AN ORDER PRESERVING INEQUALITY VIA FURUTA INEQUALITY *

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Abstract. Using Furuta's inequality, we can get that if $1 < p < t, A_1 > 0, A_2 \geq B > 0$ and $A_1^t \natural_{\frac{1-t}{p-t}} A_2^p \leq A_2$, then

$$(B^r A_1{}^t B^r)^{\frac{\alpha+2r}{t+2r}} \geq (B^r A_2{}^p B^r)^{\frac{\alpha+2r}{p+2r}}.$$

holds for any $0 \leq \alpha \leq \min\{2p-1, t\}$ and $r \geq 0$. We can also get that if $1 \leq p \leq 2p < t, A_1 > 0, A_2 \geq B > 0$ and $A_1^t \natural_{\frac{2p-t}{p-t}} A_2^p \leq A_2^{2p}$, then

$$(B^r A_1{}^t B^r)^{\frac{\alpha+2r}{t+2r}} \geq (B^r A_2{}^p B^r)^{\frac{\alpha+2r}{p+2r}}.$$

holds for any $0 \leq \alpha \leq 2p$ and $r \geq 0$.

1. INTRODUCTION

In what follows, H means a complex Hilbert space. A bounded linear operator T on H is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for any x$\in H$. Also an operator T is strictly positive (in symbol: $T > 0$) if T is positive and invertible. Furuta's inequality means the following results.

Theorem F (Furuta inequality)

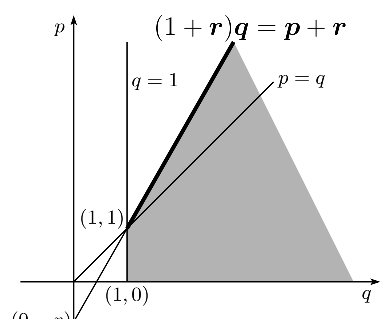
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



Figure

We remark that Theorem F yields the following famous Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above.

Theorem L-H. $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.

Alternative proofs of Theorem F are given in [1][12] and also an elementary one-page proof in [6]. It is shown in [14] that the domain drawn for p, q and r in Figure is the best

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possible for Theorem F.

It is known that (i) and (ii) in Theorem F remain valid for some negative numbers p, q and r in case A and B are invertible. By a simple observation, the problem to find real numbers p, q and r for which (i) or (ii) holds is reduced to the case $p \geq 0, q > 0$ and $r \in R$ for (ii). Here we put $r = -t \leq 0$ and q minimum in (ii),then the following results are known.

Theorem A([2][13][15][16]).If $A \geq B \geq 0$ with $A > 0$, then the following inequalites hold:
(I) $A^{1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}}$ for $1 \geq p > t \geq 0$ with $p \geq \frac{1}{2}$.
(II) $A^{-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-t}{p-t}}$ for $1 \geq t > p \geq 0$ with $\frac{1}{2} \geq p$.
(III) $A^{2p-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t}{p-t}}$ for $\frac{1}{2} \geq p > t \geq 0$.
(IV) $A^{2p-1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-1-t}{p-t}}$ for $1 \geq t > p \geq \frac{1}{2}$.

Yoshino [16] initiated an attempt to extend the domain in which the form of Theorem F holds. Afterwards, the domain given by him was enlarged to (I) by Fujii,Kamei and Furuta [2]. Kamei [13] gave simplified proofs of (I) and (III). Tanahashi [15] showed all the inequalities in Theorem A and proved that the outside exponents of (I),(II) and (IV) are best possible. Extension of Theorem A are shown in [3][4][8] and [11], and related results to Theorem A are shown in[9] and [10].

The following $\natural_s$ for any $s > 0$ is defined in [7] by

$$A \natural_s B = A^{1/2}(A^{-1/2} B A^{-1/2})^s A^{1/2}$$

for an invertible positive A and a positive operator B.

