ITERATIVE APPROXIMATION OF FIXED POINTS OF NONEXPANSIVE MAPS

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Received September 20, 1999; revised December 26, 2000

Abstract. We use the notion of a T-regular set to establish the convergence of the iterates of the bisection mapping to a fixed point of a nonexpansive map T in the context of metrizable topological vector spaces.

1. Introduction

We generalize a well-known fixed point theorem of Veeramani ([8], Theorem 1.1) from the uniformly convex Banach space setting to metrizable topological vector spaces. As an application of our result, we prove the convergence of the sequence of iterates of the bisection map $F(x) = (x + Tx)/2$, to a fixed point of the nonexpansive map T. Our result extends Theorem 2.3 of Khan and Siddiqui [5] from convex set to a T-regular set (see also Edelstein [3]). We derive a similar result for the weak convergence of iterates under various conditions. Further we study conditions for an iteration process for a finite family of nonexpansive maps to converge to a common fixed point of the family; which extends Theorem 1 of Kuhfittig [6].

2. Preliminaries

Let (E, τ) be a topological vector space (TVS). We assume that the topology τ is generated by an F-norm q which has the properties given below.

(i) $q(x) \geq 0$ and $q(x) = 0$ if and only if $x = 0 \quad (x \in E)$.

(ii) $q(x + y) \leq q(x) + q(y)$, for all x, y in E.

(iii) $q(\lambda x) \leq q(x)$ for all (real or complex) scalars λ with $|\lambda| \leq 1$.

(iv) If $q(x_n) \to 0$, then $q(\lambda x_n) \to 0$ for all scalars λ.

(v) If $\lambda_n \to 0$, then $q(\lambda_n x) \to 0$ for all $x \in E$.

The relation $d(x, y) = q(x-y)$ defines a metric on E. The space E is said to be uniformly convex if there corresponds to each pair of positive numbers (ϵ, r) a positive number δ such that if x and y lie in E with $q(x - y) \geq \epsilon$, $q(x) < r + \delta$, $q(y) < r + \delta$, then $q\left(\dfrac{x+y}{2}\right) < r$.

The set $\mathbb{R}$ of real numbers under the F-norm $q(x) = \dfrac{|x|}{1+|x|}$, where $|\cdot|$ is the usual absolute value on $\mathbb{R}$, is a uniformly convex metric linear space.

2000 *Mathematics Subject Classification.* 41A65, 46A16, 47H10.

Key words and phrases. Fixed point, topological vector space, bisection mapping, T-regular set, sequence of iterates.

The space E is said to be strictly convex if whenever $r > 0$, $x, y \in E$ with $x \neq y$, $q(x) \leq r$ and $q(y) \leq r$, then $q\left(\frac{x+y}{2}\right) < r$.

It follows from above definitions that a uniformly convex metric linear space is strictly convex but the converse is not true (cf. [1]).

The map T on a subset M of E is said to be nonexpansive if $q(Tx - Ty) \leq q(x-y)$ for all $x, y \in M$. A mapping $T : M \to E$ is said to be demiclosed at $y \in E$ if, for any sequence $\{x_n\}$ in M, the conditions $x_n \to x \in M$ weakly and $T(x_n) \to y$ strongly imply $T(x) = y$. For a nonexpansive mapping T defined on closed bounded convex set in a uniformly convex Banach space, $I - T$ is demiclosd (see [10], Prop. 10.9).

The space E satisfies Opial's condition if and only if for every sequence $\{x_n\}$ in E weakly convergent to $y \in E$,

$$\liminf_n q(x_n - y) < \liminf_n q(x_n - x)$$

holds for all $x \neq y$ in E.

It follows from the polar identity that each Hilbert space satisfies Opial's condition.

Definition 2.1. A subset M of the space E is said to be T-regular if and only if

(i) $T : M \to M$.

(ii) $\frac{x+Tx}{2} \in M$ for each $x \in M$.

Example 2.2. Let E be a nonzero vector space and $a, b \in E$ with $a \neq b$. Put $c = \frac{a+b}{2}$ and $M = \{a, b, c\}$. Define $T : M \to M$ by $T(a) = b$, $T(b) = a$ and $T(c) = c$. Then the set M is T-regular but not convex. Define $F(x) = \frac{x+Tx}{2}$. As $\frac{a+F(a)}{2} = \frac{a+c}{2} \notin M$ so M is not F-regular. Let $A = \{a, b, c, d, e\}$ where $d = \frac{a+c}{2}$, $e = \frac{b+c}{2}$. Define $T : A \to A$ by $T(a) = b$, $T(b) = a$, $T(c) = c$, $T(d) = d$ and $T(e) = e$. From $F(x) = \frac{x+Tx}{2}$, we obtain $F(a) = F(b) = F(c) = c$, $F(d) = d$ and $F(e) = e$. Also note that $F = TF = FT$ on the set A. So the set A is T-regular and F-regular (and hence TF regular) but not a convex set in E.

