THERE ARE NO CODIMENSION 1 LINEAR ISOMETRIES ON THE BALL AND POLYDISK ALGEBRAS

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Abstract. Let A be the ball algebra or the polydisk algebra in $\mathbb{C}^n$. When $n > 1$, there are no codimension 1 linear isometries on A. (via LaTeX2e)

1. Introduction.

Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex-valued continuous functions on X with the supremum norm. A uniformly closed subalgebra of $C(X)$ is called a function algebra on X if it separates the points of X and contains the constants. Let B_n be the open unit ball of $\mathbb{C}^n$ and S_n be the boundary of B_n. Let D and T stand for B_1 and S_1 respectively. Let $A(S_n)$ be the space of all $f \in C(S_n)$ which can be extended holomorphically on B_n. The algebra $A(S_n)$ is called the ball algebra. When $n = 1$, the algebra $A(T)$ is called the disk algebra.

Let D^n be the unit polydisk and T^n be the torus. Let $A(T^n)$ be the space of all $f \in C(T^n)$ which can be extended holomorphically on D^n. The algebra $A(T^n)$ is called the polydisk algebra. We note that $A(S_n)$ is a function algebra on S_n and $A(T^n)$ is a function algebra on T^n.

Let $H^\infty(D)$ be the Banach algebra of all bounded holomorphic functions on D. Let H^∞ be the space of radial limits of functions in $H^\infty(D)$. Let L^∞ be the algebra of all essentially bounded measurable functions on T. Then H^∞ is an essential supremum norm closed subalgebra of L^∞. A closed subalgebra of L^∞ containing H^∞ is said to be a Douglas algebra.

Let E be a Banach space. A linear isometry $T : E \to E$ is said to be of codimension 1 if the range of T has codimension 1 in E. In [2], Araujo and Font studied codimension 1 linear isometries on function algebras and on Douglas algebras. And they conjectured that there are no codimension 1 linear isometries on proper Douglas algebras. In [4], Izuchi gave a characterization of codimension 1 linear isometries of Douglas algebras. Also in [8], Takayama and Wada characterized codimension 1 linear isometries on the disk algebra.

In this paper, we studied codimension 1 linear isometries on the ball and polydisk algebras. Our theorem is the following.

Theorem Let A be $A(S_n)$ or $A(T^n)$. When $n > 1$, there are no codimension 1 linear isometries on A.

2. Proof.

Suppose that $T : A \to A$ is a codimension 1 linear isometry. We denote by ∂A the Shilov boundary of A. We say that the range of T separates strongly the points of ∂A, if

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for given two elements of ∂A, x_1 and x_2, there exists $f \in T(A)$ such that $|f(x_1)| \neq |f(x_2)|$. By [2, p.2277], Araujo and Font classfied codimension 1 linear isometries T on function algebras into three types:

Type I. The range of T separates strongly the points of ∂A, except two of them.
Type II. The range of T separates strongly the points of ∂A and there exists an element $x_0 \in \partial A$ such that $f(x_0) = 0$ for all $f \in T(A)$.
Type III. The range of T separates strongly the points of ∂A and, for each $x \in \partial A$, there exists $f \in T(A)$ such that $f(x) \neq 0$.

We shall prove that T is a codimension 1 linear isometry of type III. To prove this, suppose first that T is a codimension 1 linear isometry of type I and let x_1 and x_2 be the points which cannot be separated strongly. Then by [1, Corollary 5.1 and Lemma 2.1], ∂A is homeomorphic to a quotient space of ∂A identifing with x_1 and x_2 in ∂A. But S_n and T^n do not satisfy this condition. This is a contradiction.

Next suppose that T is a codimension 1 linear isometry of type II. Let x_0 be the point in ∂A such that $f(x_0) = 0$ for all $f \in T(A)$. By [2, Theorem 6.1], x_0 is isolated in ∂A. This is absurd.

