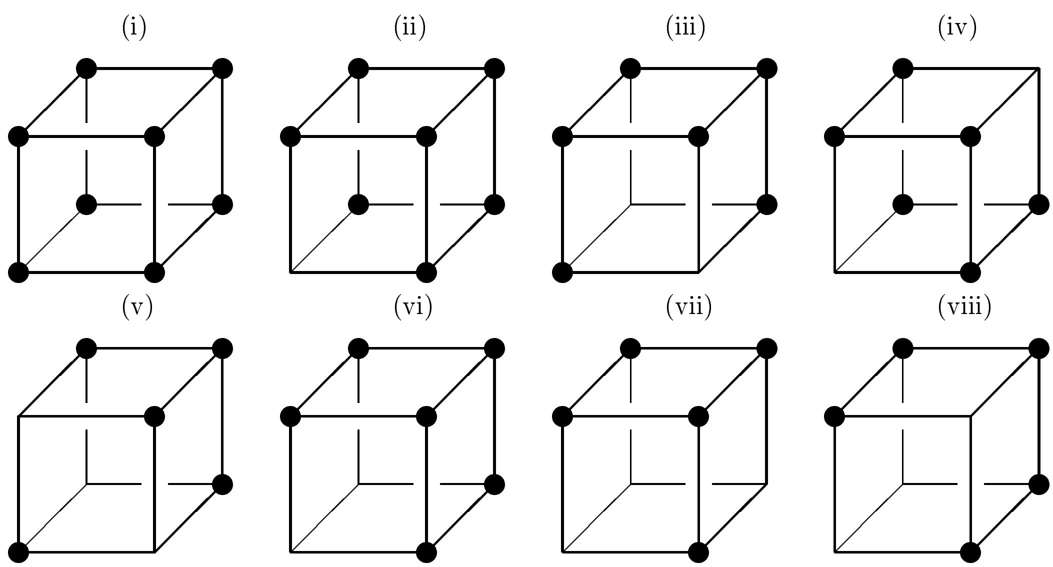


Figure 2. $SD(\boldsymbol{x}) = \{2, 3, 4, -2, -3\}$.



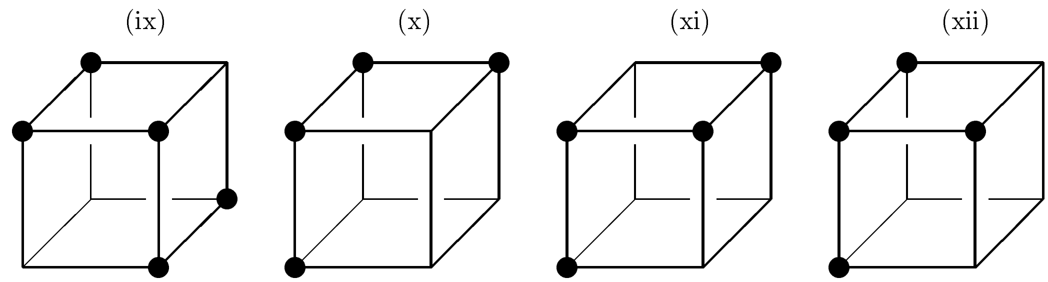


Figure 3. Patterns of summary diagrams of an intersection point.

For $\boldsymbol{x}_0 \equiv (x_0^1, x_0^2, x_0^3)^T \in I$, $SD(\boldsymbol{x}_0)$ coincides with one of patterns illustrated in Figure 3, where we identify summary diagrams in diagram form if they are the same pattern by rotation.

If the pattern of $SD(\boldsymbol{x}_0)$ is one of (i)-(v) in Figure 3, then the $D_\varepsilon(\boldsymbol{x}_0) = \{\boldsymbol{x}_0\}$ for $\varepsilon > 0$. In this case, $\boldsymbol{x}_0 \in E(D)$ by Lemma 2. If the pattern of $SD(\boldsymbol{x}_0)$ is (xii) in Figure 3, then the interior of $D_\varepsilon(\boldsymbol{x}_0)$ is not empty. In this case, $\boldsymbol{x}_0 \notin E(D)$ by Lemma 2. If the pattern of $SD(\boldsymbol{x}_0)$ is one of (vi)-(xi) in Figure 3, then $\boldsymbol{x}_0 \notin E(D)$ if and only if there exists $i \in M$ satisfying one of conditions in Table 1, where it is assumed that $\boldsymbol{d}_i - \boldsymbol{x}_0$, $i \in M$ are rotated to fit the pattern of the summary diagram. In Table 1, we put