Associated with (I) and (III) of Theorem A, it is shown in [3] and [15]that the following inequalities hold when $A \geq B > 0$:

$$(A^t \natural_{\frac{1-t}{p-t}} B^p) \leq B, \ for\ 0 \leq t < p\ and\ \frac{1}{2} \leq p \leq 1, \tag{1.1}$$

$$(A^t \natural_{\frac{2p-t}{p-t}} B^p) \leq B^{2p}, \ for\ 0 \leq t < p \leq \frac{1}{2}. \tag{1.2}$$

In this paper,we remark that $A^\alpha \geq B^\alpha$ holds for any $\alpha \in [0, 2p-1]$, when (1.1)holds for p and t such that $1 < p < 2p - 1 < t$. But $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 2p-1]$ does not always ensure (1.1) in general. Similarly, $A^\alpha \geq B^\alpha$ holds for any $\alpha \in [0, 2p]$, when the opposite inequality of (1.2) holds for p and t such that $1 \leq p \leq 2p < t$, and $A^\alpha \geq B^\alpha$ holds for any $\alpha \in [0, 2p]$ does not always ensure the opposite inequality of (1.2) in general.

2. MAIN RESULTS

Theorem 1.Let $1 \le p < t, A_1 > 0$, and $A_2 \ge B > 0$ such that

$$A_1{}^t \natural_{\frac{1-t}{p-t}} A_2{}^p \le A_2 \ for \ p > 1 \tag{2.1}$$

and
$A_1 \ge A_2 \quad for \ p = 1.$
then

$$(B^r A_1^t B^r)^{\frac{\alpha+2r}{t+2r}} \ge (B^r A_2^p B^r)^{\frac{\alpha+2r}{p+2r}}.$$

holds for any $0 \le \alpha \le \min\{2p-1, t\}$ and $r \ge 0$.

Theorem 2.Let $1 \le p \le 2p < t, A_1 > 0$, and $A_2 \ge B > 0$ such that

$$A_1^t \natural_{\frac{2p-t}{p-t}} A_2^p \ge A_2{}^{2p}, \tag{2.2}$$

then

$$(B^r A_1^t B^r)^{\frac{\alpha+2r}{t+2r}} \ge (B^r A_2^p B^r)^{\frac{\alpha+2r}{p+2r}}.$$

holds for any $0 \le \alpha \le 2p$ and $r \ge 0$.

3. PROOFS OF THE MAIN RESULTS

We need the following lemma.
Lemma ([7]).For invertible positive operators A and invertible operator B,

$$(BAB^*)^s = BA^{1/2}(A^{1/2}B^*BA^{1/2})^{s-1}A^{1/2}B^*$$

holds for any real number s.

Proof of Theorem 1.
(i)When $p = 1, A_1 \ge A_2 \ge B > 0$ implies

$$(B^r A_1{}^t B^r)^{\frac{1+2r}{t+2r}} \ge B^r A_1 B^r \ge B^r A_2 B^r$$

for $t \ge 1$ and $r \ge 0$ by Furuta inequality, so that the theorem is proved by $L\ddot{o}$wner-Heinz theorem for $\alpha \in [0, 1]$.
(ii)When $p > 1$, by (2.1) and the Lemma, we have

$$A_2^{\frac{p}{2}}(A_2^{\frac{p}{2}} A_1^{-t} A_2^{\frac{p}{2}})^{\frac{1-p}{p-t}} A_2^{\frac{p}{2}} = A_1^{\frac{t}{2}}(A_1^{\frac{-t}{2}} A_2^p A_1^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A_1^{\frac{t}{2}} = A_1^t \natural_{\frac{1-t}{p-t}} A_2^p \le A_2,$$

that is,

$$(A_2^{-\frac{p}{2}} A_1^t A_2^{-\frac{p}{2}})^{\frac{p-1}{t-p}} \ge A_2^{p-1}. \tag{3.1}$$

Applying Furuta inequality to $A_2 \geq B > 0$, we also have for each $r \geq 0$,

$$(A_2^{\frac{p}{2}} B^{2r} A_2^{\frac{p}{2}})^{\frac{p-1}{p+2r}} \leq A_2^{p-1}. \tag{3.2}$$

