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# COMPLEX ROTUNDITY OF MUSIELAK-ORLICZ SEQUENCE SPACES EQUIPPED WITH THE ORLICZ NORM

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ABSTRACT. The criteria for complex rotundity, complex local uniformly rotund points, complex local uniform rotundity or complex uniform rotundity in complex Musielak-Orlicz sequence spaces equipped with the Orlicz norm are given.

## 0. Introduction

In the recent years, many mathematicians have developed the investigations concerning the geometric theory of complex Banach spaces, because its applications are irreplaceable by the geometric theory of real Banach spaces. In 1967, E. Thorp and R. Whitley (see [13]) first investigated the structure of complex extreme points. In 1975, J. Globevnik (see [6]) investigated complex rotundity and complex uniform rotundity, and pointed out that $L_1[0,1]$ is complex uniformly rotund (real space $L_1[0,1]$ is not even rotund). Many mathematicians discussed complex convexity in general Banach spaces (see [1]-[2], [4]-[6], [8], [10], [12], [14]). It is well known that into the class of Musielak-Orlicz spaces include a lot of classical spaces such as $L_p(1 \leq p \leq \infty)$, Orlicz spaces etc.. At the end of 1980's, H. Sun and C. Wu discussed complex extreme points, complex rotundity and complex uniform rotundity (see [15]-[19]) in Musielak-orlicz spaces. Next T. Wang introduced the concepts of complex locally uniformly rotund points and complex local uniform rotundity, and obtained criteria for them in Musielak-Orlicz spaces. But the above discussion in Musielak-Orlicz spaces was proceeded in the case of the Luxemburg norm. For the Orlicz norm, only one result on complex extreme points in Musielak-Orlicz sequence spaces was given by C. Wu and H. Sun (see [15]) in 1991. In this paper, we discuss complex rotundity, complex locally uniformly rotund points, complex local uniform rotundity and complex uniform rotundity in Musielak-Orlicz sequence spaces equipped with the Orlicz norm. The conclusions that we get seem to be clear and they are much different from the corresponding results concerning the Luxemburg norm.

Let $\mathcal{N}$ denote the set of natural numbers, $\mathcal{R}$, $\mathcal{R}_+$ and $\mathcal{C}$ denote the sets of real, nonnegative real and complex numbers, respectively. Let $(X, \|\cdot\|)$ be a complex Banach space and $S(X)$ be the unit sphere of $X$. Let $l^0$, $l^c$ be the space of all real or complex sequences, respectively.

A point $x$ in $S(X)$ is called a complex extreme point if for any $y \in X$ with $y \neq 0$ there holds $\max_{|\lambda|\leq 1} \|x + \lambda y\| > 1$. A complex Banach space $X$ is called complex rotund (**CR** for short) if every point $x$ in $S(X)$ is a complex extreme point. A point $x$ in $S(X)$ is called a complex locally uniformly rotund point (**CLUR** point for short) if for any $\varepsilon > 0$ there exists a positive constant $\delta = \delta(x, \varepsilon)$ such that for all $y$ in $X$ satisfying $\|y\| > \varepsilon$, there holds $\max_{|\lambda|\leq 1} \|x + \lambda y\| \geq 1 + \delta$. A complex Banach space $X$ is called complex locally uniformly

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rotund (**CLUR** for short) if every point $x$ in $S(X)$ is a **CLUR** point. A complex Banach space $X$ is called complex uniformly rotund (**CUR** for short) if for any $\varepsilon > 0$ there exists a positive constant $\delta = \delta(\varepsilon)$ such that $\max_{|\lambda| \le 1} \|x + \lambda y\| \ge 1 + \delta$ holds for all $x$ in $S(X)$ and $y$ in $X$ satisfying $\|y\| > \varepsilon$.

A mapping $M = (M_i)_{i=1}^{\infty} : \mathcal{R} \times \mathcal{N} \to [0, +\infty]$ is called a Musielak-Orlicz function if for every $i \in \mathcal{N}$, $M_i$ is an Orlicz function, i.e. $M_i : \mathcal{R} \to [0, +\infty]$ is even, convex, vanishing at zero, left continuous on $\mathcal{R}_+$, $M_i(\infty) = +\infty$ and not identical equal to zero and infinity.

For any Musielak-Orlicz $M = (M_i)_{i=1}^{\infty}$, we define the complementary function $N = (N_i)_{i=1}^{\infty}$ of $M$ by

$$N_i(v) = \sup_{u \ge 0} \{u|v| - M_i(u)\} \ (\forall i \in \mathcal{N},\ \forall v \in \mathcal{R}).$$

$N$ is also a Musielak-Orlicz function (see [3] and [11]).

