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CUT ELIMINATION THEOREMS AND A CANONICAL MODEL CONSTRUCTION FOR SOME IMPLICATIONAL SUBSTRUCTURAL LOGICS

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ABSTRACT. We introduce several restricted versions of Gentzen's structural and logical rules, and investigate cut-elimination property, theorem-equivalence, Ackermann's property, decidability and variable sharing property among implicational sequent calculi having the rules. These results include new cut-elimination theorems for the implicational fragments of the following: relevant logic E of entailment, EW, strict implication S4, S4W and full Lambek logic FL. Next we give Kripke type semantics for the implicational fragments of EW, S4W and related logics (e.g., BCK, BCI and BB′I). Further we prove the completeness theorems for the semantics by using Ishihara's canonical model construction method.

1. Introduction.

Examples of *substructural logics* are as follows. Relevant logics R, E, T, RW, EW, TW and strict implication S4 are studied in the area of philosophical logic and of artificial intelligence. BCK-logic is closely related to the theory of BCK-algebras in mathematics. Linear logics are discussed in computer science. Lambek calculi are important in linguistics. (See, e.g., [4], [1]).

These substructural logics are defined by Gentzen-type sequent calculi in which applications of structural rules are restricted. These structural rules alone correspond to Hilbert-style axiom schemes which consist of the implication connective. Thus the essential parts of these logics are their implicational fragments, which are called *implicational substructural logics*. Hence these implicational substructural logics are very important for essential discussion.

Examples of these implicational substructural logics are as follows. BCIW(or $R_{\rightarrow}$), $E_{\rightarrow}$, BB′IW(or $T_{\rightarrow}$), BCK, BCI(or $R_{\rightarrow}-W$), $E_{\rightarrow}-W$, BB′I(or $T_{\rightarrow}-W$), $S4_{\rightarrow}$ or $FL_{\rightarrow}$ is the implicational fragment of R, E, T, BCK-logic, linear logic, EW, TW, S4 or FL(full Lambek logic) respectively. In the area of relevant logic, $R_{\rightarrow}$, $E_{\rightarrow}$, $T_{\rightarrow}$, $R_{\rightarrow}-W$, $E_{\rightarrow}-W$, $T_{\rightarrow}-W$, $S4_{\rightarrow}$ are important in formalizing "relevant (or strict) implication" in pure human reasoning. To evaluate the notion "relevant implication", some interesting properties such as variable sharing property and Ackermann's property were proposed. Furthermore, studies of decision problem and of other problems for these logics are still active. For example, the decision problem for $T_{\rightarrow}$ remains open, and P−W problem for $T_{\rightarrow}-W$ has been studied. (See, e.g., [1], [4]). BCK and BCI are important in type-assignment to λ-term and in combinatory

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logic. Typability for BCK λ-term and BCI λ-term, and some results for uniquness of normal proof in BCK are interesting results. Also, proof finding algorithms for many implicational substructural logics are studied in theoretical computer science. (See, e.g., [6], [2]).

To discuss or to solve some of the properties or the problems, various sequent calculi and different definitions of semantics have been proposed for these logics. In particular, various kinds of sequent calculi with cut-elimination property have been introduced. The calculi with the property are useful to show the fundamental properties such as decidability, variable sharing property and interpolation property.

This paper is intended to investigate the cut-elimination theorems (for new sequent calculi) and a canonical model construction method (with respect to Kripke-type semantics) for the implicational substructural logics mentioned above. As a result, we can derive Ackermann's property, decidability, variable sharing property and completeness result for many logics.

Prior to the precise discussion, we introduce our language. *Formulas* are constructed from propositional variables and $\rightarrow$(implication). Small letters $p, q, ...$ are used for propositional variables, Greek small letters $\alpha, \beta, ...$ are used for formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used for finite (possibly empty) sequences of formulas. A *sequent* is an expression of the form $\Gamma \Rightarrow \alpha$. If a formula α is of the form $\alpha_1 \rightarrow \alpha_2$, then α is said to be an *implication* and denoted by $\overrightarrow{\alpha}$. If Δ is a (possibly empty) sequence of implications, then Δ is often denoted by $\overrightarrow{\Delta}$. Moreover, $\overrightarrow{\Gamma \Rightarrow \alpha}$ means that α must be an implication if Γ is empty. If a sequent S is provable in a system L, then we write $L \vdash S$.

In this paper, the notion of "cut-elimination theorem" is precisely defined as follows.

Definition. (Admissible Rule, Derivable Rule and Cut-Elimination Theorem) A rule R is said to be *admissible* in a system L if the following condition is satisfied: for any instance

$$\frac{S_1 \quad S_2}{S}$$

of R, if $L \vdash S_i$ for any $i \in \{1, 2\}$, then $L \vdash S$. Moreover, R is said to be *derivable* in L if there is a derivation from S_1 and S_2 to S in L. Note that derivability implies admissibility. By "the *cut-elimination theorem* for a system L", we mean that the rule cut:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \beta}{\Delta, \Gamma, \Sigma \Rightarrow \beta} \text{ (cut)}$$

(or a slight modification of this form) is admissible in the cut-free part of L.

The cut-elimination theorem for a system L says that the rule cut in L is redundant in L (i.e., for any sequent S in L, if $L \vdash S$, then S is provable in L without the rule cut).

We give a precise definition of the system $\mathrm{FL}_{\rightarrow}$. The initial sequents of $\mathrm{FL}_{\rightarrow}$ are of the form $\alpha \Rightarrow \alpha$. The rules of inferences of $\mathrm{FL}_{\rightarrow}$ are as follows.