Since $(1+p)\frac{p+2r}{p-1} \geq 2r+p$.Let$X = (A_2^{\frac{p}{2}} B^{2r} A_2^{\frac{p}{2}})^{\frac{p-1}{p+2r}}$, and $Y = (A_2^{-\frac{p}{2}} A_1^t A_2^{-\frac{p}{2}})^{\frac{p-1}{t-p}}$, then by (3.1) and (3.2), we see

$$X \leq A_2^{p-1} \leq Y. \tag{3.3}$$

Applying Furuta inequality again to $Y \geq X > 0$, we also have

$$(X^{\frac{p+2r}{2(p-1)}} Y^{\frac{t-p}{p-1}} X^{\frac{p+2r}{2(p-1)}})^{\frac{\alpha+2r}{t+2r}} \geq X^{\frac{\alpha+2r}{p-1}} \tag{3.4}$$

since $\frac{t-p}{p-1}, \frac{p+2r}{p-1} \geq 0, \frac{t+2r}{\alpha+2r} \geq 1$ and $(1+\frac{p+2r}{p-1})\frac{t+2r}{\alpha+2r} \geq \frac{t-p}{p-1} + \frac{p+2r}{p-1}$.
Applying the Lemma several times and (3.4), then for each $r \geq 0$ we have the following results

$$\begin{aligned}
& (B^r A_1^t B^r)^{\frac{\alpha+2r}{t+2r}} \\
= \;& B^r A_1^{t/2} (A_1^{t/2} B^{2r} A_1^{t/2})^{\frac{\alpha-t}{t+2r}} A_1^{t/2} B^r \qquad \text{by the lemma} \\
= \;& B^r A_1^{t/2} (A_1^{-t/2} A_2^{p/2} (A_2^{-p/2} B^{-2r} A_2^{-p/2}) A_2^{p/2} A_1^{-t/2})^{\frac{t-\alpha}{t+2r}} A_1^{t/2} B^r \\
= \;& B^r A_2^{p/2} (A_2^{-p/2} B^{-2r} A_2^{-p/2})^{1/2} \{(A_2^{-p/2} B^{-2r} A_2^{-p/2})^{1/2} A_2^{p/2} A_1^{-t} A_2^{p/2} \\
& (A_2^{-p/2} B^{-2r} A_2^{-p/2})^{1/2}\}^{-\frac{\alpha+2r}{t+2r}} (A_2^{-p/2} B^{-2r} A_2^{-p/2})^{1/2} A_2^{p/2} B^r \qquad \text{by the lemma} \\
= \;& B^r A_2^{p/2} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} \{(A_2^{p/2} B^{2r} A_2^{p/2})^{1/2} A_2^{-p/2} A_1^t A_2^{-p/2} \\
& (A_2^{p/2} B^{2r} A_2^{p/2})^{1/2}\}^{\frac{\alpha+2r}{t+2r}} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} A_2^{p/2} B^r \\
= \;& B^r A_2^{p/2} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} \{X^{\frac{p+2r}{2(p-1)}} Y^{\frac{t-p}{p-1}} X^{\frac{p+2r}{2(p-1)}}\}^{\frac{\alpha+2r}{t+2r}} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} A_2^{p/2} B^r \\
\geq \;& B^r A_2^{p/2} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} X^{\frac{\alpha+2r}{p-1}} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} A_2^{p/2} B^r \qquad by (3.4) \\
= \;& B^r A_2^{p/2} (A_2^{p/2} B^{2r} A_2^{p/2})^{\frac{\alpha+2r}{p+2r}-1} A_2^{p/2} B^r \\
= \;& (B^r A_2^p B^r)^{\frac{\alpha+2r}{p+2r}} \qquad \text{by the lemma.}
\end{aligned}$$

Hence the proof of Theorem 1 is complete.

