DIMENSION ESTIMATE FOR A SET OBTAINED FROM A THREE-DIMENSIONAL NON-PERIODIC SELF-AFFINE TILING

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Abstract. Some computational results on the three-dimensional Pisot tiling generated by the roots of $x^4-x^3-x^2-x-1=0$ are shown. We show an upper bound of the Hausdorff dimension of the set which is a projection of the intersection of three-tiles to the plane.

1 Introduction A *tile* is a compact subset of $\mathbf{R}^n$ which is equal to the closure of its interior. A set of tiles $\mathcal{T}$ is a *tiling* of $\mathbf{R}^n$, if $\mathcal{T}$ is a covering of $\mathbf{R}^n$ such that the intersection of interiors of any two tiles in $\mathcal{T}$ is empty. A tiling $\mathcal{T}$ is called *self-affine*, if there is an affine map such that the image of a tile in $\mathcal{T}$ is a union of tiles of $\mathcal{T}$.

Self-affine tilings are of special interest because of their relation to several topics of recent research, Markov partitions for toral automorphisms [11, 4, 1], wavelet theory [7, 10] and real quasi-crystal [8, 13]. There are numerous studies on self-affine tilings. All of the examples used there are about one or two-dimensional cases. Explicit examples of higher than or equal to three-dimensional cases have not been studied.

In this paper, we show some explicit computational results on the 3-dimensional non-periodic self-affine tiling generated by the roots of the equation $x^4-x^3-x^2-x-1=0$. We construct a Mauldin-Williams graph [9] of intersection of three tiles. The tiling we treat here is a Pisot tiling. Here we do not give the precise and general definition of the Pisot tilings. For further details, see [2, 15, 14, 3]. The tiling is constructed as follows: The polynomial $x^4-x^3-x^2-x-1$ is irreducible over $\mathbf{Q}$ and has four distinct roots,

$$\begin{aligned}\beta &= 1.92756197548292530426190 5\cdots,\\ \gamma &= -0.77480411321543385409240 32\cdots,\\ \alpha &= -0.07637893113374572508475\cdots - 0.81470364717038652684 16\cdots i,\\ \overline{\alpha} &= -0.07637893113374572508475\cdots + 0.81470364717038652684 16\cdots i.\end{aligned}$$

So β is a *Pisot* number, that is, β is an algebraic integer greater than 1 and all of its Galois conjugates over $\mathbf{Q}$ are strictly inside the unit circle. Let $w=d_{-l}d_{-l+1}\cdots d_{-1}$ be a word over $\{0,1\}$. A tile $T(w)$ is defined as follows:

$$T(w)=\left\{ (\sum_{i=-l}^{\infty} a_i\alpha^i, \sum_{i=-l}^{\infty} a_i\gamma^i) \ : \ \begin{array}{l} a_i\in\{0,1\}, a_i\times a_{i+1}\times a_{i+2}\times a_{i+3}=0, \\ a_{-l}a_{-l+1}\cdots a_{-1}=w \end{array} \right\},$$

which is a subset of $\mathbf{C}\times\mathbf{R}\simeq\mathbf{R}^3$. The *tiling* $\mathcal{T}$ is defined by

$$\mathcal{T}:=\{T(w) : w\in\{0,1\}^*\},$$

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where $\{0,1\}^*$ denotes the set of all of the words over $\{0,1\}$(including the empty word ϵ).

An automaton is a (directed labeled) graph. For an automaton M, we denote by $\mathcal{V}(M)$ the vertex set of M and by $\mathcal{E}(M)$ the edge set of M. Every edge $e \in \mathcal{E}(M)$ has the *starting point* $s(e) \in \mathcal{V}(M)$ and the *end point* $t(e) \in \mathcal{V}(M)$, and carry a label $l(e) \in \Sigma(M)$ where $\Sigma(M)$ is a finite set called the *alphabet* of M. A sequence of edges $e_1 \cdots e_l$ is called a *path* of M if $t(e_i) = s(e_{i+1})$. An automaton has a special vertex i_M called the *initial state* of M. Σ^* denotes the set of all of the words over an alphabet Σ. A word $w = a_1 \cdots a_l \in \Sigma^*$ is *accepted* by M if there exists a path $p = e_1 \cdots e_l$ starting from i_M such that $l(e_1) \cdots l(e_l) = w$. An infinite word $(a_i)_{i \geq 0}$ over Σ is accepted by M if $a_0 \cdots a_h$ is accepted by M for all $h \geq 0$. We denote by $L(M, i)$ the set of infinite words accepted by M with the initial state i.

Example 1 *The automaton shown in Figure 1 accepts the words over* $\{0,1\}$ *which does not include* 11 *as a subword.*

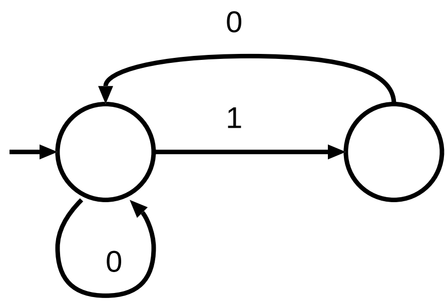

Figure 1:

The intersection of tiles are determined by automata. See [12] for the proof of the following theorem.