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \beta}{\Delta, \Gamma, \Sigma \Rightarrow \beta} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \gamma}{\Delta, \alpha\rightarrow\beta, \Gamma, \Sigma \Rightarrow \gamma} (\rightarrow\text{left}) \qquad \frac{\Delta, \alpha \Rightarrow \beta}{\Delta \Rightarrow \alpha\rightarrow\beta} (\rightarrow\text{right})$$

It is known that $\mathrm{FL}_{\rightarrow}$+ (ex^{000}), $\mathrm{FL}_{\rightarrow}$+ (ex^{000})+ (we^{00}), $\mathrm{FL}_{\rightarrow}$+ (ex^{000})+ (co^{00}) or $\mathrm{FL}_{\rightarrow}$+ (ex^{000})+ (we^{00})+ (co^{00}) ($\mathrm{LJ}_{\rightarrow}$) is a system for BCI($\mathrm{R}_{\rightarrow}-$W), BCK, BCIW($\mathrm{R}_{\rightarrow}$) or $\mathrm{H}_{\rightarrow}$ respectively, where (ex^{000}), (we^{00}) and (co^{00}) are usual structural rules:

$$\frac{\Delta, \alpha, \beta, \Gamma \Rightarrow \gamma}{\Delta, \beta, \alpha, \Gamma \Rightarrow \gamma} (\text{ex}^{000}) \qquad \frac{\Delta, \Gamma \Rightarrow \gamma}{\Delta, \alpha, \Gamma \Rightarrow \gamma} (\text{we}^{00}) \qquad \frac{\Delta, \alpha, \alpha, \Gamma \Rightarrow \gamma}{\Delta, \alpha, \Gamma \Rightarrow \gamma.} (\text{co}^{00})$$

The cut-elimination theorems hold for the systems defined above.

In [10], Kashima and Kamide introduced systems $GS4_{\to}$,$GE_{\to}$,$GS4_{\to}-W$ and $GE_{\to}-W$ for logics $S4_{\to}$, $E_{\to}$, $S4_{\to}-W$ and $E_{\to}-W$ respectively, which are defined by the following: $GS4_{\to} = FL_{\to}+ (ex^{110})+ (co^{00})+ (we^{01})$, $GE_{\to} = GS4_{\to}- (we^{01})$, $GS4_{\to}-W = GS4_{\to}- (co^{00})$ and $GE_{\to}-W = GS4_{\to}-W- (we^{01})$ where the rules (ex^{110}) and (we^{01}) are of the forms:

$$\frac{\Delta, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Gamma \Rightarrow \gamma}{\Delta, \overrightarrow{\beta}, \overrightarrow{\alpha}, \Gamma \Rightarrow \gamma}\ (ex^{110}) \qquad \frac{\Delta, \overrightarrow{\Gamma \Rightarrow \gamma}}{\Delta, \alpha, \overrightarrow{\Gamma \Rightarrow \gamma}.}\ (we^{01})$$

Further, Kashima and Kamide showed the following (except for the theorem of $FL_{\to}$).

Theorem 1. (Cut-Elimination Theorem) The cut-elimination theorems hold for $GS4_{\to}$, $GE_{\to}$, $GS4_{\to}-W$, $GE_{\to} - W$ and $FL_{\to}$.

By using the theorems, we can get the known fact: $E_{\to}-W \subset BCI(R_{\to}-W)$, $E_{\to}-W \subset S4_{\to}-W \subset BCK$, $E_{\to}-W \subset E_{\to} \subset BCIW(R_{\to})$, $S4_{\to}-W \subset S4_{\to} \subset H_{\to}$ where $\subset$ denotes the proper inclusion between the sets of provable formulas.

Also Kashima and Kamide showed that the cut-elimination theorems hold for the following systems: $FL_{\to}+ (we^{11})$, $FL_{\to}+ (we^{01})$, $FL_{\to}+ (ex^{110})+ (we^{11})$, $FL_{\to}+ (ex^{110})+ (co^{10})$, $FL_{\to}+ (ex^{110})+ (co^{10})+ (we^{11})$, $FL_{\to}+ (ex^{110})+ (co^{10})+ (we^{01})$ and $FL_{\to}+ (ex^{110})+ (co^{00})+ (we^{11})$ where (we^{11}) and (co^{10}) are

$$\frac{\Delta, \overrightarrow{\Gamma \Rightarrow \gamma}}{\Delta, \overrightarrow{\alpha}, \overrightarrow{\Gamma \Rightarrow \gamma}}\ (we^{11}) \qquad \frac{\Delta, \overrightarrow{\alpha}, \overrightarrow{\alpha}, \Gamma \Rightarrow \gamma}{\Delta, \overrightarrow{\alpha}, \Gamma \Rightarrow \gamma.}\ (co^{10})$$

The present paper is organized as follows.

In Sections 2 and 3, we introduce new restriction $\overrightarrow{\Delta}$ for the systems defined above. The advantages of introducing $\overrightarrow{\Delta}$ are as follows: (1) we can formalize many of new cut-free systems for $S4_{\to}$, $E_{\to}$ and related logics, (2) the proofs of cut-elimination theorems for some systems with $\overrightarrow{\Delta}$ are simpler than those for the systems defined above, and (3) we can derive Ackermann's property for some systems (we can find new logics having Ackermann's property). Of course we can also derive decidability and variable sharing property for some systems with $\overrightarrow{\Delta}$. The results in the sections 2 and 3 are based on the author's dissertation [9].