For any $i \in \mathcal{N}$, we denote by $p_{-,i}(\cdot)$ and $p_i(\cdot)$ the left and right derivatives of $M_i(\cdot)$ on $\mathcal{R}_+$, denote by $q_{-,i}(\cdot)$ and $q_i(\cdot)$ the left and right derivatives of $N_i(\cdot)$ on $\mathcal{R}_+$, respectively. It is known that there holds the Young inequality $|uv| \le M_i(u) + N_i(v)$ and $|uv| = M_i(u) + N_i(v)$ if and only if $p_{-,i}(u) \le |v| \le p_i(u)$ or $q_{-,i}(v) \le |u| \le q_i(v)$ $(\forall i \in \mathcal{N},\ \forall u, v \in \mathcal{R})$. For the convenience, we write

$$(p_- \circ u)(i) = p_{-,i}(u(i)),\ \ (p \circ u)(i) = p_i(u(i)),$$

$$(q_- \circ v)(i) = q_{-,i}(v(i)),\ \ (q \circ v)(i) = q_i(u(i))$$

for any $u, v \in l^0$ and $i \in \mathcal{N}$. For every $i$ in $\mathcal{N}$, define

$$\begin{aligned} e(i) &= \sup\{u \ge 0 : M_i(u) = 0\}, \\ E(i) &= \sup\{u \ge 0 : M_i(u) < \infty\}, \\ a(i) &= \sup\{v \ge 0 : N_i(v) = 0\}, \\ A(i) &= \sup\{v \ge 0 : N_i(v) < \infty\}, \end{aligned}$$

$$(p \circ E)(i) = \infty,\ \ (p_- \circ u)(i) = \infty \ \text{ for } \ u > E(i),$$

$$(q \circ A)(i) = \infty,\ \ (q_- \circ v)(i) = \infty \ \text{ for } \ v > A(i).$$

Given a Musielak-Orlicz function $M = (M_i)_{i=1}^{\infty}$, if we define the convex modular $\rho_M$ on $l^c$ by $\rho_M(x) = \sum_{i=1}^{\infty} M_i(|x(i)|)$, then the linear space $\{x \in l^c : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$ equipped with the Luxemburg norm

$$\|x\|_M = \inf\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) \le 1\}$$

or with the Amemiya-Orlicz norm

$$\|x\|_M^0 = \inf_{k > 0} \frac{1}{k}(1 + \rho_M(kx)),$$

is a complex Banach space (see [3], [9] and [11]). We denote it by $l_M$ or $l_M^0$, respectively.

Note that if there exists $M$ such that $M_i(u) = M(u)$ for any $u \in \mathcal{R}$ and $i \in \mathcal{N}$, then $l_M$ becomes an Orlicz space (see [3], [9] and [11]). It is known that $\|x\|_M^0 = \sup\{\langle |x|, |y| \rangle : \rho_N(y) \le 1\}$ which is called the Orlicz norm for any $x \in l_M$, where $\langle |x|, |y| \rangle = \sum_{i=1}^{\infty} |x(i) y(i)|$ (see [3]).

The linear subspace $h_M$ of $l_M$ defined by

$$h_M = \{x = (x(i)) \in l^c : \forall_{\lambda > 0} \exists_{i_\lambda \in \mathcal{N}} \sum_{i = i_\lambda}^{\infty} M_i(\lambda |x(i)|) < \infty\}$$

equipped with the Luxemburg norm or with the Orlicz norm is also a complex Banach space. We denote it by $h_M$ or $h_M^0$, respectively.

Let $\mathcal{N}_0$ be any infinite subset of $\mathcal{N}$. We say that a Musielak-Orlicz function $M$ satisfies the condition $\delta_2^{\mathcal{N}_0}$ ($M \in \delta_2^{\mathcal{N}_0}$ for short) if for any $h > 1$, there exist $a > 0$, $k > 1$, $i_0 \in \mathcal{N}$ and a nonnegative sequence $(c_i)$ $(i \in \mathcal{N}_0, i > i_0)$ with $\sum_{i\in\mathcal{N}_0, i>i_0} c_i < \infty$ such that

$$M_i(hu) \le kM_i(u) + c_i$$

holds whenever $i \in \mathcal{N}_0$, $i > i_0$ and $M_i(u) \le a$. If $M \in \delta_2^{\mathcal{N}}$, we write simply $M \in \delta_2$.

For any $x \in l_M^0$, we define

$$\xi_M(x) = \inf\{\lambda > 0 : \exists_{i_\lambda \in \mathcal{N}} \sum_{i=i_\lambda}^{\infty} M_i(\frac{|x(i)|}{\lambda}) < \infty\},$$
$$k_x^* = \inf\{k \ge 0 : \rho_N(p \circ |kx|) \ge 1\},$$
$$k_x^{**} = \sup\{k \ge 0 : \rho_N(p \circ |kx|) \le 1\}.$$

It is known that $\|x\|_M^0 = \frac{1}{k}(1 + \rho_M(kx))$ if and only if $k_x^* \le k \le k_x^{**}$ and $0 < k < \infty$ (see [3]).