$$s_{ij} \equiv \begin{cases} + & \text{if } d_i^j - x_0^j > 0, \\ 0 & \text{if } d_i^j - x_0^j = 0, \\ - & \text{if } d_i^j - x_0^j < 0 \end{cases}$$

for $i \in M$ and $j \in J$. For example, when $SD(\boldsymbol{x}_0) = \{1, 2, 3, 4, -3, -4\}$ whose pattern is (vi), $\boldsymbol{x}_0 \notin E(D)$ if and only if there exists $i \in M$ such that $(s_{i1}, s_{i2}, s_{i3}) = (0, +, +)$ or $(+, +, +)$ or $(-, +, +)$.

Given $\boldsymbol{x}_0 \in I$, the pattern of $SD(\boldsymbol{x}_0)$ can be determined in $O(m)$ computational time by comparing components of $\boldsymbol{x}_0$ and each $\boldsymbol{d}_i$, $i \in M$. Namely, checking that $\boldsymbol{x}_0 \in E(D)$ or not requires $O(m)$ computational time.

Table 1. Necessary and sufficient conditions of $\boldsymbol{x}_0 \notin E(D)$ in patterns (vi)-(xi).

Pattern	(s_{i1}, s_{i2}, s_{i3})				
(vi)	(a) $(0,+,+)$	(b) $(+,+,+)$	(c) $(-,+,+)$		
(vii)	(a) $(+,0,+)$	(b) $(0,+,+)$	(c) $(+,+,+)$	(d) $(+,-,+)$	(e) $(-,+,+)$
(viii)	(a) $(-,0,+)$	(b) $(-,+,0)$	(c) $(-,+,+)$		
(ix)	(a) $(+,+,0)$	(b) $(+,0,+)$	(c) $(+,+,+)$		
(x)	(a) $(0,-,+)$	(b) $(+,-,+)$	(c) $(-,-,+)$		
(xi)	(a) $(+,0,+)$	(b) $(+,+,+)$	(c) $(+,-,+)$		

Remark. In view of the fact that the frame of $E(D)$ is the union of all one-dimensional boxes in $E(D)$, which is connected, we can construct a connected graph (V, E), where $V = I \bigcap E(D)$ and E is the set of all arcs in the graph. Given $\boldsymbol{x}_1$, $\boldsymbol{x}_2 \in I \bigcap E(D)$, the arc $a(\boldsymbol{x}_1, \boldsymbol{x}_2)$ which connects $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is in E if and only if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are adjacent and efficient in (P). This concept will be the guide for describing an algorithm to locate the frame of $E(D)$.

In the Frame Generating Algorithm, we put

$$V_i \equiv \{(d_{[k]}^1, d_{[\ell]}^2, d_{[i]}^3)^T \in \boldsymbol{R}^3 : k \in \{1, 2, \cdots, m_1\}, \ell \in \{1, 2, \cdots, m_2\}\},\ i = 1, 2, \cdots, m_3.$$

We use r as a counter which represents the number of iterations. For each $r \in \{1, 2, \cdots, m_3\}$, the Frame Generating Algorithm finds one-dimensional boxes in the frame of $E(D)$, which are connected with any initial point $\boldsymbol{d}_{k_r} \in V_r$ through only intersection points in $V_r \bigcap E(D)$. $L_r \subset V_r$ is the set of checked intersection points which are connected with the initial point through only intersection points in $V_r \bigcap E(D)$. $S_r \subset L_r$ is the set of intersection points which have been checked that one-dimensional boxes connected with them are contained $E(D)$ or not. D_r, $G_r \subset V_r$ are the sets of checked intersection points which are efficient and not efficient in (P), respectively. We use T as the union of one-dimensional boxes in $E(D)$ which have been checked before. Moreover, for $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^3$, we put $[\boldsymbol{x}, \boldsymbol{y}] \equiv \{(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} : \lambda \in [0,1]\}$.

The Frame Generating Algorithm

Step 0. Set $S_i = \emptyset$, $G_i = \emptyset$, $D_i = D \bigcap V_i$, $i = 1, 2, \cdots, m_3$. For each $i \in \{1, 2, \cdots, m_3\}$, choose any $\boldsymbol{d}_{k_i} \in D_i$ and set $L_i = \{\boldsymbol{d}_{k_i}\}$. Set $T = \emptyset$ and $r = 1$.