Hence, T is a codimension 1 linear isometry of type III. By [2, theorem A], there exists a homeomorphism φ of ∂A onto ∂A and a continuous map $\psi : \partial A \to \mathbb{C}$ such that $|\psi(x)| = 1$ for all $x \in \partial A$, and

$$(Tf)(x) = \psi(x)f(\varphi(x)) \text{ for all } x \in \partial A \text{ and all } f \in A. \tag{1}$$

Since $T1 = \psi \in A$, ψ is an inner function in A.

Case $A = A(S_n)$

Since there is no non-constant inner function extends continuously to S_n, $TA = A \circ \varphi \subset A$. Since the codimension of $A \circ \varphi$ in A is 1,

$$A = A \circ \varphi + \mathbb{C}g \text{ for some } g \notin A \circ \varphi.$$

Therefore

$$A \circ \varphi^{-1} = A + \mathbb{C}g \circ \varphi^{-1}, \qquad g \circ \varphi^{-1} \notin A. \tag{2}$$

By the above, $A \circ \varphi^{-1}$ is a function algebra on S_n and A is a proper subalgebra of $A \circ \varphi^{-1}$. For a function f on S_n and $\zeta \in S_n$, put $f_\zeta(\lambda) = f(\lambda\zeta)$, $\lambda \in T$. Since $g \circ \varphi^{-1} \notin A$, there exists a point ζ_0 in S_n such that $(g \circ \varphi^{-1})_{\zeta_0} \notin A(T)$, see [6, p.6]. Put $(A \circ \varphi^{-1})_{\zeta_0} = \{f_{\zeta_0}(\lambda) : f \in (A \circ \varphi^{-1})\}$. Then $(A \circ \varphi^{-1})_{\zeta_0}$ is a closed subalgebra of $C(T)$. Since $A(S)_{\zeta_0} = A(T)$, $A(T) \subsetneqq (A \circ \varphi^{-1})_{\zeta_0} \subset C(T)$. By Wermer's maximality theorem [3, p.214], $C(T) = (A \circ \varphi^{-1})_{\zeta_0}$. Therefore

$$C(T) = A(T) + \mathbb{C}(g \circ \varphi^{-1})_{\zeta_0}. \tag{3}$$

Hence $\bar{z} = h_1 + a(g \circ \varphi^{-1})_{\zeta_0}$ and $\bar{z}^2 = h_2 + b(g \circ \varphi^{-1})_{\zeta_0}$ for some $h_1, h_2 \in A(T)$, $a, b \in \mathbb{C}$. Since $(g \circ \varphi^{-1})_{\zeta_0} \notin A(T)$, $a \neq 0$. Then $\bar{z}^2 - \frac{b}{a}\bar{z} = h_2 - \frac{b}{a}h_1$. The right-hand side belongs to $A(T)$, but the left-hand side does not. This is a contradiction. Hence, when $n > 1$, there are no codimension 1 linear isometries on $A(S_n)$.

Case $A = A(T^n)$

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_+$ the set of all nonnegative integers. Let $\mathbb{Z}^n$ and $\mathbb{Z}_+^n$ be the sets of all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Z}$ and $\alpha_i \in \mathbb{Z}_+$ for every $1 \le i \le n$, respectively. Let $\hat{f}(k)$ be the k-th Fourier coefficient of a function f on T^n, that is

$$\hat{f}(k) = \int_{T^n} f(w)\bar{w}^k dm_n(w) \qquad (k \in \mathbb{Z}^n)$$

where $\bar{w}^k = \bar{w_1}^{k_1} \ldots \bar{w_n}^{k_n}$ and $dm_n = \frac{1}{(2\pi)^n} d\theta_1 \ldots d\theta_n$.

By (1),

$$TA = \psi(A \circ \varphi) \subset A. \tag{4}$$

Furthermore

$$A \circ \varphi \subset A. \tag{5}$$

To prove this, let $\mathcal{B}$ be a closed subalgebra of $C(T^n)$ generated by A and $A \circ \varphi$. By (4), $\psi\mathcal{B} \subset A$. Suppose $A \circ \varphi \not\subset A$. Then there exists a function f_0 in $\mathcal{B}$ such that f_0 does not belong to A. By [5, Theorem 2.2.1], there exists $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \backslash \mathbb{Z}_+^n$ such that $\hat{f}_0(k) \neq 0$. We may assume $k_1 < 0$. Then there exists a point $w_0 \in T^{n-1}$ such that $f_0(\lambda, w_0) \notin A(T)$. To see this, suppose that $f_0(\lambda, w) \in A(T)$ for all $w \in T^{n-1}$. Then $\int_T f_0(\lambda, w)\bar{\lambda}^{k_1} dm_1(\lambda) = 0$. Now integrate this with respect to w and conclude that $\hat{f}_0(k_1, \ldots, k_n) = 0$. This is a contradiction.