Clearly for a family $\{A_\alpha : \alpha \in I\}$ of T-regular sets, the sets $\bigcap_{\alpha \in I} A_\alpha$ and $\bigcup_{\alpha \in I} A_\alpha$ are T-regular.

If T is a nonexpansive map on a T-regular subset M of the space E, then in general $F(x) = \frac{x+Tx}{2}$ may not be nonexpansive. However if the space E is strictly convex, then we have the nonexpansiveness of F. For, the relation $q(Tx - Ty) \leq q(x-y)$ for all $x, y \in M$ implies $q\left(\frac{Tx - Ty + x - y}{2}\right) < q(x-y)$ for all $x \neq y \in M$ i.e. $q(Fx - Fy) \leq q(x-y)$ for all $x, y \in M$.

3. Approximation of Fixed Points

Lemma 3.1. *Let E be a metrizable uniformly convex TVS and M a bounded T-regular subset of E. Then either each point of M is a fixed point of T or there exists a nondiametral point u of M.*

Proof. In case there is a point $x \in M$ such that $x \neq Tx$, we have by the T-regularity of M that $u = \dfrac{x+Tx}{2} \in M$.

We now show that u is the desired nondiametral point.

For any $y \in M$, $q(y - Tx) \leq \delta(M)$, diameter of M and $q(y-x) \leq \delta(M)$. As $x - Tx \neq 0$ so $q(x - Tx) > 0$ and hence $q(x - Tx) \geq \epsilon$ for some $\epsilon > 0$. The uniform convexity of E implies the existence of a number γ, $0 < \gamma < 1$ such that $q(u - y) \leq \gamma\delta(M)$ which gives that $\delta(u, M) = \sup\{q(u-y) : y \in M\} \leq \gamma\delta(M) < \delta(M)$. This completes the proof.

Remark 3.2. There exists a sequence of nonexpansive maps on the closed unit ball M of the Banach space C_0 such that none of these maps has a fixed point. Recall that M is not weakly compact (cf. [4], Ex. 2.1).

An application of the above lemma gives the following positive result.

Theorem 3.3. *Let M be a nonempty weakly compact T-regular subset of a complete metrizable uniformly convex TVS E. Suppose for each weakly closed T-regular subset K of M with $\delta(K) > 0$, there exists some γ, $0 < \gamma < 1$, such that for all x, y in K,*

$$q(Tx - Ty) \leq \max\{q(x-y), \gamma\delta(K)\}. \tag{1}$$

Then T has a fixed point in M.

Proof. Let $\mathcal{H}$ be the collection of all nonempty weakly closed T-regular subsets of M. By Zorn's Lemma, we may obtain a minimal element say K of $\mathcal{H}$. We wish to prove that K contains a fixed point of T. Suppose not; so there exists $x \in K$ such that $x \neq Tx$. Since K is a bounded and T-regular set, by Lemma 3.1, there exists $u \in K$ and $0 < \gamma < 1$ such that

$$\delta(u, K) \leq \gamma\delta(K) \tag{$*$}$$

By (1) and (*), there is a real number $0 < \lambda < 1$ such that

$$q(Tx - Ty) \leq \lambda\delta(K). \tag{$**$}$$

Let $\beta = \max\{\gamma, \lambda\}$

$$\begin{aligned} G &= \{x \in E : \delta(x, K) \leq \beta\delta(K)\} \\ H &= G \cap K \end{aligned}$$

By (*), $u \in G$ and so $u \in H$. Also H is weakly closed. If $x \in H$, then by (**), $q(Tx - Ty) \leq \beta\delta(K)$ for all $y \in K$. Hence $T(K)$ is contained in a closed ball B of centre Tx and radius $\beta\delta(K)$. It now follows that $T(K \cap B) \subset K \cap B$ and so $K \cap B$ is a T-regular set. The minimality of K gives that $T(H) \subset H$. This together with the definition of H implies that H is a T-regular set; consequently $H \in \mathcal{H}$. However $\delta(H) < \delta(K)$, which is a contradiction to the minimality of K. This completes the proof.