Proof of Theorem 2.
By (2.2) and the Lemma ,we have $A_2^{\frac{p}{2}} (A_2^{\frac{p}{2}} A_1^{-t} A_2^{\frac{p}{2}})^{\frac{p}{p-t}} A_2^{\frac{p}{2}} = A_1^{\frac{t}{2}} (A_1^{\frac{-t}{2}} A_2^p A_1^{\frac{-t}{2}})^{\frac{2p-t}{p-t}} A_1^{\frac{t}{2}} = A_1^t \natural_{\frac{2p-t}{p-t}} A_2^p \leq A_2^{2p}$,
that is

$$(A_2^{-\frac{p}{2}} A_1^t A_2^{-\frac{p}{2}})^{\frac{p}{t-p}} \geq A_2^p \tag{3.5}$$

Applying Furuta inequality to $A_2 \geq B > 0$, we also have for each $r \geq 0$,

$$(A_2^{\frac{p}{2}} B^{2r} A_2^{\frac{p}{2}})^{\frac{p}{p+2r}} \leq A_2^p \tag{3.6}$$

since $(1+p)\frac{p+2r}{p} \geq 2r+p$. Let $X = (A_2^{\frac{p}{2}} B^{2r} A_2^{\frac{p}{2}})^{\frac{p}{p+2r}}$, and $Y = (A_2^{-\frac{p}{2}} A_1^t A_2^{-\frac{p}{2}})^{\frac{p}{t-p}}$, then by (3.5) and (3.6) we see

$$X \leq A_2^p \leq Y \tag{3.7}$$

Applying Furuta inequality again to $Y \geq X > 0$, we also have

$$(X^{\frac{p+2r}{2p}} Y^{\frac{t-p}{p}} X^{\frac{p+2r}{2p}})^{\frac{\alpha+2r}{t+2r}} \geq X^{\frac{\alpha+2r}{p}} \tag{3.8}$$

since $\frac{t-p}{p}, \frac{p+2r}{p} \geq 0, \frac{t+2r}{\alpha+2r} \geq 1$ and $(1+\frac{p+2r}{p})\frac{t+2r}{\alpha+2r} \geq \frac{t-p}{p} + \frac{p+2r}{p}$. Applying the Lemma several times and (3.8), then for each $r \geq 0$ we have the following :

$$\begin{aligned}
& (B^r A_1^t B^r)^{\frac{\alpha+2r}{t+2r}} \\
= \ & B^r A_1^{t/2} (A_1^{t/2} B^{2r} A_1^{t/2})^{\frac{\alpha-t}{t+2r}} A_1^{t/2} B^r \qquad \text{by the lemma} \\
= \ & B^r A_1^{t/2} (A_1^{-t/2} A_2^{p/2} (A_2^{-p/2} B^{-2r} A_2^{-p/2}) A_2^{p/2} A_1^{-t/2})^{\frac{t-\alpha}{t+2r}} A_1^{t/2} B^r \\
= \ & B^r A_2^{p/2} (A_2^{-p/2} B^{-2r} A_2^{-p/2})^{1/2} \{ (A_2^{-p/2} B^{-2r} A_2^{-p/2})^{1/2} A_2^{p/2} A_1^{-t} A_2^{p/2} \\
& (A_2^{-p/2} B^{-2r} A_2^{-p/2})^{1/2} \}^{-\frac{\alpha+2r}{t+2r}} (A_2^{-p/2} B^{-2r} A_2^{-p/2})^{1/2} A_2^{p/2} B^r \qquad \text{by the lemma} \\
= \ & B^r A_2^{p/2} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} \{ (A_2^{p/2} B^{2r} A_2^{p/2})^{1/2} A_2^{-p/2} A_1^t A_2^{-p/2} \\
& (A_2^{p/2} B^{2r} A_2^{p/2})^{1/2} \}^{\frac{\alpha+2r}{t+2r}} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} A_2^{p/2} B^r \\
= \ & B^r A_2^{p/2} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} \{ X^{\frac{p+2r}{2p}} Y^{\frac{t-p}{p}} X^{\frac{p+2r}{2p)}} \}^{\frac{\alpha+2r}{t+2r}} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} A_2^{p/2} B^r \\
\geq \ & B^r A_2^{p/2} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} X^{\frac{\alpha+2r}{p}} (A_2^{p/2} B^{2r} A_2^{p/2})^{-1/2} A_2^{p/2} B^r \qquad by (3.8) \\
= \ & B^r A_2^{p/2} (A_2^{p/2} B^{2r} A_2^{p/2})^{\frac{\alpha+2r}{p+2r}-1} A_2^{p/2} B^r \\
= \ & (B^r A_2^p B^r)^{\frac{\alpha+2r}{p+2r}} \qquad \text{by the lemma.}
\end{aligned}$$

Hence the proof of Theorem 2 is complete.

