ON THE STRONG EXTREME POINTS OF THE SET $N_0(\Delta)$ *

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ABSTRACT. Suppose H is a Hilbert space , and $N_0(\Delta)$ is the set
$N_0(\Delta) = \{f(z) = I + B(1)z + B(2)z^2 + \cdots | f(z)$ is an analytic operator function on the open uint disk $\Delta, Ref(z) \geq 0$, where $\{B(n)\}_{n=1}^{\infty}$ are normal operators on H,and $B_nB_m = B_mB_n$ for every positive integers n,m$\}$.
The note proves that Holland F's extreme point theorem can be generalized to
1. $N_0(\Delta)$ has not any strong extreme point,when $dimH > 1$.
2. The sub-strong extreme points of $N_0(\Delta)$ have the following forms

$$f(z) = (I + Uz)(I - Uz)^{-1}, z \in \Delta,$$

where U is an unitary operator on H.

§1.Preliminaries

Let H be a Hilbert space,and $\mathcal{L}(H)$, be the space of all the bounded linear operators on H ,I be the identity operator on H. By an operator function of f on the open unit disk Δ, we mean that $f(z) \in \mathcal{L}(H)$ for every $z \in \Delta$. An operator function of f on Δ is said to be analytic if $\varphi(f(z))$ is analytic on Δ in the classical sense for every $\varphi \in \mathcal{L}(H)^*$, the conjugate space of $\mathcal{L}(H)$.

A set G consisted of some analytic operator functions on the unit disk Δ is said to be strong convex set , if for any $f, g \in G$ and any operator A, commuting with $f, g, 0 \leq A \leq I$, we have $Af + (I - A)g \in G$.

Definition 1. Let G be a strong convex set , $f \in G$, we say f(z) is a strong extreme point of G if $f(z) = Ag(z) + (I - A)h(z)$ for some $g, h \in G, z \in \Delta$ and $0 \leq A \leq I (A \neq 0, I)$ commuting with g, h, then $g = h = f$.

Definition 2. In the definition 1, if using $0 < A < I$ in place of $0 \leq A \leq I$ $(A \neq 0, I)$, at the moment, we say $f(z)$ is a sub-strong extreme point.

Remark: Let M be a closed linear subspace of H,and $f(z)M \subseteq M, \forall z \in \Delta$, if $f(z)$ is a strong extreme point on H, so is $f(z)|_M$ on M.

In [1], Holland's extreme point theorem points out the set $P_0(\Delta) = \{f(z) = 1 + b_1z + b_2z^2 + \cdots | f(z)$ is an analytic function on the open unit disk $\Delta, Ref(z) \geq 0\}$ has extrem points as $f(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}$, where $\theta \in [0, 2\pi]$.

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Now, considering the following set([3]), $N_0(\Delta)$=$\{f(z) = I + B(1)z + B(2)z^2 + \cdots | f(z)$ is an analytic operator function on the open uint disk $\Delta, Ref(z) \geq 0$, where $\{B(n)\}_{n=1}^{\infty}$ are normal operators on H, and $B_nB_m = B_mB_n$, for every positive integers n, m$\}$.

We can easy see $N_0(\Delta)$ is a strong convex set. In this paper, we prove the following theorem.

Theorem 1. $N_0(\Delta)$ has not any strong extreme point, when $dimH > 1$.

Theorem 2. The sub-strong extreme points of $N_0(\Delta)$ have the following forms

$$f(z) = (I + Uz)(I - Uz)^{-1}, z \in \Delta,$$

where U is an unitary operator on H.

Remark: Let H be complex number field , then Theorem 2 turns into Holland F's extreme point theorem.

§2. The Proofs Of Main Results

First, we give some lemmas.

Lemma 1.[1] Let $f(z) = I + B(1)z + B(2)z^2 + \cdots \in N_0(\Delta)$, then $\|B(k)\| \leq 2$.

Lemma 2. If $f(z) = I + B(1)z + B(2)z^2 + \cdots \in N_0(\Delta), n \in N$, let

$$4iu(z) = \sum_{k=0}^{\infty}\{B^*(n)B(n+k) - B(n)B(k-n)\}z^k$$

where $B(0) = 2I, B(-k) = B^*(k), k \geq 1$, then $f \pm u \in N_0(\Delta)$.

