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AN INTEGRAL PRESERVED BY A TRANSLATION ON THE SPACE $\Gamma_0(D) \bigoplus M_0(D)$

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Abstract. In this paper, we introduce a translation invariant integral, called the $(E.R.\mathcal{T})$-integral, such that the (E.R)-integrable function defined by using the Cantor set by Kunugi is integrable.

1 Introduction In our paper [9] , we defined the space $\Gamma_0(D) \bigoplus M_0(D)$ of generalized functions on an interval D. A generalized function is expressed by a pair of elements in $\Gamma_0(D)$ and $M_0(D)$. The set $\Gamma_0(D)$ is the singular part of $\Gamma_0(D) \bigoplus M_0(D)$ in the sense that it contains the δ-function together with it's higher derivatives. The set $M_0(D)$ consists of all real valued measurable functions on D, which is the regular part of $\Gamma_0(D) \bigoplus M_0(D)$. In our papers [9] and [6], the translation invariant $(E.R.\mathcal{M})$-integral over this space was defined. This integral was defined for a function g such that there exists a Cauchy sequence $(V(g_n, \varepsilon_n, A_n))$ satisfying $\bigcap_{n=1}^{\infty} V(g_n, \varepsilon_n, A_n) \ni g$, where the sets A_n are restricted to the sets of the form $(-n, n) \setminus \bigcup_{k=1}^{m} B_k$ for some open intervals B_k with length $1/n$. By this restriction to the sets A_n, the $(E.R)$-integrable function (Kunugi[1]), mentioned in Section 4, defined by using the Cantor set is not $(E.R.\mathcal{M})$-integrable .

In this paper, we introduce another translation invariant integral called the $(E.R.\mathcal{T})$-integral. The definition of the integral is independent of the above restriction. The above function due to Kunugi is $(E.R.\mathcal{T})$-integrable . In Section 2 , we recall some terminologies and notations containing the definition of the $(E.R.\Lambda)$-integral in the paper [9]. In Section 3, we give the definition of the $(E.R.\mathcal{T})$-integral. In Section 4, the $(E.R.\mathcal{T})$-integral is shown to be an extention of the $(E.R)$-integral.

2 Terminologies and notations Let $M_0(D)$ be the set of all real valued Lebesgue measurable functions defined on a finite or an infinite interval D. In what follows, we suppose that the set $M_0(D)$ is classified by the usual equivalence relation $f(x) = g(x)$ a.e. We denote measurable functions by symbols $f(x), g(x), ...$ and a class in $M_0(D)$ containing a measurable function $g(x)$ by the same symbol $g(x)$ or g. For each Lebesgue measurable subset A of D and $\varepsilon > 0$, we define a pre-neighbourhood $V(f, \varepsilon, A)$ as

$$V(f, \varepsilon, A) = \{g \in M_0(D) : \int_A |f(x) - g(x)| dx \leq \varepsilon\}.$$

We denote $V(f, \varepsilon, A)$ by $V(f)$ if there is no fear of confusion.

Definition 1 *A sequence* $(V(f_n)) = (V(f_n.\varepsilon_n, A_n))$ *of preneighbourhoods in* $M_0(D)$ *is called a Cauchy sequence if*
(i) $V(f_1) \sqsupseteq V(f_2) \sqsupseteq ...$, *and*
(ii) $\varepsilon_n \to 0$.

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For a Cauchy sequence $(V(f_n, \varepsilon_n, A_n))$ on D, we consider the following two conditions:

(T_1) $\mathrm{m}((D \setminus A_n) \cap [-1/\varepsilon_n, 1/\varepsilon_n]) \leq \varepsilon_n$.[1]

(T_2) $f_n(x)$ is decomposed into a sum of measurable functions $f_{1n}(x)$ and $f_{2n}(x)$ on D, where supp $f_{1n} \subseteq D \setminus A_n$, and

$$\int_{D \setminus A_n} |f_{2n}(x)| dx \leq \varepsilon_n.$$

If $(V(f_n)) = (V(f_n, \varepsilon_n, A_n))$ is a Cauchy sequence which satisfies conditions (T_1) and (T_2), the Cauchy sequence is called a G_0-Cauchy sequence on D. Let $G_0(D)$ be the set of sequences (f_n) such that there exists a G_0-Cauchy sequence $(V(f_n))$ with $0 \in \bigcap_{n=1}^{\infty} V(f_n)$.