In Section 4, we give Kripke type semantics for contraction-less logics such as $E_{\to}-W$, $T_{\to}-W$, $S4_{\to}-W$, BCI and BCK. Further we prove the completeness theorems for the semantics by using Ishihara's canonical model construction method.

2. Cut-elimination theorems.

In this section, we introduce new sequent systems $LS4_{\to}$, $LE_{\to}$, $LS4_{\to}-W$, $LE_{\to}-W$ and $LFL_{\to}$, and show theorem equivalence between cut-free parts of these systems and that of $GS4_{\to}$, $GE_{\to}$, $GS4_{\to}-W$, $GE_{\to}-W$ and $FL_{\to}$ respectively. As a result, the cut-elimination theorems hold for these new systems. Hence the systems $LS4_{\to}$, $LE_{\to}$, $LS4_{\to}-W$, $LE_{\to}-W$ and $LFL_{\to}$ are another formulations of the systems for $S4_{\to}$, $E_{\to}$, $S4_{\to}-W$, $E_{\to}-W$ and $FL_{\to}$ respectively. Moreover we show that there are many of cut-free systems for the logics.

Initial sequents of $LFL_{\to}$ are of the form: $\alpha \Rightarrow \alpha$. The rules of inferences of $LFL_{\to}$ are as follows:

$$\frac{\Gamma \Rightarrow \alpha \quad \overrightarrow{\Delta}, \alpha, \Sigma \Rightarrow \gamma}{\overrightarrow{\Delta}, \Gamma, \Sigma \Rightarrow \gamma,}\ (cut^*)$$

$$\frac{\overrightarrow{\Delta}, \alpha \Rightarrow \beta}{\overrightarrow{\Delta} \Rightarrow \alpha \to \beta,}\ (\to\text{right*}) \qquad \frac{\Gamma \Rightarrow \alpha \quad \overrightarrow{\Delta}, \beta, \Sigma \Rightarrow \gamma}{\overrightarrow{\Delta}, \alpha \to \beta, \Gamma, \Sigma \Rightarrow \gamma.}\ (\to\text{left*})$$

We consider the following structural rules:

$$\frac{\overrightarrow{\Delta}, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Gamma \Rightarrow \gamma}{\overrightarrow{\Delta}, \overrightarrow{\beta}, \overrightarrow{\alpha}, \Gamma \Rightarrow \gamma,}\ (\text{ex}^{110}*) \qquad \frac{\overrightarrow{\Delta}, \overrightarrow{\Gamma \Rightarrow \gamma}}{\overrightarrow{\Delta}, \alpha, \overrightarrow{\Gamma \Rightarrow \gamma},}\ (\text{we}^{01}*) \qquad \frac{\overrightarrow{\Delta}, \alpha, \alpha, \Gamma \Rightarrow \gamma}{\overrightarrow{\Delta}, \alpha, \Gamma \Rightarrow \gamma.}\ (\text{co}^{00}*)$$

We define the following systems: $\text{LS4}_{\to} = \text{LFL}_{\to} +$ (ex^{110*})+ (co^{00*})+ (we^{01*}), $\text{LE}_{\to} = \text{LS4}_{\to} -$ (we^{01*}), $\text{LS4}_{\to}-\text{W} = \text{LS4}_{\to} -$ (co^{00*}) and $\text{LE}_{\to}-\text{W} = \text{LE}_{\to} -$ (co^{00*}).

Obviously, $\text{LS4}_{\to} \subseteq \text{GS4}_{\to}$, $\text{LE}_{\to} \subseteq \text{GE}_{\to}$, $\text{LS4}_{\to}-\text{W} \subseteq \text{GS4}_{\to}-\text{W}$, $\text{LE}_{\to}-\text{W} \subseteq \text{GE}_{\to}-\text{W}$, and $\text{LFL}_{\to} \subseteq \text{FL}_{\to}$ ($\subseteq$ denotes the inclusion between the sets of provable sequents). Next, we show that the cut-free part of $\text{LS4}_{\to}$, $\text{LE}_{\to}$, $\text{LS4}_{\to}-\text{W}$, $\text{LE}_{\to}-\text{W}$ or $\text{LFL}_{\to}$is theorem equivalent to that of $\text{GS4}_{\to}$, $\text{GE}_{\to}$, $\text{GS4}_{\to}-\text{W}$, $\text{GE}_{\to}-\text{W}$ or $\text{FL}_{\to}$ respectively.

Before the proof, we must prove the following lemma.

Lemma 2. (Key Lemma). Let $L = \text{LS4}_{\to}$, $\text{LE}_{\to}$, $\text{LS4}_{\to}-\text{W}$, $\text{LE}_{\to}-\text{W}$ or LFL. If there is a cut-free proof P of $\Phi, \Psi \Rightarrow \psi$ in L and if $\Psi \Rightarrow \psi$ is an implication, then there are a sequence Φ^- and a proof P^- which satisfy the following conditions. (1) P^- is a cut-free proof of $\Phi^-, \Psi \Rightarrow \psi$ in L. (2) Φ^- is a (possibly empty) sequence of implications. (3) The rule of inference

$$\frac{\overrightarrow{\Gamma}, \Phi^-, \overrightarrow{\Delta \Rightarrow \alpha}}{\overrightarrow{\Gamma}, \Phi, \overrightarrow{\Delta \Rightarrow \alpha}}\ (\mathcal{D}_{\Phi}^{\Phi^-})$$

is cut-free derivable in L. (The sequence Φ, which is a component of the last sequent of the given proof P, will be called a redex.)

Proof The proof of the theorem is similar to that of the key lemma in [10]. Q.E.D.

Remark that this lemma does not hold for systems with (ex^{000}) or (we^{00}).