Step 1. If $L_r = S_r$, then set $r = r + 1$. If $r > m_3$, then stop. (T is the frame of $E(D)$.)

Step 2. Choose any $\boldsymbol{x}_0 = (d^1_{[k]}, d^2_{[\ell]}, d^3_{[r]})^T \in L_r \setminus S_r$ and set $S_r = S_r \bigcup \{\boldsymbol{x}_0\}$.

Step 3. Set $W = \emptyset$.

(a) If $k > 1$, then set $\boldsymbol{x}_{-1} = (d^1_{[k-1]}, d^2_{[\ell]}, d^3_{[r]})^T$ and $W = W \bigcup \{\boldsymbol{x}_{-1}\}$.

(b) If $k < m_1$, then set $\boldsymbol{x}_1 = (d^1_{[k+1]}, d^2_{[\ell]}, d^3_{[r]})^T$ and $W = W \bigcup \{\boldsymbol{x}_1\}$.

(c) If $\ell > 1$, then set $\boldsymbol{x}_{-2} = (d^1_{[k]}, d^2_{[\ell-1]}, d^3_{[r]})^T$ and $W = W \bigcup \{\boldsymbol{x}_{-2}\}$.

(d) If $\ell < m_2$, then set $\boldsymbol{x}_2 = (d^1_{[k]}, d^2_{[\ell+1]}, d^3_{[r]})^T$ and $W = W \bigcup \{\boldsymbol{x}_2\}$.

Step 4. If $W = \emptyset$, then go to Step 6, otherwise choose any $\boldsymbol{x}_\eta \in W$ and set $W = W \setminus \{\boldsymbol{x}_\eta\}$.

Step 5. If $[\boldsymbol{x}_0, \boldsymbol{x}_\eta] \subset T$, then go to Step 4.

(a) If $\boldsymbol{x}_\eta \in D_r$, then set $T = T \bigcup [\boldsymbol{x}_0, \boldsymbol{x}_\eta]$, and if $\boldsymbol{x}_\eta \notin L_r$ then $L_r = L_r \bigcup \{\boldsymbol{x}_\eta\}$, and go to Step 4.

(b) If $\boldsymbol{x}_\eta \notin G_r$, then check that $\boldsymbol{x}_\eta \in E(D)$ or not by using its summary diagram. If $\boldsymbol{x}_\eta \in E(D)$, then set $T = T \bigcup [\boldsymbol{x}_0, \boldsymbol{x}_\eta]$, $D_r = D_r \bigcup \{\boldsymbol{x}_\eta\}$ and $L_r = L_r \bigcup \{\boldsymbol{x}_\eta\}$, otherwise set

$$G_r = \begin{cases} G_r \bigcup \{(d^1_{[p]}, d^2_{[\ell]}, d^3_{[r]})^T : p = 1, 2, \cdots, k-1\} & \text{if } \eta = -1, \\ G_r \bigcup \{(d^1_{[p]}, d^2_{[\ell]}, d^3_{[r]})^T : p = k+1, \cdots, m_1\} & \text{if } \eta = 1, \\ G_r \bigcup \{(d^1_{[k]}, d^2_{[p]}, d^3_{[r]})^T : p = 1, 2, \cdots, \ell-1\} & \text{if } \eta = -2, \\ G_r \bigcup \{(d^1_{[k]}, d^2_{[p]}, d^3_{[r]})^T : p = \ell+1, \cdots, m_2\} & \text{if } \eta = 2. \end{cases}$$

Go to Step 4.

Step 6. If $r < m_3$, then set $\boldsymbol{x}_3 = (d^1_{[k]}, d^2_{[\ell]}, d^3_{[r+1]})^T$, otherwise go to Step 1.

(a) If $\boldsymbol{x}_3 \in D_{r+1}$, then set $T = T \bigcup [\boldsymbol{x}_0, \boldsymbol{x}_3]$ and go to Step 1.