Proof.1) Suppose $\sum_{k=0}^{\infty} \|B(k)\| < +\infty$. We can see that $f(z) \pm u(z)$ is analytic in Δ, and continuous on $\bar{\Delta}$. For any $z \in \bar{\Delta}$, we have

$$4iu(z) = [B^*(n)z^{-n} - B(n)z^n]f(z) - [B(n)P^*(\bar{z}^{-1})z^n + B^*(n)z^{-n}P(z)] + B^*(n)B(n)$$

where $P(z) = I + \sum_{k=1}^{n} B(k)z^k$. If $|z| = 1$, then

$$\begin{aligned} Re[f(z) \pm u(z)] &= Re[I \pm Im(\tfrac{B(n)z^n}{2})^*]f(z) \\ &= [I \pm Im(\tfrac{B(n)z^n}{2})^*]Ref(z) \\ &\geq 0. \end{aligned}$$

So, $Re[f(z) \pm u(z)] \geq 0, z \in \Delta$, therefore,$f(z) \pm u(z) \in N_0(\Delta)$.
2) Let $\delta \in (0, 1)$, we can use $B(k)\delta^{|k|}$ in place of $B(k), f_\delta(z), u_\delta(z)$ in place of $f(z), u(z)$ repectively, then $f_\delta(z) = f(\delta z) \in N_0(\Delta)$. Using the consequence of 1), we have $f_\delta \pm u_\delta \in N_0(\Delta)$. Let $\delta \to 1$, we can see $f(z) \pm u(z) \in N_0(\Delta)$.

Lemma 3. $\frac{1+z}{1-z}I$ is not a strong extreme point of $N_0(\Delta)$, when $dimH > 1$.

Proof. We may assume $dimH = \infty$. Let $e_1, e_2, \cdots$ be an orthonormal basis on H, we can define an operator A on H such that $Ae_1 = e_1, Ae_j = 0$ $(j \geq 2)$, and define

$$g(z) = I + B(1)z + B(2)z^2 + \cdots, h(z) = I + C(1)z + C(2)z^2 + \cdots$$

where $B(n), C(n) \in \mathcal{L}(H)$ and such that

$$B(n)e_1 = 2e_1, B(n)e_2 = (-1)^n 2e_2, B(n)e_j = 0 (j \geq 3), n = 1, 2, \cdots$$
$$C(n)e_1 = (-1)^n 2e_1, C(n)e_2 = 2e_2, C(n)e_j = 2e_j (j \geq 3)\ n = 1, 2, \cdots$$

we have $\frac{1+z}{1-z}I = Ag(z) + (I - A)h(z)$, and for $\forall x = \sum_{k=1}^{\infty} \lambda_n e_n \in H, re^{i\theta} \in \Delta$,

$$\begin{aligned}(Reg(re^{i\theta})x, x) &= |\lambda_1|^2(1 + 2r\cos\theta + 2r^2\cos 2\theta + \cdots) \\ &+ |\lambda_2|^2(1 - 2r\cos\theta + 2r^2\cos 2\theta - \cdots) \\ &+ |\lambda_3|^2 + |\lambda_4|^2 + \cdots \\ &\geq 0.\end{aligned}$$

So $g(z) \in N_0(\Delta)$. Similarly, $h(z) \in N_0(\Delta)$.

The proof of theorem 1

Let $f(z) = I + B(n)z^n + B(n+1)z^{n+1} + \cdots \in N_0(\Delta)$, and it is a strong extreme point of $N_0(\Delta)$, so by [1], $f(z) \not\equiv I$, and by Lemma 2

$$f(z) = \frac{1}{2}[(f(z) + u(z)) + (f(z) - u(z))].$$

We have $u(z) \equiv 0$, so

$$B^*(n)B(n+k) = B(n)B(k-n), k = 0, 1, 2, \cdots \quad (1)$$

By $B(n)$ is a normal operator, we obtain

$$KerB(n) = KerB^*(n).$$

1) Suppose $KerB(n) = KerB^*(n) = \{0\}$. Using (1), we have
$B^*(n)B(j) = 0, B(j) = 0\ (j \neq nm, m = 1, 2, 3 \cdots)$ and

$$B^*(n)B(nm) = B(n)B((m-2)n) \quad (2)$$
$$B^*(n)B(2n) = 2B(n) \quad (3)$$

From (2),(3),

$$B(nm) = \frac{1}{2}B(2n)B((m-2)n).$$

So, $B(2kn) = (\frac{1}{2})^{k-1}B^k(2n), B((2k+1)n) = (\frac{1}{2})^k B^k(2n)B(n)$.
Let $B(n) = UB^+(n)$ be the polar decomposition of B(n), by B(n) is normal,we see $KerB(n) = KerB^+(n) = \{0\}, B^+(n) > 0$, and $B^+(n)U^*B(2n) = B^*(n)B(2n) = 2B(n) = 2UB^+(n) = B^+(n)(2U)$
so, $B(2n) = 2U^2$ and

$$f(z) = I + B^+(n)(Uz^n + (Uz^n)^3 + \cdot) + 2((Uz^n)^2 + (Uz^n)^4 + \cdots).$$