We can now prove the following theorem.

Theorem 3. (Cut-Free Equivalence). Let L^* be the sequent system $\text{LS4}_{\to}$ ($\text{LE}_{\to}$, $\text{LS4}_{\to}-\text{W}$, $\text{LE}_{\to}-\text{W}$ or $\text{LFL}_{\to}$) without (cut*), and L the system $\text{GS4}_{\to}$ ($\text{GE}_{\to}$, $\text{GS4}_{\to}-\text{W}$, $\text{GE}_{\to}-\text{W}$ or $\text{FL}_{\to}$ respectively) without (cut). Then L^* is theorem equivalent to L . That is, we have the following. $L^* \vdash \Gamma \Rightarrow \alpha$ if and only if $L \vdash \Gamma \Rightarrow \alpha$.

Proof ($\Rightarrow$) Obvious. ($\Leftarrow$) By induction on the proof P of $\Gamma \Rightarrow \alpha$ in L. We distinguish cases according to the last inference in P. We show only the following case.
(Case ($\to$left)): The last inference of P is

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \gamma}{\Delta, \alpha \to \beta, \Gamma, \Sigma \Rightarrow \gamma.}\ (\to\text{left})$$

We show that ($\to$left) is admissible in L^*. We assume that $\Gamma \Rightarrow \alpha$ and $\Delta, \beta, \Sigma \Rightarrow \gamma$ are provable in L^* and these proofs are of the forms

$$\begin{array}{c} \vdots\ Q \\ \Gamma \Rightarrow \alpha, \end{array} \qquad \begin{array}{c} \vdots\ R \\ \Delta, \beta, \Sigma \Rightarrow \gamma. \end{array}$$

We apply the Key Lemma 2 to R in which the redex is Δ; and we get a sequence Δ^- of implications, a proof R^- and derivability of the rule $\mathcal{D}_{\Delta}^{\Delta^-}$. Then $\Delta^-, \beta, \Sigma \Rightarrow \gamma$ is provable in L^* and

$$\dfrac{\dfrac{\begin{matrix} \vdots\, Q & \vdots\, R^- \\ \Gamma \Rightarrow \alpha & \Delta^-, \beta, \Sigma \Rightarrow \gamma \end{matrix}}{\Delta^-, \alpha{\to}\beta, \Gamma, \Sigma \Rightarrow \gamma}\ (\to\text{left*})}{\Delta, \alpha{\to}\beta, \Gamma, \Sigma \Rightarrow \gamma.}\ (\mathcal{D}_{\Delta}^{\Delta^-})$$

Q.E.D.

We can prove the following cut-elimination theorems.

Theorem 4. (Cut-Elimination Theorem). The cut elimination theorems hold for $\mathrm{LS4}_{\to}$, $\mathrm{LE}_{\to}$, $\mathrm{LS4}_{\to}{-}\mathrm{W}$, $\mathrm{LE}_{\to}{-}\mathrm{W}$ and $\mathrm{LFL}_{\to}$. That is, the rule (cut*) is admissible in cut-free $\mathrm{LS4}_{\to}$, $\mathrm{LE}_{\to}$, $\mathrm{LS4}_{\to}{-}\mathrm{W}$, $\mathrm{LE}_{\to}{-}\mathrm{W}$ and $\mathrm{LFL}_{\to}$.

Proof By using the fact that (cut*) is an instance of (cut), and using Theorem 1 (cut elimination theorem for $\mathrm{GS4}_{\to}$ and its subsystems) and Theorem 3 (cut-free equivalence). Q.E.D.

Moreover we note the following facts. Let $\mathrm{IS4}_{\to}$ ($\mathrm{IE}_{\to}$, $\mathrm{IS4}_{\to}{-}\mathrm{W}$ or $\mathrm{IE}_{\to}{-}\mathrm{W}$) be an arbitrary system having an arbitrary combination of the restriction $\overrightarrow{\Delta}$ and satisfying $\mathrm{LS4}_{\to} \subseteq \mathrm{IS4}_{\to} \subseteq \mathrm{GS4}_{\to}$ ($\mathrm{LE}_{\to} \subseteq \mathrm{IE}_{\to} \subseteq \mathrm{GE}_{\to}$, $\mathrm{LS4}_{\to}{-}\mathrm{W} \subseteq \mathrm{IS4}_{\to}{-}\mathrm{W} \subseteq \mathrm{GS4}_{\to}{-}\mathrm{W}$ or $\mathrm{LE}_{\to}{-}\mathrm{W} \subseteq \mathrm{IE}_{\to}{-}\mathrm{W} \subseteq \mathrm{GE}_{\to}{-}\mathrm{W}$ respectively) where $\subseteq$ denotes the inclusion between the sets of provable sequents. The cut elimination theorems hold for $\mathrm{IS4}_{\to}$, $\mathrm{IE}_{\to}$, $\mathrm{IS4}_{\to}{-}\mathrm{W}$ and $\mathrm{IE}_{\to}{-}\mathrm{W}$.

Further, we consider the system $\mathrm{KE}_{\to} = \mathrm{FL}_{\to}-$ ($\to$right)+ ($\to$right*)+ (ex^{110})+ (co^{00}). We introduce a rule (mix) which is of the form

$$\frac{\Phi \Rightarrow \phi \quad \Psi \Rightarrow \psi}{\Psi^{\star} \Rightarrow \psi}\ (\text{mix})$$

where $\Psi^{\star}$ is a sequence of formulas obtained from Ψ by replacing arbitrary occurrences of ϕ by Φ. By using (mix), we can directly prove the cut-elimination theorem for $\mathrm{KE}_{\to}$, which is proved without the key lemma. The proof is simple.