(b) Check that $\boldsymbol{x}_3 \in E(D)$ or not by using its summary diagram. If $\boldsymbol{x}_3 \in E(D)$, then set $T = T \bigcup [\boldsymbol{x}_0, \boldsymbol{x}_3]$ and $D_{r+1} = D_{r+1} \bigcup \{\boldsymbol{x}_3\}$, otherwise set $G_p = G_p \bigcup \{(d^1_{[k]}, d^2_{[\ell]}, d^3_{[p]})^T\}$, $p = r+1, \cdots, m_3$. Go to Step 1.

In Step 0, for each $j \in J$, we can obtain $d^j_{[1]}$, $d^j_{[2]}$, $\cdots$, $d^j_{[m_j]}$ by sorting m real numbers d^j_1, d^j_2, $\cdots$, d^j_m which requires $O(m \log m)$ computational time [1]. Then V_i, D_i, $i = 1, 2, \cdots, m_3$ are determined. Thus, D_i, $i = 1, 2, \cdots, m_3$ can be determined in $O(m \log m)$ computational time. In rth iteration, the Frame Generating Algorithm checks that intersection points adjacent to each intersection points in $L_r \subset V_r$ are efficient in (P) or not by using their summary diagrams. The number of iterations is $O(m)$. The number of intersection points in L_r is $O(m^2)$ and the number of intersection points adjacent to an intersection point, which should be checked, is at most five. Checking that an intersection point is efficient in (P) or not by using its summary diagram requires $O(m)$ computational time. Therefore, the Frame Generating Algorithm requires $O(m^4)$ computational time.

Finally, we consider an example problem for $\boldsymbol{d}_1 = (3, 0, 4)^T$, $\boldsymbol{d}_2 = (4, 2, 0)^T$, $\boldsymbol{d}_3 = (2, 1, 3)^T$, $\boldsymbol{d}_4 = (0, 4, 5)^T$ and $\boldsymbol{d}_5 = (1, 5, 2)^T$. Applying the Frame Generating Algorithm for the multicriteria location problem (P), we have the frame of $E(D)$ illustrated in Figure 4.