The following results will play a leading role in this paper.

**Lemma 0.1** (see [19], Proposition 5.17). *Let $i$ be a complex number satisfying $i^2 = -1$. For any $\varepsilon > 0$ there exists a positive constant $\delta \in (0, \frac{1}{2})$ such that if $u, v \in \mathcal{C}$ with*

$$|v| \ge \frac{\varepsilon}{8} \max_j |u + jv|,$$

*then*

$$|u| \le \frac{1 - 2\delta}{4} \sum_j |u + jv|,$$

*where*

$$\sum_j |u + jv| := |u + v| + |u - v| + |u + iv| + |u - iv|,$$
$$\max_j |u + jv| := \max\{|u + v|, |u - v|, |u + iv|, |u - iv|\}.$$

**Lemma 0.2** (see [15], Theorem 1). *Let $0 \ne x \in l_M^0$.*

*(1) If $\rho_N(A\chi_{S_x}) > 1$, then the only form for $\|x\|_M^0$ is $\|x\|_M^0 = \frac{1}{k}(1 + \rho_M(kx))$,*

*(2) If $\rho_N(A\chi_{S_x}) \le 1$, then $\|x\|_M^0 = \langle |x|, A\rangle$ and if $\rho_N(A\chi_{S_x}) < 1$, then it is the only form for $\|x\|_M^0$, where $S_x = \{i \in \mathcal{N} : x(i) \ne 0\}$ and $\chi_{S_x}$ is the characteristic function on $S_x$.*

## 1. Results

**Theorem 1.1**. *The space $l_M^0$ is complex rotund if and only if $e(j) > 0$ implies*

$$\rho_N(A\chi_{\mathcal{N}\setminus\{j\}}) < 1 \ \text{or}\ \rho_N(A\chi_{\mathcal{N}\setminus\{j\}}) = 1 \ \text{and}\ \rho_M(q_- \circ A\chi_{\mathcal{N}\setminus\{j\}}) = \infty.$$

*Proof.* Necessity. Let first $e(j) > 0$ and $\rho_N(A\chi_{\mathcal{N}\setminus\{j\}}) > 1$. We can find $x \in S(l_M^0)$ with $S_x = \{i \in \mathcal{N} : i \ne j\}$. By Lemma 0.2, there exists $k > 0$ such that $\|x\|_M^0 = \frac{1}{k}(1 + \rho_M(kx))$.

But $k|x(j)| = 0 < e(j)$. So $x$ is not a complex extreme point (see Theorem 2 in [15]), i.e. if $l_M^0$ is **CR** and $e(j) > 0$, then

$$\rho_N(A\chi_{\mathcal{N}\setminus\{j\}}) \leq 1. \tag{1}$$

Assume that $e(j) > 0$, $\rho_N(A\chi_{\mathcal{N}\setminus\{j\}}) = 1$ and $\rho_M(q_- \circ A\chi_{\mathcal{N}\setminus\{j\}}) < \infty$. Define $x$ with coordinates

$$x(i) = (q_- \circ A)(i) \text{ for } \mathcal{N}\setminus\{j\} \text{ and } x(j) = 0.$$

Then $x \in l_M^0$. From the Young inequality, we have

$$\|x\|_M^0 \leq 1 + \rho_M(x) = \rho_N(A\chi_{\mathcal{N}\setminus\{j\}}) + \rho_M(q_- \circ A\chi_{\mathcal{N}\setminus\{j\}}) = \langle |x|, A\chi_{\mathcal{N}\setminus\{j\}}\rangle \leq \|x\|_M^0.$$

So, $\|x\|_M^0 = 1 + \rho_M(x)$ and $1 \in K(x)$. But $x(j) = 0 < e(j)$. Hence $x$ is not a complex extreme point, which means that if $l_M^0$ is **CR** and $e(j) > 0$, then

$$\rho_N(A\chi_{\mathcal{N}\setminus\{j\}}) \neq 1 \text{ or } \rho_M(q_- \circ A\chi_{\mathcal{N}\setminus\{j\}}) = \infty. \tag{2}$$

So the necessity is proved, which follows by inequalities (1) and (2).