Definition 2 *A decomposition $f_n = f_{1n} + f_{2n}$ in (T_2) for a G_0-Cauchy sequence $(V(f_n))$ is called an associated decomposition of f_n.*

If (f_n) and (g_n) have associated decompositions $f_{1n} + f_{2n}$ and $g_{1n} + g_{2n}$ of f_n and g_n respectively such that there is an $n_0 \in \mathbf{N}$ satisfying $f_{1n} = g_{1n}$ a.e. for each $n \geq n_0$, we say that (f_n) and (g_n) are equivalent. Let $\Gamma_0(D)$ be the quotient space of $G_0(D)$ classified by this equivalence relation, whose element containing (f_n) is denoted by $[f_n]$.

The following set is the underling space of our whole theory:

$$\Gamma_0(D) \bigoplus M_0(D) = \{([f_n], g); [f_n] \in \Gamma_0(D), g \in M_0(D)\}.$$

In what follows, we denote the pair $([f_n], g)$ by $[f_n] \oplus g$.

Let $\Lambda = (\lambda_n)$ be a sequence of finite absolutely continuous measures on $\mathbf{R}$. A Cauchy sequence $(V(g_n, \varepsilon_n, A_n))$ is called an L_0-Cauchy sequence if it satisfies the following three conditions on D:

(K_1) if B is a Lebesgue measurable subset of D with $\lambda_n(D \setminus A_n) \geq \lambda_n(B)$, then $\mathrm{m}(B \cap [-1/\varepsilon_n, 1/\varepsilon_n]) \leq \varepsilon_n$.

(K_2) if $\mathrm{m}(D \setminus A_n) > 0$ for all n , there exist $k,\ k' > 0$ such that

$$k \leq \lambda_n(D \setminus A_n) \leq k'$$

for all n.

(K_3) if B is a Lebesgue measurable subset of D with $\lambda_n(D \setminus A_n) \geq \lambda_n(B)$, then

$$\int_B |g_n(x)| dx \leq \varepsilon_n.$$

Let $\mathbf{F}_0(\Lambda)$ be the set of L_0-Cauchy sequences on D and let $L_0(\Lambda)$ be the set of sequences (g_n) in $L^1(D)$ such that there exists an L_0-Cauchy sequence $(V(g_n))$.

Definition 3 *A sequence $(V(g_n)) \in \mathbf{F}_0(\lambda)$ is called on L_0-Cauchy sequence for g if $\bigcap_{n=1}^{\infty} V(g_n) = \{g\}$.*

Definition 4 *Let (g_n) be a sequence in $L_0(\Lambda)$ with an L_0-Cauchy sequence for $g \in M_0(D)$. If*

$$\lim_{n \to \infty} \sup \int_D g_n(x) dx = \lim_{n \to \infty} \inf \int_D g_n(x) dx,$$

[1] We denote the Lebesgue measure of the set A by m(A) .

this common value is denoted by

$$I(g, \Lambda) = (E.R.\Lambda) \int_D g(x)dx$$

and $I(g, \Lambda)$ is called the $(E.R.\Lambda)$-integral of g on D. If $-\infty < I(g, \Lambda) < \infty$, g is called to be $(E.R.\Lambda)$-integrable on D.

Now we give the definition of the $(E.R.\Lambda)$-integration on $\Gamma_0(D) \bigoplus M_0(D)$.