Finally, we mention traditional sequent systems for $\mathrm{S4}_{\to}$. Kripke's system for $\mathrm{S4}_{\to}$ is $\mathrm{FL}_{\to}-$ ($\to$right)+ ($\to$right*)+ (ex^{000})+ (co^{00})+ (we^{00}), Anderson and Belnap's system for $\mathrm{S4}_{\to}$ is defined by using "merge operation" and Došen's system for $\mathrm{S4}_{\to}$ is similar to $\mathrm{GS4}_{\to}$ in the present paper. As for the details of the systems, see [10] [1].

3. Applications of the cut-elimination theorems.

We consider the following new structural rules:

$$\frac{\overrightarrow{\Delta}, \overrightarrow{\Gamma \Rightarrow \gamma}}{\overrightarrow{\Delta}, \overrightarrow{\alpha}, \overrightarrow{\Gamma \Rightarrow \gamma},}\ (\mathrm{we}^{11*}) \qquad \frac{\overrightarrow{\Delta}, \overrightarrow{\alpha}, \overrightarrow{\alpha}, \Gamma \Rightarrow \gamma}{\overrightarrow{\Delta}, \overrightarrow{\alpha}, \Gamma \Rightarrow \gamma.}\ (\mathrm{co}^{10*})$$

We define the following systems: $\mathrm{LFLw}''_{\to} = \mathrm{LFL}_{\to}+$ (we^{11*}), $\mathrm{LFLw}'_{\to} = \mathrm{LFL}_{\to}+$ (we^{01*}), $\mathrm{LFLe}''_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*}) $= \mathrm{LE}_{\to}{-}\mathrm{W}$, $\mathrm{LFLe}''\mathrm{w}''_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*})+ (we^{11*}), $\mathrm{LFLe}''\mathrm{w}'_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*})+ (we^{01*}) $= \mathrm{LS4}_{\to}{-}\mathrm{W}$, $\mathrm{LFLe}''\mathrm{c}'_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*})+ (co^{10*}), $\mathrm{LFLe}''\mathrm{c}_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*})+ (co^{00*}) $= \mathrm{LE}_{\to}$, $\mathrm{LFLe}''\mathrm{c}'\mathrm{w}''_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*})+ (co^{10*})+ (we^{11*}), $\mathrm{LFLe}''\mathrm{c}'\mathrm{w}'_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*})+ (co^{10*})+ (we^{01*}), $\mathrm{LFLe}''\mathrm{cw}''_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*})+ (co^{00*})+ (we^{11*}) and $\mathrm{LFLe}''\mathrm{cw}'_{\to} = \mathrm{LFL}_{\to}+$ (ex^{110*})+ (co^{00*})+ (we^{01*}) $= \mathrm{LS4}_{\to}$.

We have the following new results.

Theorem 5. (Cut-Elimination Theorem). The cut elimination theorem holds for $\mathrm{LFLx}_{\rightarrow}$ (x $\in$ {null, w$''$, w$'$, e$''$, e$''$w$''$, e$''$w$'$, e$''$c$'$, e$''$c, e$''$c$'$w$''$, e$''$c$'$w$'$, e$''$cw$''$, e$''$cw$'$}).

By using Theorem 5, we can show the following.

Theorem 6. (Ackermann's Property). For any propositional variable p and formulas α and β, the sequent $p \Rightarrow \alpha\rightarrow\beta$ is not provable in $\mathrm{LFLx}_{\rightarrow}$ (x $\in$ {null, w$''$, e$''$, e$''$w$''$, e$''$c$'$, e$''$c, e$''$c$'$w$''$, e$''$cw$''$}).

Ackermann's property means that a non-necessitive proposition can never entail a necessitive one. This avoids fallacies of modality. (As for the details of the property, see [1].) The property does not hold for the other systems defined above. A counterexample for $\mathrm{LFLw}'_{\rightarrow}$ is $p \Rightarrow q\rightarrow q$ where p and q are propositional variables. (Failure of Ackermann's property for $\mathrm{R}_{\rightarrow}$ is well-known.) Ackermann's property for $\mathrm{E}_{\rightarrow}$ is well-known, but the syntactical proof is new and the results for the systems with restricted weakening are also new interesting results. In the proof, the restriction $\overrightarrow{\Delta}$ makes critical role.

Theorem 7. (Decidability and Variable Sharing Property). (1) $\mathrm{LFLx}_{\rightarrow}$ (x$\in${null, w$''$, w$'$, e$''$, e$''$w$''$, e$''$w$'$}) is decidable. (2) If $\alpha \Rightarrow \beta$ is provable in $\mathrm{LFLx}_{\rightarrow}$ (x $\in$ {null, e$''$, e$''$c$'$, e$''$c}), then there exists some propositional variable p that occur in both α and β.

We do not know the (direct)proofs of decidability for the other systems. The variable sharing property does not hold for the other systems defined above. A counterexample for $\mathrm{LFLw}''_{\rightarrow}$ is $p\rightarrow p \Rightarrow q\rightarrow q$ where p and q are distinct propositional variables. Further we note that there are logics without the variable sharing property and with Ackermann's property.