Sufficiency. Let $x \in S(l_M^0)$ and $\|x\|_M^0 = \frac{1}{k}(1+\rho_M(kx))$. If there exists a natural number $j$ (without loss of generality we assume that $j = 1$) such that $0 \leq k|x(1)| < e(1)$, then there exists a positive constant $\varepsilon$ such that the inequality $k(1+\eta)|x(1)| < e(1)$ holds for all $0 \leq \eta \leq \varepsilon$. Then

$$1 \leq \rho_N(p \circ (1+\eta)k|x|) = \rho_N(p \circ (1+\eta)k|x|\chi_{\mathcal{N}\setminus\{1\}}) \leq \rho_N(A\chi_{\mathcal{N}\setminus\{1\}}) \leq 1.$$

So $\rho_N(A\chi_{\mathcal{N}\setminus\{1\}}) = 1$ and $p\circ(1+\eta)k|x|)(i) = A(i)$ $(i \in \mathcal{N}\setminus\{1\})$. Since $\eta \in (0,1)$ is arbitrary and $p_i(\cdot)$ is right continuous, we get $p \circ k|x|(i) = A(i)$ $(i \in \mathcal{N}\setminus\{1\})$. Therefore,

$$\begin{aligned}\infty &= \rho_M(q_- \circ A\chi_{\mathcal{N}\setminus\{1\}}) = \rho_M(q_- \circ (p \circ k|x|)\chi_{\mathcal{N}\setminus\{1\}}) \\ &= \rho_M(k|x|\chi_{\mathcal{N}\setminus\{1\}}) = \rho_M(kx) = k-1,\end{aligned}$$

which is a contradiction. This completes the proof. □

**Theorem 1.2**. *The following assertions are equivalent:*
*(1)* $l_M^0$ *is* **CUR**,
*(2)* $l_M^0$ *is* **CLUR**,
*(3)* $l_M^0$ *is* **CR** *and if* $\rho_N(A) > 1$, *then* $M \in \delta_2$.

*Proof.* The implications *(1)* ⇒ *(2)* and *(2)* ⇒" $l_M^0$ is **CR**" are trivial. Assume that $l_M^0$ is **CLUR** and $\rho_N(A) > 1$ but $M \notin \delta_2$.

Take $i_0$ large enough such that $\rho_N(A\chi_{\{i\leq i_0\}}) > 1$. Take $x$ with coordinates

$$x(i) > 0 \text{ for } 1 \leq i \leq i_0 \text{ and } x(i) = 0 \text{ for } i > i_0$$

such that $x \in S(l_M^0)$. By Lemma 0.2, there exists a constant $k > 1$ such that $\|x\|_M^0 = \frac{1}{k}(1+\rho_M(kx))$. Since $M \notin \delta_2$, there exists a real sequence $z \in l_M^0$ such that $\rho_M(z) \leq 1$ and $\xi_M(z) = 1$ (see [7]). Define $y_n$ with

$$y_n(i) = 0 \text{ for } 1 \leq i \leq n \text{ and } y_n(i) = \frac{z(i)}{k} \text{ for } i > n \ (\forall n \in \mathcal{N}).$$

Then for $n > i_0$, there holds

$$\|x+\lambda y_n\|_M^0 \leq \frac{1}{k}(1+\rho_M(k(x+\lambda y_n))) = \frac{1}{k}(1+\rho_M(kx\chi_{\{i\leq n\}})+\rho_M(z\chi_{\{i>n\}}))$$
$$= \|x\|_M^0 + \frac{1}{k}\rho_M(z\chi_{\{i>n\}}) \to 1.$$

But $\|y_n\|_M^0 \geq \frac{1}{k}\cdot\xi_M(z) = \frac{1}{k}$ ($\forall n \in \mathbb{N}$). This means that $x$ is not a **CLUR** point. So the implication *(2)* $\Rightarrow$ *(3)* is proved.

*(3)* $\Rightarrow$ *(1)*. Otherwise, there exist two sequences $(x_n)$ and $(y_n)$ in $l_M^0$ satisfying $\|x_n\|_M^0 = 1$ and $\|y_n\|_M^0 > \varepsilon > 0$, but

$$\|x_n + ty_n\| \leq 1 + \frac{1}{n} \ (n \in \mathbb{N}, |t| \leq 1).$$

If

$$E_n = \{i \in \mathbb{N} : |y_n(i)| \geq \frac{\varepsilon}{8}\max_t |x_n(i)+ty_n(i)|\},$$

then by Lemma 0.1, for $i \in E_n$ there holds

$$|x_n(i)| \leq (1-2\delta)\frac{1}{4}\sum_t |x_n(i)+ty_n(i)|.$$

Similarly, we can prove that $\|y_n\chi_{\mathbb{N}\setminus E_n}\|_M^0 < \frac{2}{3}\varepsilon$ and $\|y_n\chi_{E_n}\|_M^0 > \frac{\varepsilon}{3}$ ($n \geq 3$). In the remaining part of the proof we discuss three cases.