Definition 5 *Suppose that a sequence (f_n) in $G_0(D)$ has an associated decomposition $f_{1n} + f_{2n}$ of f_n such that the value*

$$I([f_n]; D) = \lim_{n\to\infty} \int_D f_{1n}(x)dx$$

exists and the $(E.R.\Lambda)$-integral $I(g, \Lambda)$ of $g \in M_0(D)$ exists, where the values of these integrals may be finite or infinite. Then, if $I([f_n]; D) + I(g, \Lambda)$ has a meaning, this sum is denoted by

$$(E.R.\Lambda) \int_D [f_n] \oplus g dx = (E.R.\Lambda) \int_D (f_n(x)) \oplus g(x)dx,$$

and the common value is called the $(E.R.\Lambda)$-integral of $[f_n] \oplus g$ on D.

3 The $(E.R.\mathcal{T})$-integral . Our integral is considered on a finite or an infinite open interval D. We fix two increasing sequences $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ of real numbers with $\lim_{n\to\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \beta_n = \infty$, and a decreasing sequences (J_n) of measurable subsets with $J_n \subseteq [-\beta_n, \beta_n]$ and $\lim_{n\to\infty} m(J_n) = 0$. Now, we define a sequence (μ_n) of finite measures on $\mathbf{R}$ as the following :

(1) Let ν_n be absolutely continuous measure on $\mathbf{R}$ such that

$$(1.1) \qquad \nu_n(E_n) = \exp(-\alpha_n) \ ,$$

where $E_n = \mathbf{R} \setminus [-\beta_n, \beta_n]$, and , if $J_n \neq \phi$ for n=1,2,3,...,

$$\nu_n(J_n) = \exp(-\alpha_n) \ .$$

We fix ν_n in the following .

(2) Denote $J_n + a = \{x + a; x \in J_n\}$ by J_n^a. For any Lebesgue measurable subset E of $\mathbf{R}$ and for any mutually diffrent points $a_1, a_2, ..., a_l \in D$, we set

$$(1.2) \qquad \mu_n^0(E) = \sum_{i=1}^{l} \nu_n((E \cap J_n^{a_i}) - a_i) + \nu_n(E \cap E_n)$$

$$+ \ \mathrm{m}(E \cap (CE_n \setminus \cup_{i=1}^{l} J_n^{a_i})) \ .^2$$

(3) Put, for n=1,2,3,...,

$$(1.3) \qquad \mu_n = \mu_n^0 \backslash \exp(-\alpha_n) \ .$$

Then (μ_n) is called a sequence of measures defined for $a_1, a_2, .., a_l$. We denote (μ_n) by $T((a_i)_1^l)$ or $T(a_1, a_2, ..., a_l)$. If $J_{n_0} = \phi$ for some number $n_0 \in \mathbf{N}$, for $n \geq n_0$, the measure μ_n is independent of the choice of a finite number of points $a_1, a_2, ..., a_l$.

As mentioned above, we fix sequences $(J_n), (\alpha_n)$, and (ν_n) in the following.

Let $\mathcal{T}$ be the set of all sequences $T((a_i)_1^l)$ of measures . The set $\mathcal{T}$ is a direct set with respect to the order $T((a_i)_1^l) \leq T((b_i)_1^k)$ defined by $\{a_1, a_2, ..., a_l\} \subseteq \{b_1, b_2, ..., b_k\}$.

[2] We denote $\mathbf{R} \setminus A$ by CA .

Definition 6 *Suppose that a sequence (g_n) of functions in $M_0(D)$ satisfies the following condition which is called the $(*)$-condition for $a_1, a_2, ..., a_l$:*

For any $a \in D$ with $a \neq a_i$ $(i = 1, 2, ..., l)$,

$$\lim_{n\to\infty} \int_{J_n^a \cap D} |g_n(x)| dx = 0 \ .$$

Let $L_0^*(T((a_i)_1^l))$ (or $L_0^*(T(a_1, a_2, ..., a_l))$)be the set of all sequences (g_n) in $L^1(D)$ satisfying the $(*)$-condition for $a_1, a_2, ..., a_l$ for which an L_0-Cauchy sequence $(V(g_n))$ exists.