Corollary 3.4. *Let M be a nonempty weakly compact T-regular subset of a complete metrizable unifmorly convex TVS E. If $T : M \to M$ is a nonexpansive map, then T has a fixed point.*

The following is well known.

Lemma 3.5. *A strictly convex metric linear space is locally convex.*

Example 3.6. Let $E = C(-1, 1)$ be the space of continuous real-valued functions on $I = (-1, 1)$. Let Δ be the family of all compact subsets of I. Define for $K \in \Delta$, $p_K(f) =$

$\max_{x \in K} |f(x)|$ $(f \in E)$. The topology generated by the family $\{p_K : K \in \Delta\}$ of semi-norms makes E a strictly convex complete metric linear space which is also a locally convex space.

Let M be the closed unit ball in the usual plane and T a rotation through the angle π. Obviously T is a nonexpansive map. For any $y_0 \neq 0$ in M, the sequence $\{T^n y_0\}$ is not convergent. For $F(x) = \dfrac{x + Tx}{2}$, the sequence of iterates $\{F^n(y_0)\}$ converges to a fixed point 0 of T for all $y_0 \in M$ (see [10], p. 481).

We shall follow the argument used by Khan and Siddiqui [5] to prove the following positive result.

Theorem 3.7. *Let M be a weakly compact T-regular subset of a uniformly convex complete metrizable TVS E. Suppose $T : M \to M$ is a nonexpansive mapping and $T(M)$ is contained in a compact subset M_1 of M. Let $F : M \to M$ be a mapping defined by*

$$F(x) = \frac{x + Tx}{2}. \tag{2}$$

Then the sequence $\{F^n(x)\}$ of iterates converges to a fixed point of T for all $x \in M$.

Proof. The use of T-regularity of M is two-fold :

(i) The function F is well defined.

(ii) The sequence $\{F^n(x)\}$ lies in M.

The existence of a fixed point of T follows from Corollary 3.4. Now $q(Tx - Ty) \leq q(x - y)$ implies by the strict convexity of E that

$$q\left(\frac{Tx - Ty + x - y}{2}\right) < q(x - y) \tag{3}$$

and so

$$q(F(x) - y) < q(x - y) \text{ for } Ty = y \text{ and } Tx \neq x.$$

Obviously $\{F^n(x)\}$ is a subset of the closed convex hull of $M_1 \cup \{x\}$. By Lemma 3.5, E is a locally convex space. Since M_1 and $\{x\}$ are compact and hence closed, the convex hull of $M_1 \cup \{x\}$ must be compact. Since a sequence in a compact set has a convergent subsequence, $\{F^n(x)\}$ has a subsequence $\{F^{n_i}(x)\}$ such that

$$\lim_{i \to \infty} F^{n_i}(x) = p \text{ for some } p \in M \text{ and each } x \in M. \tag{4}$$

The conclusion of the theorem will follow if we can show that $Tp = p$. In fact once we show that $Tp = p$, we obtain

$$\lim_{n \to \infty} F^n(x) = p$$

as follows. We note that if $Tp = p$, then $F(p) = p$. Putting $y = p$ in (3), we have

$$q(F(x) - F(p)) = q(F(x) - p) < q(x - p), \tag{5}$$

The application of F on both sides of (5) for n times gives

$$q(F^{n+1}(x) - F^{n+1}(p)) < q(F^n(x) - F^n(p)).$$

Hence

$$q(F^n(x) - p) = q(F^n(x) - F^n(p)) < q(F^{n_i}(x) - F^{n_i}(p)) = q(F^{n_i}(x) - p).$$

This implies by (4) that

$$\lim_{n\to\infty} F^n(x) = p. \tag{6}$$

Now suppose $Tp \neq p$. It follows that $F(p) \neq p$; that is p is not a fixed point of T and hence for F. Suppose that T has some fixed point, say $F^k(x)$ ($\neq p$) so that $T(F^k(x)) = F^k(x)$.

It follows from the definition of F with $x = F^k(x)$ that

$$F^{k+i}(x) = F^k(x) \qquad i = 1, 2, \ldots$$

This gives by (4), $p = \lim_{i\to\infty} F^{k+i}(x) = F^k(x)$. Consequently any such fixed point of T coincides with p. Application of F to $q(F(x) - y) < q(x - y)$ implies

$$q(F^{k+1}(x) - y) < q(F^k(x) - y). \tag{7}$$

It can be easily proved that F is continuous. Let

$$r = \frac{1}{2}\{q(p - y) - q(F(p) - y)\}. \tag{8}$$

Define $B_r(F(p)) = \{w | q(w - F(p)) < r\}$. By the continuity of F there exists an open ball $B_{r_1}(p)$ with $r_1 < r$ such that