I. $\|(\frac{1}{4}\sum_t |x_n+ty_n|)\|_M^0 = \langle \frac{1}{4}\sum_t |x_n+ty_n|, A\rangle$ ($n \in \mathbb{N}$). Then for $n$ large enough, there holds

$$1 = \|x_n\|_M^0 \leq \langle |x_n|, A\rangle = \langle |x_n|, A\chi_{\mathbb{N}\setminus E_n}\rangle + \langle |x_n|, A\chi_{E_n}\rangle$$
$$\leq \langle \frac{1}{4}\sum_t |x_n+ty_n|, A\chi_{\mathbb{N}\setminus E_n}\rangle + (1-2\delta)\langle \frac{1}{4}\sum_t |x_n+ty_n|, A\chi_{E_n}\rangle$$
$$= \langle \frac{1}{4}\sum_t |x_n+ty_n|, A\rangle - 2\delta\langle \frac{1}{4}\sum_t |x_n+ty_n|, A\chi_{E_n}\rangle$$
$$\leq \frac{1}{4}\cdot 4(1+\frac{1}{n}) - 2\delta\|y_n\chi_{E_n}\|_M^0 < 1 + \frac{1}{n} - 2\delta\cdot\frac{\varepsilon}{3} < 1. \tag{3}$$

This is a contradiction.

II. $\|(\frac{1}{4}\sum_t |x_n+ty_n|)\|_M^0 = \frac{1}{k_n}(1+\rho_M(\frac{k_n}{4}\sum_t |x_n+ty_n|))$ ($n \in \mathbb{N}$) and $k_n \to \infty$. Then for $n$ large enough, there holds

$$1 = \|x_n\|_M^0 \leq \frac{1}{k_n}(1+\rho_M(k_nx_n\chi_{\mathbb{N}\setminus E_n}) + \rho_M(k_nx_n\chi_{E_n}))$$
$$\leq \frac{1}{k_n}(1+\rho_M(\frac{k_n}{4}\sum_t |x_n+ty_n|\chi_{\mathbb{N}\setminus E_n}) + (1-2\delta)\rho_M(\frac{k_n}{4}\sum_t |x_n+ty_n|\chi_{E_n}))$$
$$\leq \|(\frac{1}{4}\sum_t |x_n+ty_n|)\|_M^0 - \frac{2\delta}{k_n}(1+\rho_M(\frac{k_n}{4}\sum_t |x_n+ty_n|\chi_{E_n})) + \frac{2\delta}{k_n}$$
$$\leq \|(\frac{1}{4}\sum_t |x_n+ty_n|)\|_M^0 - 2\delta\|y_n\chi_{E_n}\|_M^0 + \frac{2\delta}{k_n} \leq 1 + \frac{1}{n} - 2\delta\cdot\frac{\varepsilon}{3} + \frac{2\delta}{k_n} < 1, \tag{4}$$

which is a contradiction.

III. $\|(\frac{1}{4}\sum_t |x_n+ty_n|)\|_M^0 = \frac{1}{k_n}(1+\rho_M(\frac{k_n}{4}\sum_t |x_n+ty_n|))$ ($n \in \mathbb{N}$) and $k_n \to k < \infty$.

If $\rho_N(A) \leq 1$, the proof can be proceeded in the same way as the proof in case I.

If $\rho_N(A) > 1$, then $M \in \delta_2$. So, there exist $D > 1$ and $a > 0$, $i_0 \in \mathcal{N}$ and a nonnegative sequence $(c_i)$ $(i > i_0)$ with $\sum_{i>i_0} c_i \leq \frac{1}{2}$ such that

$$M_i(\frac{24}{\varepsilon k}u) \leq DM_i(u) + c_i \ \ (i > i_0, M_i(u) \leq a).$$

Take $\theta > 0$ such that the sequence $x$ with coordinates

$$x(i) = \theta \text{ for } 1 \leq i \leq i_0 \text{ and } x(i) = 0 \text{ for } i > i_0$$

satisfies $\|x\|_M^0 < \frac{\varepsilon}{6}$. Then for any $z \in l_M^0$, there holds $\|z\chi_F\|_M^0 < \frac{\varepsilon}{6}$, where $F = \{i \in \mathcal{N} : 1 \leq i \leq i_0, |z(i)| < \theta\}$. Define

$$F_n = \{i \in \mathcal{N} : 1 \leq i \leq i_0 \text{ and } |y_n(i)| \geq \theta \text{ or } i > i_0 \text{ and } |y_n(i)| \geq \frac{\varepsilon}{8}\max_t |x_n(i) + ty_n(i)|\}.$$

Then $E_n \backslash F_n = \{i : 1 \leq i \leq i_0 \text{ and } |y_n(i)| < \theta\}$ and $\|y_n\chi_{E_n\backslash F_n}\|_M^0 < \frac{\varepsilon}{6}$, $\|y_n\chi_{F_n}\|_M^0 > \frac{\varepsilon}{6}$ for any $n \geq 3$. By (4), there holds

$$\|x_n\|_M^0 \leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\rho_M(\frac{k_n}{4}\sum_t |x_n + ty_n|\chi_{F_n}). \tag{5}$$