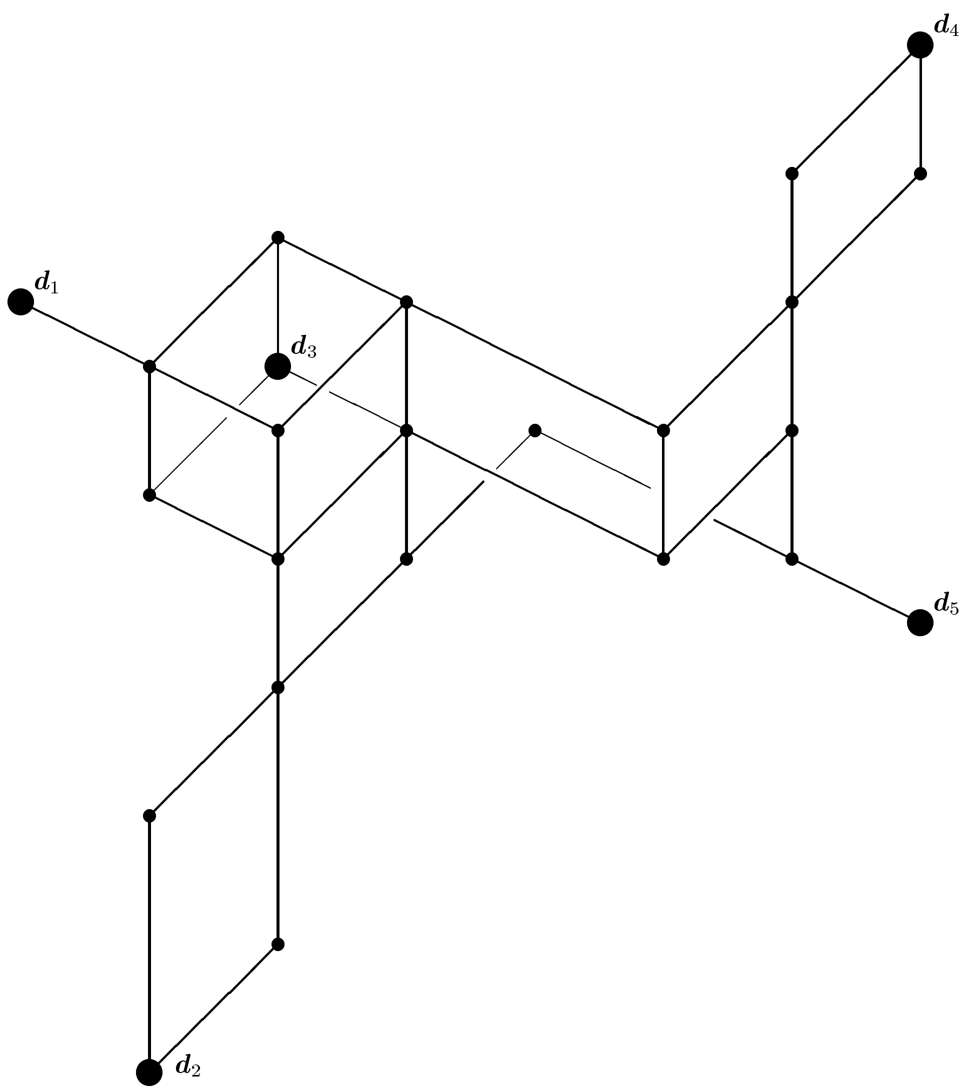


Figure 4. The frame of $E(D)$. ($\bullet$: intersection points in $E(D)$)

5. Conclusions. We dealt with multicriteria and minisum location problems with rectilinear norm in $\boldsymbol{R}^3$. Our main interest was to find $E(D)$. First, as corollaries of Theorem 2 and 3, we obtained characterizations of efficient solutions of (P) by using optimal solutions of $(\mathrm{P}_{\boldsymbol{\lambda}})$. They guarantee that $E(D)$ can be determined by the frame of $E(D)$ and that the frame of $E(D)$ is connected. Next, we introduced the concept of the summary diagram to check that an intersection point is efficient in (P) or not. We can check that an intersection point is efficient in (P) or not according to the pattern of its summary diagram. Finally, based on these results, we proposed the Frame Generating Algorithm to find the frame of $E(D)$. The Frame Generating Algorithm generates the frame of $E(D)$ by tracing one-dimensional boxes in $E(D)$.

References

[1] A. V. Aho, J. E. Hopcroft and J. D. Ullman, *The design and analysis of computer algorithms*, Addison-Wesley, Reading MA (1974).

[2] L. G. Chalmet, R. L. Francis and A. Kolen, *Finding efficient solutions for rectilinear distance location problems efficiently*, Eur. J. Oper. Res., **6** (1981), 117-124.

[3] Z. Drezner and G. O. Wesolowsky, *The asymmetric distance location problem*, Trans. Sci., **23** (1989), 201-207.

[4] M. Fukushima, *Introduction to mathematical programming*(in Japanese), Asakura Syoten, Japan (1996).

[5] J. -B. Hiriart-Urruty and C. Lemaréchal, *Convex analysis and minimization algorithms I: Fundamen- tals*, Springer-Verlag, Berlin, 1993.

[6] M. Kon, *Efficient solutions for multicriteria location problems under the block norm*, Mathematica Japonica, **47** (1998), 295-303.

[7] M. Kon and S. Kushimoto, *On efficient solutions of multicriteria location problems with the block norm*, Scientiae Mathematicae, **2** (1999), 245-254.

[8] T. J. Lowe, J. -F. Thisse, J. E. Ward and R. E. Wendell, *On efficient solutions to multiple objective mathematical programs*, Manage. Sci., **30** (1984), 1346-1349.

[9] O. L. Mangasarian, *Nonlinear programming*, McGraw-Hill, New York, 1969.

[10] T. Matsutomi and H. Ishii, *Fuzzy facility location problem with asymmetric rectilinear distance*(in Japanese), Journal of Japan Society for Fuzzy Theory and Systems, **8** (1996), 57-64.

[11] B. Pelegrin and F. R. Fernandez, *Determination of efficient points in multiple-objective location prob-lems*, Nav. Res. Logist. q., **35** (1988), 697-705.

[12] J. E. Ward and R. E. Wendell, *Characterizing efficient points in location problems under one-infinity norm*, in, J. -F. Thisse and H. G. Zoller, Eds., *Locational analysis of public facilities*, North Holland, Amsterdam (1983), 413-429.

[13] R. E. Wendell, A. P. Hurter, Jr. and T. J. Lowe, *Efficient points in location problems*, AIIE Trans., **9** (1977), 238-246.

[14] P. L. Yu and M. Zeleny, *The set of all nondominated solutions in linear cases and a multicriteria simplex method*, J. Math. Anal. Appl., **49** (1975), 430-468.

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