Remark 1 *In the paper* [9] *and* [6] *, we give a concrete expression for measures ν_n and sets A_n. However, in this paper, their expressions are more general.*

Proposition 1 *If $(g_n) \in L_0^*(T(a_1, a_2, ..., a_l))$, then $(g_n) \in L_0^*(T(a, a_1, ..., a_l))$ for any $a \in D$ with $a \neq a_i$ $(i = 1, 2, .., l)$.*

Proof. Let $J_n \neq \phi$ for $n = 1, 2, 3, ...,$ and let $(g_n) \in L_o^*(T(a_1, a_2, ..., a_l))$. Then there exists an L_0-Cauchy sequence $(V(g_n, \varepsilon_n, A_n)) \in \mathbf{F}_0(T(a_1, a_2, ..., a_l))$ and (g_n) satisfies the $(*)$-condition. Let $(\mu_n) = T(a_1, a_2, ..., a_l)$. Then we have $\mu_n = \mu_n^0 / \exp(-\alpha_n)$ by (3). By virtue of (K_2), there exist $k_1, k_2 > 0$ such that

$$(1.4) \qquad k_1 \leq \mu_n(CA_n) \leq k_2 \ .$$

There exists an integer $c > 1$ such that

$$(1.5) \qquad (k_2 + l)/k_1 \ < c \ .$$

Put $\rho_n = c \ \varepsilon_n + \mathrm{m}(J_n) + \eta_n$, where

$$\eta_n = \sup_{k \geq n} \int_{J_k^a \cap D} |g_k(x)| dx.$$

We will show that

$$(V(g_n))_N^\infty = (V(g_n, \rho_n, B_n))_N^\infty \in \mathbf{F}_0(T(a_1, a_2, ..., a_l))$$

for a sufficiently large N, where $B_n = A_n \cap (D \setminus {J_n}^a)$. Let $(\tau_n) = T(a, a_1, ..., a_l)$. Then we see that $\tau_n = \tau_n^0 / \exp(-\alpha_n)$ by (3). By (1.1), (1.2), and (1.4), it follows that

$$(1.6) \qquad \exp(-\alpha_n) \leq \tau_n^0(J_n^a) \leq \tau_n^0(CB_n) \leq \mu_n^0(CA_n \cap CJ_n^a) + {\tau_n}^0(J_n^a) \leq (k_2 + l) \exp(-\alpha_n) \ .$$

Hence $(V(g_n))$ satisfies (K_2).

Next we will show that $(V(g_n))$ satisfies (K_1) for $T(a, a_1, ..., a_l)$. Let B be a subset of D such that $\tau_n^0(CB_n) \geq \tau_n^0(B)$. By $(1.4), (1.5), and (1.6)$, we find that

$$(1.7) \qquad \tau_n^0(CB_n) \leq (k_2 + l) \exp(-\alpha_n) \leq ((k_2 + l)/k_1) \ \mu_n^0(CA_n) < c \ \mu_n^0(CA_n).$$

Therefore we have

$$(1.8) \qquad \mu_n^0(B \cap CJ_n^a) = \tau_n^0(B \cap CJ_n^a) \leq \tau_n^0(B) < c \ \mu_n^0(CA_n) \ .$$

Since $(V(g_n, \varepsilon_n, A_n))$ satisfies (K_1) for $T(a_1, a_2, ..., a_l)$, we have $\mathrm{m}(B_0 \cap [-1/\varepsilon_n, 1/\varepsilon_n]) \leq \varepsilon_n$ for B_0 with $\mu_n^0(CA_n) \geq \mu_n^0(B_0)$. Hence, from $\rho_n > \varepsilon_n$, we obtain

$$\mathrm{m}(B \cap CJ_n^a \cap [-1/\rho_n, 1/\rho_n]) \leq c\ \varepsilon_n. \tag{1.9}$$

Moreover, we get

$$\mathrm{m}(B \cap J_n^a \cap [-1/\rho_n, 1/\rho_n]) \leq \mathrm{m}(J_n^a) = \mathrm{m}(J_n)\ , \tag{1.10}$$

so that $\mathrm{m}(B \cap [-1/\rho_n, 1/\rho_n]) \leq \rho_n$. Thus $(V(g_n))$ satisfies (K_1) .