If $\{i : 1 \leq i \leq i_0, e(i) = 0\} \neq \emptyset$, denote $d_1 = \min\{M_i(\frac{k\theta}{2}) : 1 \leq i \leq i_0, e(i) = 0\}$ and

if $\{i : 1 \leq i \leq i_0, e(i) > 0\} \neq \emptyset$, denote $d_2 = \min\{M_i(\frac{e(i)}{1-\delta}) : 1 \leq i \leq i_0, e(i) > 0\}$. It is obvious that $d_1, d_2 > 0$. Notice that $F_n \neq \emptyset$ for any $n \in \mathcal{N}$. Define

$$\begin{aligned}
\mathcal{N}_1 &= \{n \in \mathcal{N} : M_{i_n}(k_n|y_n(i_n)|) > a \text{ for some } i_n \in F_n\},\\
\mathcal{N}_2 &= \{n \in \mathcal{N} : M_{i_n}(k_n|y_n(i_n)|) \leq a \ (\forall i_n \in F_n) \text{ and } i_n > i_0 \text{ for some } i_n \in F_n\},\\
\mathcal{N}_3 &= \{n \in \mathcal{N} : i_n \leq i_0 \ (\forall i_n \in F_n) \text{ and } e(i_n) = 0 \text{ for some } i_n \in F_n\},\\
\mathcal{N}_4 &= \{n \in \mathcal{N} : i_n \leq i_0 \text{ and } e(i_n) > 0 \ (\forall i_n \in F_n)\}.
\end{aligned}$$

If $n \in \mathcal{N}_1$, then there exists $i_n \in F_n$ such that $M_{i_n}(k_n|y_n(i_n)|) > a$. So

$$\rho_M(\frac{k_n}{4}\sum_t |x_n + ty_n|\chi_{F_n}) \geq M_{i_n}(k_n|y_n(i_n)|) > a. \tag{6}$$

If $n \in \mathcal{N}_2$, then from $\|y_n\chi_{F_n}\|_M \geq \frac{1}{2}\|y_n\chi_{F_n}\|_M^0 \geq \frac{\varepsilon}{12}$ (see [3]), we have $\rho_M(\frac{12}{\varepsilon}y_n\chi_{F_n}) \geq 1$. For $n$ large enough, we get

$$\begin{aligned}
1 &\leq \rho_M(\frac{12}{\varepsilon}y_n\chi_{F_n}) \leq \rho_M(\frac{24}{\varepsilon k}k_ny_n\chi_{F_n})\\
&\leq D\rho_M(k_ny_n\chi_{F_n}) + \sum_{i\in F_n} c_i \leq D\rho_M(k_ny_n\chi_{F_n}) + \frac{1}{2}.
\end{aligned}$$

Therefore,

$$\rho_M(k_ny_n\chi_{F_n}) \geq \frac{1}{2D}. \tag{7}$$

If $n \in \mathcal{N}_3$, then there exists $i_n \in F_n$ satisfying $i \leq i_0$ and $e(i_n) = 0$. So

$$\rho_M(\frac{k_n}{4}\sum_t |x_n + ty_n|\chi_{F_n}) \geq M_{i_n}(k_n\theta) \geq M_{i_n}(\frac{k\theta}{2}) \geq d_1. \tag{8}$$

Consider the case $n \in \mathcal{N}_4$. Without loss of generality we assume that $\mathcal{N}_4$ is an infinite subset of $\mathcal{N}$ and $i_n \in F_n$ $(n \in \mathcal{N}_4)$. For the convenience, we assume that $\mathcal{N}_4 = \mathcal{N}$. Write $i^{'} = i_n$. Now, we prove that

$$\liminf_{n\to\infty} k_n|x_n(i^{'})| \geq e(i^{'}). \tag{9}$$

From

$$1 + \frac{1}{n} \geq \frac{1}{k_n}(1 + \rho_M(\frac{k_n}{4}\sum_t |x_n + ty_n|)) \geq \frac{1}{k_n}(1 + \rho_M(k_n x_n)) \geq \|x_n\|_M^0 = 1,$$

we have

$$\lim_{n\to\infty} \frac{1}{k_n}(1 + \rho_M(k_n x_n)) = 1.$$

Let $\|x_n\|_M^0 = \frac{1}{h_n}(1 + \rho_M(h_n x_n))$ for some $h_n > 0$ (we write $h_n = \infty$ if $\|x_n\|_M^0 = \langle |x_n|, A\rangle$).