4. A canonical model construction.

In this section, we discuss Kripke type semantics for the implicational logics: $\mathrm{E}_{\rightarrow}-\mathrm{W}$, $\mathrm{S4}_{\rightarrow}-\mathrm{W}$, BCK, $\mathrm{R}_{\rightarrow}-\mathrm{W}$(or BCI), $\mathrm{T}_{\rightarrow}-\mathrm{W}$(or BB$'$I) and $\mathrm{E5}_{\rightarrow}-\mathrm{W}$. The semantic framework is due to Došen [3] and Ono and Komori [11]. The semantics for the logics (except for $\mathrm{S4}_{\rightarrow}-\mathrm{W}$ and $\mathrm{E5}_{\rightarrow}-\mathrm{W}$) are already discussed by Došen [3]. Ishihara [8] proposed that a canonical model construction method for propositional intuitionistic substructural logics including Corsi's logic F, FL-family and positive relevant logics. We apply the method to the implicational logics above. The present paper's result is a slight refinment of Došen [3], Ishihara [7] (the manuscript shows completeness for logic BIs equivalent to the implicational fragment of Lambek calculus L) and Ono [12] (the manuscript shows completeness for the logic BB$'$I).

Traditionally, operational semantics for the logics $\mathrm{E}_{\rightarrow}$, $\mathrm{S4}_{\rightarrow}$, $\mathrm{R}_{\rightarrow}$ and $\mathrm{T}_{\rightarrow}$ (and $\mathrm{E5}_{\rightarrow}$) were given by Urquhart [13] in 1972 (the semantics for $\mathrm{E5}_{\rightarrow}$ is due to Fine [5]). These results give us good semantics in the sense that these have optimal structures. But each of the semantics has different structure and canonical model construction method. Our motivation is to give a uniform framework with a common structure and a common way of constructing canonical model. This succseeds for the contraction-less logics but not yet for the logics with contraction.

First, we introduce Hilbert-style systems for the logics (see [1][10][5]). $\mathrm{E5}_{\rightarrow}-\mathrm{W}$ is the contraction-less part of the logic $\mathrm{E5}_I$ in [5]. A Hilbert-style system H which consists of axiom schemes $A_1, ..., A_n$ and rules $R_1, ..., R_m$ will be denoted by $\langle A_1, ..., A_n, R_1, ..., R_m\rangle$. We adopt the convention of association to the right for omitting parentheses in the following.

Our system $\mathrm{E}_{\rightarrow}-\mathrm{W}$ is ⟨B,B$'$,I,mp,mp2⟩ where the axiom schemes B, B$'$ and I are B: $(\beta\rightarrow\gamma)\rightarrow(\alpha\rightarrow\beta)\rightarrow\alpha\rightarrow\gamma$, B$'$: $(\alpha\rightarrow\beta)\rightarrow(\beta\rightarrow\gamma)\rightarrow\alpha\rightarrow\gamma$, I: $\alpha\rightarrow\alpha$, and the rules of inferences (mp) and (mp2) are

$$\frac{\alpha \quad \alpha\to\beta}{\beta,}\ (\mathrm{mp}) \qquad \frac{\beta \quad \alpha\to\beta\to\gamma}{\alpha\to\gamma.}\ (\mathrm{mp2})$$

This axiomatization ⟨B,B′,I,mp,mp2⟩ of $\mathrm{E}_{\to}-\mathrm{W}$ is theorem equivalent to the standard axiomatization ⟨B,C′,I,mp⟩ of $\mathrm{E}_{\to}-\mathrm{W}$ in [1] where C′ is $(\alpha\to\overrightarrow{\beta}\to\gamma)\to\overrightarrow{\beta}\to\alpha\to\gamma$.

Further, we consider the following axiom schemes:

C: $(\alpha\to\beta\to\gamma)\to\beta\to\alpha\to\gamma$, C′′: $(\alpha\to\beta\to\overrightarrow{\gamma})\to\beta\to\alpha\to\overrightarrow{\gamma}$, K: $\alpha\to\beta\to\alpha$ and F: $\alpha\to\beta\to\beta$.

We can define the following logics: $\mathrm{S4}_{\to}-\mathrm{W} = \mathrm{E}_{\to}-\mathrm{W}+\mathrm{F}$, $\mathrm{T}_{\to}-\mathrm{W}$(or BB′I) $= \mathrm{E}_{\to}-\mathrm{W}-$(mp2), $\mathrm{R}_{\to}-\mathrm{W}$(or BCI) $= \mathrm{E}_{\to}-\mathrm{W}+\mathrm{C}$, BCK $= \mathrm{S4}_{\to}-$ W+K and $\mathrm{E5}_{\to}-\mathrm{W} = \mathrm{E}_{\to}-\mathrm{W}+\mathrm{C}''$.

The axiomatization ⟨B,B′,I,F,mp,mp2⟩ of $\mathrm{S4}_{\to}-\mathrm{W}$ is theorem equivalent to the axiomatization ⟨B,C′,I,K′,mp⟩ of $\mathrm{S4}_{\to}-\mathrm{W}$ where K′ is $\overrightarrow{\beta}\to\gamma\to\overrightarrow{\beta}$.

A *Kripke frame* for $\mathrm{E}_{\to}-\mathrm{W}$ is a structure $\langle M,\leq,\cdot,\varepsilon\rangle$ satisfying the following conditions: (1) $\langle M,\leq\rangle$ is a poset, (2) $\cdot$ is a binary operation on M and $\varepsilon\in M$ such that (C1) $\varepsilon\cdot x = x$ for all $x\in M$, (C2) $x\cdot\varepsilon\leq x$ for all $x\in M$, (C3) $x\leq y$ implies $x\cdot z\leq y\cdot z$ for all $x,y,z\in M$, (C4) $x\cdot(y\cdot z)\leq(x\cdot y)\cdot z$ for all $x,y,z\in M$, (C5) $x\cdot(y\cdot z)\leq(y\cdot x)\cdot z$ for all $x,y,z\in M$.