Suppose $f_1(z) = I + B^+(n)(z + z^3 + \cdot) + 2(z^2 + z^4 + \cdots)$, because $f(z)$ is a strong extreme point, so is $f_1(z)$, but

$$f_1(z) = (I - \frac{B^+(n)}{2})\frac{1+z^2}{1-z^2} + \frac{B^+(n)}{2}\frac{1+z}{1-z}.$$

Therefore, $B^+(n) = 2I$ and $f_1(z) = \frac{1+z}{1-z}I$, but lemma3 has proved that $f_1(z)$ is not a strong extreme point, this is a contradiction.
2) Suppose $KerB(n) = KerB^*(n) \neq \{0\}$.
Let $M = KerB(n)$, and $P_M(P_{M^\perp})$ be the projection on $M(M^\perp)$, then $f(z) = P_{M^\perp}(I + B(n)P_{M^\perp}z + B(n+1)P_{M^\perp}z^2 + \cdots) + (I - P_{M^\perp})(I + B(n)P_M z + B(n+1)P_M z^2 + \cdots)$.
So, f(z) is not a strong extreme point.

Lemma 4. $\frac{1+z}{1-z}I$ is a sub-strong extreme point of $N_0(\Delta)$.

Proof. Let
$$\frac{1+z}{1-z}I = Ag(z) + (I-A)h(z)$$
where, $g(z) = I + B_1 z + B_2 z^2 + \cdots \in N_0(\Delta), h(z) = I + C_1 z + C_2 z^2 + \cdots \in N_0(\Delta)$ and $0 < A < I$, commuting with h,g. Then

$$2I = AB_j + (I-A)C_j, (j \geq 1)$$

If for some j, $B_j \neq 2I$, then

$$C_j = (I-A)^{-1}(2I - AB_j) = 2I + A(I-A)^{-1}(2I - B_j)$$

$$C_j^* C_j = 4I + 2A(I-A)^{-1}\{4I - B_j^* - B_j\} + (I-A)^{-2}A^2(2I-B_j)^*(2I-B_j) \geq 4I$$

and, $C_j^* C_j \neq 4I$, so $\|C_j\| > 2$, this contracdicts to lemma 1.

The proof of theorem 2

We will follow the proof by Holland [1].
Let $f(z) = I + B(n)z^n + B(n+1)z^{n+1} + \cdots \in N_0(\Delta)$, and it be a sub-strong extreme point of $N_0(\Delta), B(n) \neq 0$ and $M = KerB(n), M^\perp = R(B(n))$, then

$$f(z)P_{M^\perp} = IP_{M^\perp} + B(n)P_{M^\perp}z^n + \cdots$$

is a sub-strong extreme point of $N_0(\Delta)$ on $M^\perp$. Because $KerB(n)P_{M^\perp} = \{0\}$, as in the proof of Theorem 1, we can see $\frac{1}{2}B(n)P_{M^\perp}$ must be an unitary operator U_0 on $M^\perp$, so

$$f(z)P_{M^\perp} = [IP_{M^\perp} + U_0 z^n][IP_{M^\perp} - U_0 z^n]^{-1}$$

But, from[4], there is a self-adjoint opertor A on $M^\perp$, such that $U_0 = exp(iA)$. Let $U_1 = exp(\frac{iA}{n})$ and $\omega = exp(\frac{i2\pi}{n})$, then

$$f(z)P_{M^\perp} = \frac{1}{n}\sum_{j=0}^{n-1}(IP_{M^\perp} + U_1\omega^j z)(IP_{M^\perp} - U_1\omega^j z)^{-1}.$$

Because $f(z)P_{M^\perp}$ is a sub-strong extreme point , we must have $n = 1$, so

$$f(z)P_{M^\perp} = (IP_{M^\perp} + U_0 z)(IP_{M^\perp} - U_0 z)^{-1}.$$

where U_0 is an unitary operator on $M^\perp$. On the other hand,

$$f(z)P_M = IP_M + B(n+1)P_M z^{n+1} + \cdots$$

But $n+1 \neq 1$, and when $M \neq \{0\}, IP_M$ is not an extreme point on M, so $M = \{0\}$ and $f(z) = (I + Uz)(I - Uz)^{-1}$, wherre U is an unitary operator on H.

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