If (9) does not hold, then there exists $\eta > 0$ such that $(1+\eta)k_n|x_n(i^{'})| \leq e(i^{'})$. Since $l_M^0$ is **CR**, so $x_n$ is a complex extreme point $(n \in \mathcal{N})$. Therefore $e(i^{'}) \leq h_n|x_n(i^{'})|$. Thus

$$\begin{aligned}
\frac{1}{n} &\geq \frac{1}{k_n}(1 + \rho_M(k_n x_n)) - \frac{1}{h_n}(1 + \rho_M(h_n x_n)) \\
&\geq \frac{1}{k_n}(1 + \rho_M(k_n x_n)) - \frac{1}{(1+\eta)k_n}(1 + \rho_M((1+\eta)k_n x_n)) \\
&= \frac{\eta}{(1+\eta)k_n}(1 - \frac{1+\eta}{\eta}(\rho_M((1+\eta)k_n x_n) - \rho_M(k_n x_n)) + \rho_M((1+\eta)k_n x_n)) \\
&\geq \frac{\eta}{(1+\eta)k_n}(1 - \frac{1+\eta}{\eta}\langle \eta k_n|x_n|, p\circ(1+\eta)k_n|x_n|\rangle + \rho_M((1+\eta)k_n x_n)) \\
&= \frac{\eta}{(1+\eta)k_n}(1 - \rho_N(p\circ(1+\eta)k_n|x_n|)).
\end{aligned}$$

In virtue of $k_n \to k < \infty$, we get $\rho_N(p\circ(1+\eta)k_n|x_n|) \to 1$. From

$$1 \geq \rho_N(A\chi_{\mathcal{N}\setminus\{i'\}}) \geq \rho_N(p\circ(1+\eta)k_n|x_n|\chi_{\mathcal{N}\setminus\{i'\}}) = \rho_N(p\circ(1+\eta)k_n|x_n|) \to 1,$$

we get $\rho_N(A\chi_{\mathcal{N}\setminus\{i'\}}) = 1$ and $(p\circ(1+\eta)k_n|x_n|)(i) \to A(i)$ $(i \in \mathcal{N}\setminus\{i^{'}\})$. Moreover,

$$1 + \frac{1}{n} \geq \frac{1}{k_n}(1 + \rho_M(k_n x_n)) \geq \frac{1}{(1+\eta)k_n}(1 + \rho_M((1+\eta)k_n x_n)).$$

Therefore

$$\rho_M((1+\eta)k_n x_n) \geq \rho_M((1+\eta)k_n x_n \chi_{\mathcal{N}\setminus\{i'\}}) \to \rho_M(q_- \circ A\chi_{\mathcal{N}\setminus\{i'\}}) = \infty$$

and

$$\rho_M((1+\eta)k_n x_n) \leq (1 + \frac{1}{n})k_n(1+\eta) - 1 \to k(1+\eta) - 1.$$

This is a contradiction. So, by (9) we have for $n$ large enough,

$$\rho_M(\frac{k_n}{4}\sum_t |x_n + ty_n|\chi_{F_n}) \geq M_{i'}(\frac{k_n x_n(i^{'})}{1-2\delta}) \geq M_{i'}(\frac{e(i^{'})}{1-\delta}) \geq d_2. \tag{10}$$

Combining (5), (6), (7), (8), (10) and $\mathcal{N} = \cup_{i=1}^4 \mathcal{N}_i$, we get for $n$ large enough

$$1 \leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\min\{a, d_1, d_2, \frac{1}{2D}\} < 1,$$

which is a contradiction finishing the proof. □

**Theorem 1.3**. *If $x \in S(l_M^0)$, then $x$ is a* **CLUR** *point if and only if for $k > 0$ satisfying $\|x\|_M^0 = \frac{1}{k}(1+\rho_M(kx))$ there holds*

*(1)* $\{i \in \mathcal{N} : k|x(i)| < e(i)\} = \emptyset$,

*(2) If there exist $s \in (0,1)$ and an infinite subset* $\mathcal{N}_0$ of $\mathcal{N}$ satisfying $\rho_M(\frac{kx}{1-s}\chi_{\mathcal{N}_0}) < \infty$, *then $M \in \delta_2^{\mathcal{N}_0}$.*

*Proof.* Necessity. The implications that "$x$ is a **CLUR** point" $\Rightarrow$ "$x$ is a complex extreme point" and "$x$ is a complex extreme point" $\Rightarrow$ *(1)* are trivial.

Assume now that *(2)* does not hold, i.e. there exist $s \in (0,1)$ and an infinite subset $\mathcal{N}_0$ of $\mathcal{N}$ with $\rho_M(\frac{kx}{1-s}\chi_{\mathcal{N}_0}) < \infty$ and $M \notin \delta_2^{\mathcal{N}_0}$. So there exists a sequence $z \in l_M^0$ with $S_z = \mathcal{N}_0$ satisfying $\rho_M(z) \le 1$ and $\xi_M(z) = 1$ (see [7]). Define $y_n$ with coordinates

$$y_n(i) = \frac{s}{k}z(i) \ \ (i \in \mathcal{N}_0, i > n), \ \ y_n(i) = 0 \ \ (i \le n).$$

Then

$$\begin{aligned}\|x+ty_n\|_M^0 &\le \frac{1}{k}(1+\rho_M(k(x+ty_n)))\\ &\le \frac{1}{k}(1+\rho_M(kx)+\rho_M(kx\chi_{\{i>n\}})+\rho_M(sz\chi_{\{i>n\}})\\ &\le 1+\frac{1}{k}((1-s)\rho_M(\frac{kx}{1-s}\chi_{\{i>n\}})+s\rho_M(z\chi_{\{i>n\}}))\to 1.\end{aligned}$$

But $\|y_n\|_M^0 \ge \frac{s}{k}\cdot\xi_M(z) = \frac{s}{k}$, a contradiction.