For a subspace L of $C(T^n)$, let $L_{w_0} = \{f(\lambda, w_0) : f \in L\}$. Then $\mathcal{B}_{w_0}$ is a closed subalgebra of $C(T)$. Since $A(T^n)_{w_0} = A(T)$, $A(T) \subsetneqq \mathcal{B}_{w_0}$. By Wermer's maximality theorem, $\mathcal{B}_{w_0} = C(T)$. Since $\psi\mathcal{B} \subset A$, then $\psi_{w_0} C(T) = A(T)$, where $\psi_{w_0}(\lambda) = \psi(\lambda, w_0)$. Since ψ is an inner function, $|\psi_{w_0}| = 1$ on T. Hence $\psi_{w_0} C(T) = C(T)$. This is a contradiction. Hence (5) holds.

First, suppose that ψ is invertible in A. Since ψ is inner, ψ is a constant function. By (4), the codimension of $A \circ \varphi$ in A is 1. Then

$$A = A \circ \varphi + \mathbb{C}g \text{ for some } g \in A, \qquad g \notin A \circ \varphi.$$

Therefore

$$A \circ \varphi^{-1} = A + \mathbb{C}g \circ \varphi^{-1}, \qquad g \circ \varphi^{-1} \notin A.$$

Hence $A \circ \varphi^{-1}$ is a function algebra on T^n, and A is a proper subalgebra of $A \circ \varphi^{-1}$. In the same way as the proof of (5), there exist a point $w_0 \in T^{n-1}$ such that

$$(A + \mathbb{C}g \circ \varphi^{-1})_{w_0} = C(T).$$

Therefore

$$A(T) + \mathbb{C}(g \circ \varphi^{-1})_{w_0} = C(T).$$

This leads a contradiction as the case $A = A(S_n)$.

Hence ψ is not invertible in A. Then there exists a point $x_0 \in \bar{D}^n \backslash T^n$ such that $\psi(x_0) = 0$. By (4) and (5), $f(x_0) = 0$ for every $f \in TA$. Let A_{x_0} be the set of all $f \in A$ such that $f(x_0) = 0$. Then $TA \subset A_{x_0}$ and A_{x_0} has codimension 1 in A. Since the codimension of TA in A is 1, $TA = A_{x_0}$. Therefore $\psi A \subset A_{x_0} = TA = \psi(A \circ \varphi) \subset \psi A$. Hence the codimension of ψA in A is 1, so that $A = \psi A + \mathbb{C}h$ for some $h \in A$ and $h \notin \psi A$.

Then

$$\bar{\psi}A = A + \mathbb{C}\bar{\psi}h,\ \bar{\psi}^2A = A + \mathbb{C}\bar{\psi}h, \text{ and } \bar{\psi}h \notin A.$$

Hence

$$\bar{\psi} = h_1 + a\bar{\psi}h, \qquad \text{for some } h_1 \in A \text{ and } a \in \mathbb{C}.$$
$$\bar{\psi}^2 = h_2 + b\bar{\psi}h, \qquad \text{for some } h_2 \in A \text{ and } b \in \mathbb{C}.$$

Since $\bar{\psi}h \notin A$, $a \neq 0$, so that

$$\bar{\psi}^2 - \frac{b}{a}\bar{\psi} = h_2 - \frac{b}{a}h_1.$$

The right-hand side belongs to A. Since ψ is an inner function in A and $\frac{b}{a} \in \mathbb{C}$, the left-hand side does not. This is a contradiction. Hence when $n > 1$, there are no codimension 1 linear isometries on $A(T^n)$. □

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