$$F(B_{r_1}(p)) \subseteq B_r(F(p)). \tag{9}$$

As $F^k(x) \in B_{r_1}(p)$, so we have by (9) that

$$F^{k+1}(x) \in B_r(F(p)) \text{ and so}$$

$$q(F^{k+1}(x) - F(p)) < r. \tag{10}$$

By using (8) and (10) we have

$$\begin{aligned} q(F^{k+1}(x) - y) &\leq q(F^{k+1}(x) - F(p)) + q(F(p) - y) \\ &< r + q(F(p) - y) \\ &< \frac{1}{2}\{q(p - y) - q(F(p) - y)\} + q(F(p) - y) \\ &< \frac{1}{2}\{q(F(p) - y) + q(p - y)\}. \end{aligned} \tag{11}$$

Now applying F to (7), we get

$$\begin{aligned} q(F^{k+2}(x) - y) &< q(F^{k+1}(x) - y) \\ &< \frac{1}{2}\{q(F(p) - y) + q(p - y)\} \qquad \text{(by 11)} \end{aligned}$$

Applying F again and again, we get

$$q(F^{k+i}(x) - y) < \frac{1}{2}\{q(F(p) - y) + q(p - y)\}. \tag{12}$$

Consider

$$q(p-y) \leq q(p-F^{k+i}(x)) + q(F^{k+i}(x)-y).$$

It now follows by (12) that $q(F^{k+i}(x)-p) > r$, that is,

$$\lim_{i\to\infty} F^{k+i}(x) \neq p$$

which contradicts (4). Thus p is a fixed point of T and by (6), $\{F^n(x)\}$ converges to p as required.

Corollary 3.8. *Let M be a compact T-regular subset of a uniformly convex complete metrizable TVS E. If $T : M \to M$ is nonexpansive, then the sequence $\{F^n(x)\}$ of iterates of F converges to a fixed point of T for all $x \in M$.*

Example 3.9. Consider the set $M = \left\{1, 2, \frac{3}{2}\right\}$ in $(\mathbb{R}, q)$ where q is the F-norm given by $q(x) = \dfrac{|x|}{1+|x|}$. Define nonexpansive map $T : M \to M$ by $T(1) = 2,\ T(2) = 1,\ T(3/2) = 3/2$. The set M is compact and T-regular and the sequence of iterates $\{F^n(x)\}$ is the constant sequence $\{3/2\}$ for all $x \in M$.

Following the technique of proof of Theorem 7 due to Browder [2] we obtain:

Theorem 3.10. *Let M be a weakly compact T-regular subset of a uniformly convex complete metrizable TVS E. If T is a nonexpansive self map on M and has at most one fixed point y in M, then the sequence $\{F^n(x)\}$ of iterates converges weakly to a fixed point of T for all $x \in M$ provided either (i) $I - T$ is demiclosed at 0 or (ii) E satisfies Opial's condition.*

Proof. For any $x_0 \in M$, put $x_n = F^n(x_0)$. Let $\{x_j\}$ be a subsequence converging weakly to $z \in M$. Since F is nonexpansive and y is also a fixed point of F, therefore we have

$$q(x_{j+1}-y) = q(F(x_j) - F(y)) \leq q(x_j - y).$$

Thus $\{q(x_j - y)\}$ is nonincreasing in j and hence $q(x_j - y)$ converges. Also

$$\begin{aligned} x_{j+1} - y &= F(x_j) - y \\ &= \frac{1}{2}(T(x_j) - T(y) + x_j - y) \end{aligned}$$

where $q(Tx_j - Ty) \leq q(x_j - y)$. The uniform convexity of E gives that

$$T(x_j) - x_j \to 0. \tag{13}$$

If (i) holds, then $(I-T)z = 0$. That is, $Tz = z$. If (ii) holds and $Tz \neq z$, we have by nonexpansiveness of T and (13) that

$$\begin{aligned} \liminf q(x_j - z) &< \liminf q(x_j - Tz) \\ &\leq \liminf q((1-T)x_j) + \liminf q(Tx_j - Tz) \\ &= \liminf q(Tx_j - Tz) \\ &\leq \liminf q(x_j - z) \end{aligned}$$

which is a contradiction. Therefore $Tz = z$. Hence $z = y$. As an arbitrary subsequence $\{x_j)$ converges weakly to y so $\{x_n\}$ converges weakly to y as required.