Sufficiency. Otherwise, there exists a sequence $(y_n)$ in $l_M^0$ with $\|y_n\|_M^0 > \varepsilon > 0$ $(n \in \mathcal{N})$ satisfying

$$\|x+ty_n\|_M^0 \le 1+\frac{1}{n} \ \ (n \in \mathcal{N}, |t| \le 1).$$

Denote

$$E_n = \{i \in \mathcal{N} : |y_n(i)| \ge \frac{\varepsilon}{8}\max_t |x(i)+ty_n(i)|\}.$$

Then $\|y_n\chi_{T\setminus E_n}\|_M^0 \le \frac{\varepsilon}{2}(1+\frac{1}{n}) < \frac{2\varepsilon}{3}$ $(n \ge 3)$. Therefore $\|y_n\chi_{E_n}\|_M^0 > \frac{\varepsilon}{3}$ $(n \ge 3)$. If $i \in E_n$, then

$$|x(i)| < (1-2\delta)\frac{1}{4}\sum_t |x(i)+ty_n(i)|,$$

where $\delta \in (0,\frac{1}{2})$. The remaining part of the proof will be discussed in three cases.

I. $\|(\frac{1}{4}\sum_t|x+ty_n|)\|_M^0 = \langle \sum_t |x+ty_n|, A\rangle$ $(n \in \mathcal{N})$. Then in virtue of (1) for $n$ large enough, we conclude that

$$1 = \|x\|_M^0 \le 1+\frac{1}{n}-2\delta\|y_n\chi_{E_n}\|_M^0 \le \frac{2\delta\varepsilon}{3}.$$

This is a contradiction.

II. $\|(\frac{1}{4}\sum_t|x+ty_n|)\|_M^0 = \frac{1}{k_n}(1+\rho_M(\frac{k_n}{4}\sum_t|x+ty_n|))$ $(n \in \mathcal{N})$ and $k_n \to \infty$. In virtue of (2) for $n$ large enough, we obtain

$$\begin{aligned}1 = \|x\|_M^0 &\le 1+\frac{1}{n}-\frac{2\delta}{k_n}\rho_M(\frac{k_n}{4}\sum_t|x+ty_n|\chi_{E_n}) \le 1+\frac{1}{n}-\frac{2\delta}{k_n}\rho_M(k_ny_n\chi_{E_n})\\ &\le 1+\frac{1}{n}-2\delta\|y_n\chi_{E_n}\|_M^0+\frac{2\delta}{k_n} \le 1-\frac{2\delta\varepsilon}{3},\end{aligned}$$

which is a contradiction.

III. $\|(\frac{1}{4}\sum_t |x+ty_n|)\|_M^0 = \frac{1}{k_n}(1+\rho_M(\frac{k_n}{4}\sum_t |x+ty_n|))$ $(n \in \mathbb{N})$ and $k_n \to k < \infty$. From

$$1+\frac{1}{n} \geq \|(\frac{1}{4}\sum_t |x+ty_n|)\|_M^0 = \frac{1}{k_n}(1+\rho_M(\frac{k_n}{4}\sum_t |x+ty_n|))$$
$$\geq \frac{1}{k_n}(1+\rho_M(k_n x)) \geq \|x\|_M^0 = 1,$$

taking $n \to \infty$, we get $1 = \|x\|_M^0 = \frac{1}{k}(1+\rho_M(kx))$.

III-1. $\inf_n \rho_M(\frac{kx}{1-\delta}\chi_{E_n}) = a > 0$. Then in virtue of (2), we get for $n$ large enough,

$$\|x\|_M^0 \leq 1+\frac{1}{n}-\frac{2\delta}{k_n}\rho_M(\frac{kx}{1-\delta}\chi_{E_n}) \leq 1+\frac{1}{n}-\frac{2\delta}{k_n}a \leq 1-\frac{2\delta a}{k},$$

which is a contradiction.

III-2. $\inf_n \rho_M(\frac{kx}{1-\delta}\chi_{E_n}) = 0$. Passing to a subsequence of $(E_n)$ if necessary we can assume that

$$\sum_{n=1}^{\infty} \rho_M(\frac{kx}{1-\delta}\chi_{E_n}) < \infty.$$

Denote $E = \cup_{n=1}^{\infty} E_n$. Then $\rho_M(\frac{kx}{1-\delta}\chi_E) < \infty$. By the assumption, we have $M \in \delta_2^E$. The remaining part of the proof is similar to the proof of case III in Theorem 1.2, so we omit it here. The proof is finished. □

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