A *valuation* $\models$ on a Kripke frame $\langle M,\leq,\cdot,\varepsilon\rangle$ for $\mathrm{E}_{\to}-\mathrm{W}$ is a mapping which assigns a subset of M, to each propositional variables, such that $x\leq y$ and $x\in\models(p)$ imply $y\in\models(p)$ for all propositional variable p and all $x,y\in M$. We will write $x\models p$ for $x\in\models(p)$. Each valuation $\models$ can be extended to a mapping from the set of all formulas to the power set of M by $x\models\alpha\to\beta$ if and only if $y\models\alpha$ implies $x\cdot y\models\beta$ for all $y\in M$.

By using the frame condition (C3), we can show the following.

Lemma 8. Let $\models$ be a valuation on a Kripke frame $\langle M,\leq,\cdot,\varepsilon\rangle$ for $\mathrm{E}_{\to}-\mathrm{W}$ and $x,y\in M$. Then, $x\leq y$ and $x\models\alpha$ imply $y\models\alpha$ for all formula α.

A *Kripke model* is a structure $\langle M,\leq,\cdot,\varepsilon,\models\rangle$ such that (1) $\langle M,\leq,\cdot,\varepsilon\rangle$ is a Kripke frame, (2) $\models$ is a valuation on $\langle M,\leq,\cdot,\varepsilon\rangle$. A formula α is *true* in a Kripke model $\langle M,\leq,\cdot,\varepsilon,\models\rangle$ if $\varepsilon\models\alpha$, and *valid* in a Kripke frame $\langle M,\leq,\cdot,\varepsilon\rangle$ if it is true for any valuation $\models$ on the Kripke frame.

By using Lemma 8 and the frame conditions (C1)–(C5), we can show the following.

Theorem 9. (Soundness for $\mathrm{E}_{\to}-\mathrm{W}$). If a formula α is provable in $\mathrm{E}_{\to}-\mathrm{W}$, then it is valid in any Kripke frame for $\mathrm{E}_{\to}-\mathrm{W}$.

Next we prove the completeness theorem. To prove the theorem, we introduce a notion L-pretheory and construct a canonical model. The notion "L-pretheory" is due to Ono and Komori [11]. Let $L := \{\alpha|\alpha \textit{ is provable in } \mathrm{E}_{\to}-\mathrm{W}\ \}$. An *$L$-pretheory* x is a subset of the set of all formulas such that if $\alpha\in x$ and $\alpha\to\beta\in L$, then $\beta\in x$. $\circ$ is a binary operation on the power set of all formulas, defined by $x\circ y := \{\beta|\ \exists\alpha\in y\ (\alpha\to\beta\in x)\}$.

Lemma 10. (L-pretheory). The following holds. (1) If x and y are L-pretheories, then so is $x\circ y$, (2) $L\circ\{\alpha\}$ is an L-pretheory.

By using Lemma 10, we can show the following.

Lemma 11. (Canonical Frame). Let $L := \{\alpha|\alpha \textit{ is provable in } \mathrm{E}_{\to}-\mathrm{W}\}$, and M_L be the set of all L-pretheories. Then $\langle M_L,\subseteq,\circ,L\rangle$ is a Kripke frame for $\mathrm{E}_{\to}-\mathrm{W}$, that is we have the following: (1) $\langle M_L,\subseteq\rangle$ is a poset, (2) M_L is closed under $\circ$, (3) $L\in M_L$, (4) $L\circ x = x$ for all $x\in M_L$, (5) $x\circ L\subseteq x$ for all $x\in M_L$, (6) $x\subseteq y$ implies $x\circ z\subseteq y\circ z$ for all $x,y,z\in M_L$, (7) $x\circ(y\circ z)\subseteq(x\circ y)\circ z$ for all $x,y,z\in M_L$, (8) $x\circ(y\circ z)\subseteq(y\circ x)\circ z$ for all $x,y,z\in M_L$.

Lemma 12. (Canonical Model). Let $L := \{\alpha | \alpha \textit{ is provable in } \mathrm{E}_{\rightarrow} - \mathrm{W}\}$ and $\models_L$ be a mapping from the set of all propositional variables to the set of all subsets of M_L (M_L is the set of all L-pretheories) defined by $x \models_L p$ if and only if $p \in x$. Then $\mathbf{M_L} := \langle M_L, \subseteq, \circ, L, \models_L \rangle$ is a Kripke model for $\mathrm{E}_{\rightarrow}-\mathrm{W}$ such that for any formula α, $\alpha \in L$ if and only if α is true in $\mathbf{M_L}$.

Proof It is easy to see that $\models_L$ is a valuation on $\langle M_L, \subseteq, \circ, L \rangle$, and hence $\mathbf{M_L}$ is a Kripke model for $\mathrm{E}_{\rightarrow}-\mathrm{W}$. It remains to show that for any formula α, $L \models_L \alpha$ if and only if $\alpha \in L$. For this, it is enough to prove the following. For any formula α and any $x \in M_L$, $x \models_L \alpha$ if and only if $\alpha \in x$. We prove this by induction on the complexity of α. We show the case $\alpha \equiv \sigma \rightarrow \tau$. ($\Rightarrow$) Suppose that $x \models_L \sigma \rightarrow \tau$. We can show that $L \circ \{\sigma\}$ is L-pretheory by Lemma 10 (2), and $\sigma \in L \circ \{\sigma\}$. Then since $L \circ \{\sigma\} \models_L \sigma$ by the induction hypothesis, $x \circ (L \circ \{\sigma\}) \models_L \tau$, and hence $\tau \in x \circ (L \circ \{\sigma\})$ by the induction hypothesis. By using the frame conditions (C5) and (C1), we have $x \circ (L \circ \{\sigma\}) \subseteq (L \circ x) \circ \{\sigma\} \subseteq x \circ \{\sigma\}$. Thus $\tau \in x \circ \{\sigma\}$ and hence $\sigma \rightarrow \tau \in x$. ($\Leftarrow$) Straightforward. Q.E.D.