Proposition 3.11 [7]. *Let E be a strictly convex metric linear space, $u \in E$ and M a subset of E. If $y_1 \neq y_2 \in P_M(u)$, then $\lambda y_1 + (1-\lambda)\, y_2 \notin M$, $0 < \lambda < 1$.*

Proposition 3.12. *Let E be a strictly convex metrizable TVS, M any subset of E and $T : M \to M$. For any $u \in E$ if $P_M(u)$ is nonempty and T-regular, then each point of $P_M(u)$ is a fixed point of T.*

Proof. Suppose for some $x \in P_M(u)$, we have $x \neq Tx$. Then by Prop. 3.11, $\dfrac{x+Tx}{2} \notin M$ and so it cannot be in $P_M(u)$. By hypothesis $P_M(u)$ is T-regular and hence $x = Tx$ must hold. Thus each best M-approximation of u is a fixed point of T.

Next we present a result concerning a more general iteration scheme and its convergence to a common fixed point of a finite family of nonexpansive maps.

Suppose $\{T_i : i = 1, 2, \ldots, k\}$ is a family of nonexpansive selfmaps of a subset M of E. Let $F_0 = I$, identity map

$$\begin{aligned} F_1 &= \frac{1}{2}I + \frac{1}{2}T_1F_0 \\ F_2 &= \frac{1}{2}I + \frac{1}{2}T_2F_1 \\ F_3 &= \frac{1}{2}I + \frac{1}{2}T_3F_2 \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \\ F_k &= \frac{1}{2}I + \frac{1}{2}T_kF_{k-1} \end{aligned}$$

The sequence $\{F_k^n(x)\}$ of iterates can be expressed in the form

$$\begin{aligned} x_1 &= \frac{1}{2}x_0 + \frac{1}{2}T_kF_{k-1}x_0 \qquad (x_0 \in M) \\ x_2 &= \frac{1}{2}x_1 + \frac{1}{2}T_kF_{k-1}x_1 \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \\ x_{n+1} &= \frac{1}{2}x_n + \frac{1}{2}T_kF_{k-1}x_n \end{aligned} \tag{14}$$

For $k = 1$, (14) gives

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}T_1x_n = F_1^{n+1}(x). \tag{15}$$

Recall that the set A in Example 2.2 is T, F and TF-regular but not convex in E.

Theorem 3.13. *Let $\{T_i : i = 1, 2, \ldots, k\}$ be a finite family of nonexpansive self maps of M with nonempty set of common fixed points where M is a compact and T_iF_{i-1}-regular subset (for all $i = 1, 2, \ldots, k$) of a uniformly convex metrizable TVS E. Then for any $x \in M$, the sequence $\{F_k^n(x)\}$ converges to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$.*

Proof. Notice that F_j and T_jF_{j-1} are nonexpansive self maps of M for all $j = 1, 2, \ldots, k$. The sequence of iterates in (14) has the same form as (15) so by Corollary 3.8, the sequence

$\{F_k^n(x)\}$ converges to a fixed point y of T_kF_{k-1}. It can be easily proved as in the proof of Theorem 1 [6] that y is a common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$.

Remark 3.14. If in addition the family $\{T_i : i = 1, 2, \ldots, k\}$ in the above theorem is commutative and the set M is T_i-regular for all $i = 1, 2, \ldots, k$, then by Corollary 3.8, the set of fixed points of each T_i is nonempty and hence the set of common fixed points is nonempty.

Following example reveals that the weak compactness of the underlying set is essential for the conclusion of Theorem 3.7.

Example 3.15 [9]. Define $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by

$$T(x,y) = \begin{cases} \dfrac{\max\{x^2, y^2\}}{x^2+y^2}(y,x), & \text{if } (x,y) \in \mathbb{R}^2_+ \setminus (0,0) \\ (0,0), & \text{if } (x,y) = (0,0). \end{cases}$$

Note that $(0,0)$ is the only fixed point of T. For any $x > 0$, $F(x,0) = \dfrac{(x,0)+T(x,0)}{2}$. Then the sequence $\{F^n(x,0)\}$ of iterates is the constant sequence $\left\{\left(\dfrac{x}{2}, \dfrac{x}{2}\right)\right\}$ with the limit $\left(\dfrac{x}{2}, \dfrac{x}{2}\right)$ which is not a fixed point of T. Observe that T is a continuous function on $\mathbb{R}^2_+$ which satisfies $\|Tx + Ty\| \leq \|x + y\|$ for all $x, y \in \mathbb{R}^2_+$.

Acknowledgment. The author A.R. Khan, gratefully acknowledges the support provided by King Fahd University of Petroleum and Minerals during this research.

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