Finally, we will show that $(V(g_n))$ satisfies (K_3). For any set $B \subseteq D$ with $\tau_n^0(CB_n) \geq \tau_n^0(B)$, we find by (1.8) that

$$\int_{B \cap CJ_n^a} |g_n(x)| dx \leq c\ \varepsilon_n\ .$$

Hence we have

$$\int_B |g_n(x)| dx \leq c\ \varepsilon_n + \eta_n \leq \rho_n,$$

which means that $(V(g_n))$ satisfies (K_3) . This completes the proof.

Proposition 2 *Let $[f_n] \oplus g \in \Gamma_0(D) \bigoplus M_0(D)$ and let (g_n) be an element in $L_o^*(T((a_i)_1^l))$ such that there exists an L_o-Cauchy sequence $(V(g_n)) \in \mathbf{F}_0(T((a_i)_1^l))$ for g. If $\{b_1, b_2, ..., b_k\}$ contains $\{a_1, a_2, ..., a_l\}$ and $[f_n] \oplus g$ is $(E.R.T((a_i)_1^l))$-integrable $(Definition\ 4)$, then $[f_n] \oplus g$ is $(E.R.T((b_i)_l^k))$-integrable and their integrals coincide.*

Proof. By Proposition 1, we obtain $(g_n) \in L_o^*(T((b_i)_1^k))$. Hence, $[f_n] \oplus g$ is $(E.R.T((b_i)_1^k))$-integrable for $T((b_i)_1^k)$. We obtain

$$\begin{aligned}(E.R.T((a_i)_1^l)) \int_D [f_n] \oplus g dx &= \lim_{n \to \infty} \int_D (f_n(x) + g_n(x)) dx \\ &= (E.R.T((b_i)_l^k)) \int_D [f_n] \oplus g dx.\end{aligned}$$

Now we define a translation invariant integral in $\Gamma_0(D) \bigoplus M_0(D)$.

Definition 7 *Let $[f_n] \oplus g \in \Gamma_0(D) \bigoplus M_0(D)$. Suppose that there exist two sequences (g_n) and $T((a_i)_1^l)$ such that (g_n) is an element in $L_o^*(T((a_i)_1^l))$ with an L_o-Cauchy sequence $(V(g_n)) \in \mathbf{F}_o(T((a_i)_1^l))$ for g. When $[f_n] \oplus g$ is $(E.R.T((a_i)_1^l))$-integrable, $[f_n] \oplus g$ is said to be $(E.R.\mathcal{T})$-integrable . The $(E.R.\mathcal{T})$-integral*

$$(E.R.\mathcal{T}) \int_D [f_n] \oplus g dx$$

of $[f_n] \oplus g$ is defined to be the $(E.R.T((a_i)_1^l))$-integral of $[f_n] \oplus g$.

Remark 2 *Let $[f_n] \oplus g \in \Gamma_0(D) \bigoplus M_0(D)$ and let (g_n) be an element in $L_0^*(T((a_i)_1^l))$ such that there exists an L_o-Cauchy sequence $(V(g_n)) \in \mathbf{F}_0(T((a_i)_1^l))$ for g. If $\{b_1, b_2, ..., b_k\}$ contains $\{a_1, a_2, ..., a_l\}$ and $[f_n] \oplus g$ is $\overline{\mathcal{P}}_c$-differentiable for $T((a_i)_1^l)$ $([9], Definition\ 8)$, we can also prove easily that $[f_n] \oplus g$ is $\overline{\mathcal{P}}_c$-differentiable for $T((b_i)_1^k)$ and they have the same derivatives.*