By using Lemma 12, we can show the following.

Theorem 13. (Completeness for $\mathrm{E}_{\rightarrow}-\mathrm{W}$). If a formula α is valid in any Kripke frame for $\mathrm{E}_{\rightarrow}-\mathrm{W}$, then it is provable in $\mathrm{E}_{\rightarrow}-\mathrm{W}$.

Let L be $\mathrm{T}_{\rightarrow}-\mathrm{W}$, $\mathrm{R}_{\rightarrow}-\mathrm{W}$, $\mathrm{E5}_{\rightarrow}-\mathrm{W}$, $\mathrm{S4}_{\rightarrow}-\mathrm{W}$ or BCK. We say *L-frame* for Kripke frame for L. A $\mathrm{T}_{\rightarrow}-\mathrm{W}$-frame is an $\mathrm{E}_{\rightarrow}-\mathrm{W}$-frame without (C2). An $\mathrm{R}_{\rightarrow}-\mathrm{W}$-frame is an $\mathrm{E}_{\rightarrow}-\mathrm{W}$-frame with the frame condition: (C6) $(x \cdot y) \cdot z \leq (x \cdot z) \cdot y$ for all $x, y, z \in M$. An $\mathrm{E5}_{\rightarrow}-\mathrm{W}$-frame is an $\mathrm{E}_{\rightarrow}-\mathrm{W}$-frame with the frame condition: (C7) $((x \cdot y) \cdot z) \cdot w \leq ((x \cdot z) \cdot y) \cdot w$ for all $x, y, z, w \in M$. An $\mathrm{S4}_{\rightarrow}-\mathrm{W}$-frame (BCK-frame) is an $\mathrm{E}_{\rightarrow}-\mathrm{W}$-frame ($\mathrm{R}_{\rightarrow}-\mathrm{W}$-frame respectively) with the frame condition: (C8) $\varepsilon \leq x$ for all $x \in M$.

By using similar method, we can show the following.

Theorem 14. (Completeness). Let L be $\mathrm{T}_{\rightarrow}-\mathrm{W}$, $\mathrm{R}_{\rightarrow}-\mathrm{W}$, $\mathrm{E5}_{\rightarrow}-\mathrm{W}$, $\mathrm{S4}_{\rightarrow}-\mathrm{W}$ or BCK. A formula α is valid in any L-frame if and only if it is provable in L.

To prove the theorems for $\mathrm{S4}_{\rightarrow}-\mathrm{W}$ and BCK, we must add the condition "L-pretheory x is nonempty" for the definition of L-pretheory.

Finally, we remark that the method for the completeness theorems works for $\mathrm{HFL}_{\rightarrow} = \mathrm{E}_{\rightarrow}-\mathrm{W}-\mathrm{B}'$, $\mathrm{HFLw}'_{\rightarrow} = \mathrm{HFL}_{\rightarrow}+\mathrm{F}$ and $\mathrm{HFLw}_{\rightarrow}$(also called BCC) $= \mathrm{HFL}_{\rightarrow}+\mathrm{K}$ (these are Hilbert-style axiomatizations for $\mathrm{FL}_{\rightarrow}$, $\mathrm{FL}_{\rightarrow}+(\mathrm{we}^{01})$ and $\mathrm{FL}_{\rightarrow}+(\mathrm{we}^{00})$ respectively).

References

[1] A.R. Anderson and N.D. Belnap, *Entailment: The Logic of Relevance and Necessity*, vol.1, Princeton University Press, Princeton, 1975.

[2] M.W. Bunder, *Proof finding algorithms for implicational logics*, Theoretical Computer Science, vol.232 (2000), pp97-105.

[3] K. Došen, *Sequent-systems and groupoid models 2*, Studia Logica, vol.48 (1988), pp41-65.

[4] K. Došen and P. S. Heister (eds.), *Substructural Logics*, Oxford University Press (1993).

[5] K. Fine, *Completeness for the S5 analogue of E_I (abstract)*, Journal of Symbolic Logic, vol.41 (1976), pp559-560.

[6] J.R. Hindley and J P. Seldin, *Introduction to Combinators and λ-calculus*, London Mathematical Society Student Texts 1, Cambridge University Press (1993).

[7] H. Ishihara, *A logic BIs*, preprint (1995).

[8] H. Ishihara, *A canonical model construction for substructural logics*, Journal of Universal Computer Science, vol.6 (2000), pp155-168.

[9] N. Kamide, *Substructural logics with restricted versions of structural rules*, Doctoral dissertation, Japan advanced institute of science and technology (2000).

[10] R. Kashima and N. Kamide, *Substructural implicational logics including the relevant logic E*, Studia Logica, vol.63 (1999), pp181-212.

[11] H. Ono and Y. Komori, *Logics without the contraction rule*, Journal of Symbolic Logic, vol.50 (1985), pp169-201.

[12] H. Ono, *Kripke-type semantics for BB′I*, preprint (1994).

[13] A. Urquhart, *Semantics for relevant logics*, Journal of Symbolic Logic, vol.37 (1972), pp159-169.

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