Putting $r = 0$ in Theorem 1 , Theorem 2, we have

Corollary 1 Let $A_1 > 0, A_2 > 0$, and $1 < p < 2p-1 < t$, if

$$A_1{}^t \natural_{\frac{1-t}{p-t}} A_2{}^p \leq A_2,$$

then $A_1^\alpha \geq A_2^\alpha$ holds for $0 \leq \alpha \leq 2p-1$

Corollary 2 Let $A_1 > 0, A_2 > 0$, and $1 \leq p < 2p < t$, if

$$A_1{}^t \natural_{\frac{2p-t}{p-t}} A_2{}^p \geq A_2^{2p},$$

then $A_1^\alpha \geq A_2^\alpha$ holds for $0 \leq \alpha \leq 2p$

4. EXAMPLES

Example 1. There exist $A_1 > 0, A_2 > 0, 1 < p < 2p-1 < t$, such that $A_1^{2p-1} \geq A_2^{2p-1}$, but

$$A_1{}^t \natural_{\frac{1-t}{p-t}} A_2{}^p \not\leq A_2.$$

Let $p = 2, t = 4$ and $A_1^3 = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}$, and $A_2^3 = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. we can get $A_1^3 \geq A_2^3$, but the computer shows

$(A_2 A_1^{-4} A_2)^{1/2}$

$= \begin{pmatrix} 0.5324196859\cdots & 0.1228447632\cdots \\ 0.1228447632\cdots & 0.3766522145\cdots \end{pmatrix}$,and

A_2^{-1}

$$= \begin{pmatrix} 0.7924017738\cdots & -0.2075982261\cdots \\ -0.2075982261\cdots & 0.7924017738\cdots \end{pmatrix},$$
then the eigenvalues of $A_2^{-1} - (A_2A_1^{-4}A_2)^{1/2}$ are $0.6773631660\cdots and -0.0016315189\cdots$, so

$$(A_2A_1^{-4}A_2)^{1/2} \not\leq A_2^{-1}.$$

Hence ${A_1}^4\natural_{\frac{3}{2}}{A_2}^2 = A_1^2(A_1^{-2}A_2^2A_1^{-2})^{\frac{3}{2}}A_1^2 = A_2(A_2A_1^{-4}A_2)^{\frac{1}{2}}A_2 \not\leq A_2$. by the Lemma.
Example 2. There exist $A_1 > 0, A_2 > 0, 1 \leq p < 2p < t$, such that $A_1^{2p} \geq A_2^{2p}$, but

$${A_1}^t\natural_{\frac{2p-t}{p-t}}{A_2}^p \not\geq {A_2}^{2p}.$$

Let $p = 2, t = 6$ and $A_1^4 = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}$, and $A_2^4 = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. we can get $A_1^4 \geq A_2^4$, but the computer shows
$(A_2A_1^{-6}A_2)^{1/2}$
$= \begin{pmatrix} 0.4383872572\cdots & 0.0755991077\cdots \\ 0.0755991077\cdots & 0.2932678545\cdots \end{pmatrix}$, and
A_2^{-2}
$$= \begin{pmatrix} 0.7236067977\cdots & -0.2763932022\cdots \\ -0.2763932022\cdots & 0.7236067977\cdots \end{pmatrix},$$
then the eigenvalues of $A_2^{-2} - (A_2A_1^{-6}A_2)^{1/2}$ are $0.7171724757\cdots and -0.0016139920\cdots$, so

$$(A_2A_1^{-6}A_2)^{1/2} \not\leq A_2^{-2}.$$

Hence ${A_1}^6\natural_{\frac{1}{2}}{A_2}^2 = A_1^3(A_1^{-3}A_2^2A_1^{-3})^{\frac{1}{2}}A_1^3 = A_2(A_2A_1^{-6}A_2)^{-\frac{1}{2}}A_2 \not\geq A_2^4$, by the Lemma.

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