Therefore, if $[f_n] \oplus g$ is $\overline{\mathcal{P}}_c$-differentiable for $T((a_i)_1^l)$, $[f_n] \oplus g$ is said to be $\overline{\mathcal{P}}_c$-differentiable for $\mathcal{T}$. The $\overline{\mathcal{P}}_c$-derivative $([f_n] \oplus g)'_{\overline{\mathcal{P}}_c, \mathcal{T}}$ for $\mathcal{T}$ is defined to be the $\overline{\mathcal{P}}_c$-derivative of $[f_n] \oplus g$ for $T((a_i)_1^l)$.

Remark 3 *In Definition in* [9] *and* [6] (*Section 4*) , *we defined a translation invariant* $(E.R.\mathcal{M})$*-integral in* $\Gamma_0(D) \bigoplus M_0(D)$. *In the similar way as Definition 7, this integral was defined for functions* $g \in M_0(D)$ *such that there exists an* L_o*-Cauchy sequence* $(V(g_n, \varepsilon_n, A_n))$ *satisfying* $\bigcap_{n=1}^{\infty} V(g_n, \varepsilon_n, A_n) \ni g$, A_n *are restricted to the form*

$$A_n = D \setminus \bigcup_{i=1}^{l} (a_i - 1/(2n), a_i + 1/(2n)).$$

Owing to this restriction, the $(E.R)$*-integrable function , mentioned in Section 4, defined by Kunugi by using Cantor set is not* $(E.R.\mathcal{M})$*-integrable . In order to remove this restriction, we use* $(*)$*-condition for* (g_n).

Example 1 *Let* ν_n *be a measure on* $\mathbf{R}$ *defined by*

$$\nu_n(E) = \int_E k_n(x)dx,$$

where

$$k_n(x) = \begin{cases} \exp(-1/x)/x^2, & on\ J_n \\ 2\exp(-2|x|), & on\ E_n \\ 1, & on\ \mathbf{R} \setminus (J_n \cup E_n). \end{cases}$$

Let c *be a number with* $0 < c < 2$. *Put* $J_n = [-1/(2n), 1/(2n)]$ *and* $a_1 = c$.*There exists a number* $n_0 \in N$ *such that* $c - 1/(2n_0), c + 1/(2n_0) \in [0,2]$. *For each* $n > n_0$, *a function* g_n *on* $D = [0,2]$ *is defined to be* $1/(x-c)$ *on* A_n *and* 0 *elsewhere, where* $A_n = D \setminus J_n^c$. *Then* $(V(g_n, 2/n, A_n))_N^{\infty} \in \mathbf{F}_0(T(a_1))$ *for a sufficiently large number* $N > n_0$. *It is easily verified that* (g_n) *satisfies* $(*)$*-condition. Hence we have*

$$(E.R.\mathcal{T}) \int_D 0 \oplus \frac{1}{x-c} dx \; = \; \log((2-c)/c).$$

4 Relation to the $(E.R)$-integral.

In the paper [1] , Kunugi defined a function by using the Cantor set as follows :

Let S_1^0 be the open middle third of $S = [0,1]$, $S_1^0 = (1/3, 2/3)$; let S_1^1 and S_2^1 be the open middle thirds of two closed intervals which make up $S \setminus S_1^0$, i.e. $S_1^1 = (1/9, 2/9)$ and $S_2^1 = (7/9, 8/9)$; let S_1^2, S_2^2, S_3^2, and S_4^2 be the open middle thirds of the four closed intervals which make up $S \setminus \bigcup_{j=1}^{2} S_j^1$ and so on ad infinitum. Putting $\bigcup_{j=1}^{2^n} S_j^n = U^n$, we have the Cantor set $S \setminus \bigcup_{n=0}^{\infty} U^n$. A function f is defined to be $(-1)^n 3^{(n+1)}/(2^n(n+1))$ on U^n for each n and 0 on $S \setminus \bigcup_{n=0}^{\infty} U^n$. It is shown by Kunugi that f is $(E.R)$-integrable.

In this section, we will show that the $(E.R.\mathcal{T})$-integral is an extension of the $(E.R)$-integral. Here, we use the definition of the $(E.R)$-integral due to Okano. In the following, D is a finite open interval.

Definition 8 [*Okano*[2]] *Let* $(V(f_n, \varepsilon_n, A_n))$ *be a Cauchy sequence on* D *satisfying the following three conditions:*

(*i*) $\mathrm{m}(CA_n) \leq \varepsilon_n$,

(*ii*) *For each* n, *there exists* $k > 0$ *such that* $k\ \mathrm{m}(CA_{n+1}) \geq \mathrm{m}(CA_n)$.

(*iii*) *For any Lebesgue measurable subset* B *of* D *with* $\mathrm{m}(CA_n) \geq \mathrm{m}(B)$,

$$\int_B |f_n(x)|dx \leq \varepsilon_n.$$

Let (f_n) be a sequence with a Cauchy sequence $(V(f_n,\varepsilon_n,A_n))$ such that $\bigcap_{n=1}^{\infty} V(f_n,\varepsilon_n,A_n) \ni f$. If

$$\lim_{n\to\infty} \sup \int_D f_n(x)dx = \lim_{n\to\infty} \inf \int_D f_n(x)dx,$$

the common value is called the $(E.R)$-integral of f on D and is denoted by

$$(E.R)\int_D f(x)dx.$$

Theorem 1 *If f is $(E.R)$-integrable on D, then f is $(E.R.\mathcal{T})$-integrable on D.*

Proof. Let $(V(f_n,\varepsilon_n,A_n))$ be a Cauchy sequence satisfying conditions $(i),(ii)$, and (iii). We may assume that $0<\varepsilon_n<1$ for every $n\in\mathbf{N}$. Let $\bigcap_{n=1}^{\infty} V(f_n,\varepsilon_n,A_n)\ni f$. Put $J_n=\phi$, $\beta_n=n$, and $\alpha_n=-\log\gamma_n$, where $\gamma_n=\mathrm{m}(CA_n)$. Let ν_n be a measure on $\mathbf{R}$ such that

$$\nu_n(E)=\int_E h_n(x)dx,$$

where

$$h_n(x)=\begin{cases} 1, & \text{on } \mathbf{R}\setminus E_n \\ -\alpha_n \exp(-\alpha_n|x|/n)/(2n), & \text{on } E_n. \end{cases}$$

Then we have

$$\mu_n^0(E)=\mathrm{m}(E\cap(-n,n))+\nu_n(E\cap E_n)$$

for any $E\subseteq\mathbf{R}$, and

$$\mu_n(E)=\mu_n^0(E)/\exp(-\alpha_n) \qquad (n=1,2,3,...).$$

There exists a number $n_0\in\mathbf{N}$ such that $[-n,n]\supset D$ for any $n\geq n_0$,

We will show that $(V(f_n))_{n_0}^{\infty}=(V(f_n,\varepsilon_n,A_n))_{n_0}^{\infty}$ is an L_0-Cauchy sequence .

Let B be any subset of D such that $\mu_n^0(CA_n)\geq\mu_n^0(B)$. Then we have, by (i),

$$\mathrm{m}(B\cap[-1/\varepsilon_n,1/\varepsilon_n])\leq\mathrm{m}(B)\leq\mathrm{m}(CA_n)\leq\varepsilon_n$$

for any $n\geq n_0$. Hence, (K_1) is satisfied . It holds that

$$\mu_n(CA_n)=\mu_n^0(CA_n)/\exp(-\alpha_n)=\mathrm{m}(CA_n)/\gamma_n=1 \ ,$$

so (K_2) is satisfied.

Moreover, from (iii), (K_3) is satisfied. This completes the proof.

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