Theorem 1 (Sadahiro) *The intersection of tiles are represented by automata: For any n tiles $T(w_1), T(w_2), \ldots, T(w_n)$, there exists an automaton m with the following property.*

$$(\sum_{i=-l}^{\infty} a_i \alpha^i, \sum_{i=-l}^{\infty} a_i \gamma^i) \in T(w_1) \cap T(w_2) \cap \cdots \cap T(w_n)$$

if and only if $a_{-l} \cdots a_{-1} = w_1$ and $a_{-l} a_{-l+1} \cdots a_h$ is accepted by m for any $h \geq 0$.

An infinite word accepted by the automaton in the theorem above determines a point in the intersection. For example, the automaton which represents $T(0) \cap T(1) \cap T(11) \cap T(111)$ is shown in Figure 2, from which we can see $T(0) \cap T(1) \cap T(11) \cap T(111)$ consists of only one point $(-1, -1) \in \mathbf{C} \times \mathbf{R}$.

In fact, the following four presentations of $(-1, -1)$ exist:

$$\begin{aligned}
(-1,-1) &= \left(\sum_{n=0}^{\infty} \alpha^{4n} (\alpha + \alpha^2 + \alpha^3),\ \sum_{n=0}^{\infty} \gamma^{4n} (\gamma + \gamma^2 + \gamma^3) \right) (\in T(0)) \\
&= \left(\frac{1}{\alpha} + \sum_{n=0}^{\infty} \alpha^{4n} (\alpha + \alpha^2 + \alpha^4),\ \frac{1}{\gamma} + \sum_{n=0}^{\infty} \gamma^{4n} (\gamma + \gamma^2 + \gamma^4) \right) (\in T(1))
\end{aligned}$$

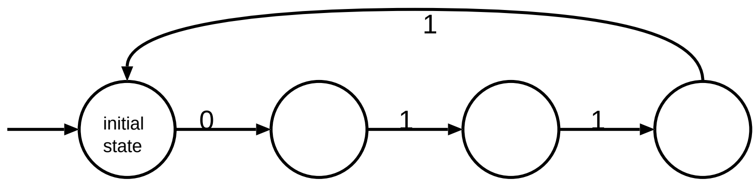

Figure 2:

$$= \left(\frac{1}{\alpha^2} + \frac{1}{\alpha} + \sum_{n=0}^{\infty} \alpha^{4n}(\alpha + \alpha^3 + \alpha^4), \right.$$
$$\left. \frac{1}{\gamma^2} + \frac{1}{\gamma} + \sum_{n=0}^{\infty} \gamma^{4n}(\gamma + \gamma^3 + \gamma^4)\right) (\in T(11))$$
$$= \left(\frac{1}{\alpha^3} + \frac{1}{\alpha^2} + \frac{1}{\alpha} + \sum_{n=0}^{\infty} \alpha^{4n}(\alpha^2 + \alpha^3 + \alpha^4), \right.$$
$$\left. \frac{1}{\gamma^3} + \frac{1}{\gamma^2} + \frac{1}{\gamma} + \sum_{n=0}^{\infty} \gamma^{4n}(\gamma^2 + \gamma^3 + \gamma^4)\right) (\in T(111)).$$

2 Dimension of $T(0) \cap T(1) \cap T(11)$ We will study the following set in **C**:

$$E = \left\{ \sum_{i=0}^{\infty} a_i \alpha^i \ : \ \left(\sum_{i=0}^{\infty} a_i \alpha^i, \sum_{i=0}^{\infty} a_i \gamma^i \right) \in T(0) \cap T(1) \cap T(11) \right\}.$$

Figure 3 shows E. The automaton which accepts words determining points in $T(0) \cap T(1) \cap T(11)$ is shown in the appendix.

A *cycle* is a directed graph H for which there is a closed path which passes into every vertex exactly once and such that every edge of H is an edge of this path. A directed graph H is *strongly connected* provided that whenever each of x and y is a vertex of H, then there is a path from x to y.

A *strongly connected component* of G is a maximal subgraph H of G such that H is strongly connected. It is clear that the strongly connected components of G are pairwise disjoint. A vertex is not considered to be strongly connected unless it is looped on itself.

The automaton which represents $T(0) \cap T(1) \cap T(11)$ is decomposed into strongly connected components as is shown in Figure 4. Every strongly connected components except a special component X consists of one cycle. Figure 5 shows the component X. All of the infinite paths which do not remain in X end up in cycles and they are a countable set. The dimension of E is equal to the dimension of the set which consists of points determined by the infinite words accepted by X fixing a vertex as the initial state. Let `a`, `b` and `c` be the states shown in Figure 5. Let $A = L(X, \mathtt{a})$, $B = L(X, \mathtt{b})$, $C = L(X, \mathtt{c})$ be the sets of the infinite words accepted by X with the initial satetes, `a`, `b`, `c`, respectively. Then A, B, C satisfy the following set-equations, namely we obtain a graph iterated function system [5].

(1)
$$\left\{ \begin{array}{lcl} A & = & f_1(A) \cup f_2(A) \cup f_3(B) \\ B & = & g_1(A) \cup g_2(B) \cup g_3(C) \\ C & = & h_1(A) \cup h_2(A) \cup h_3(C) \end{array} \right. .$$

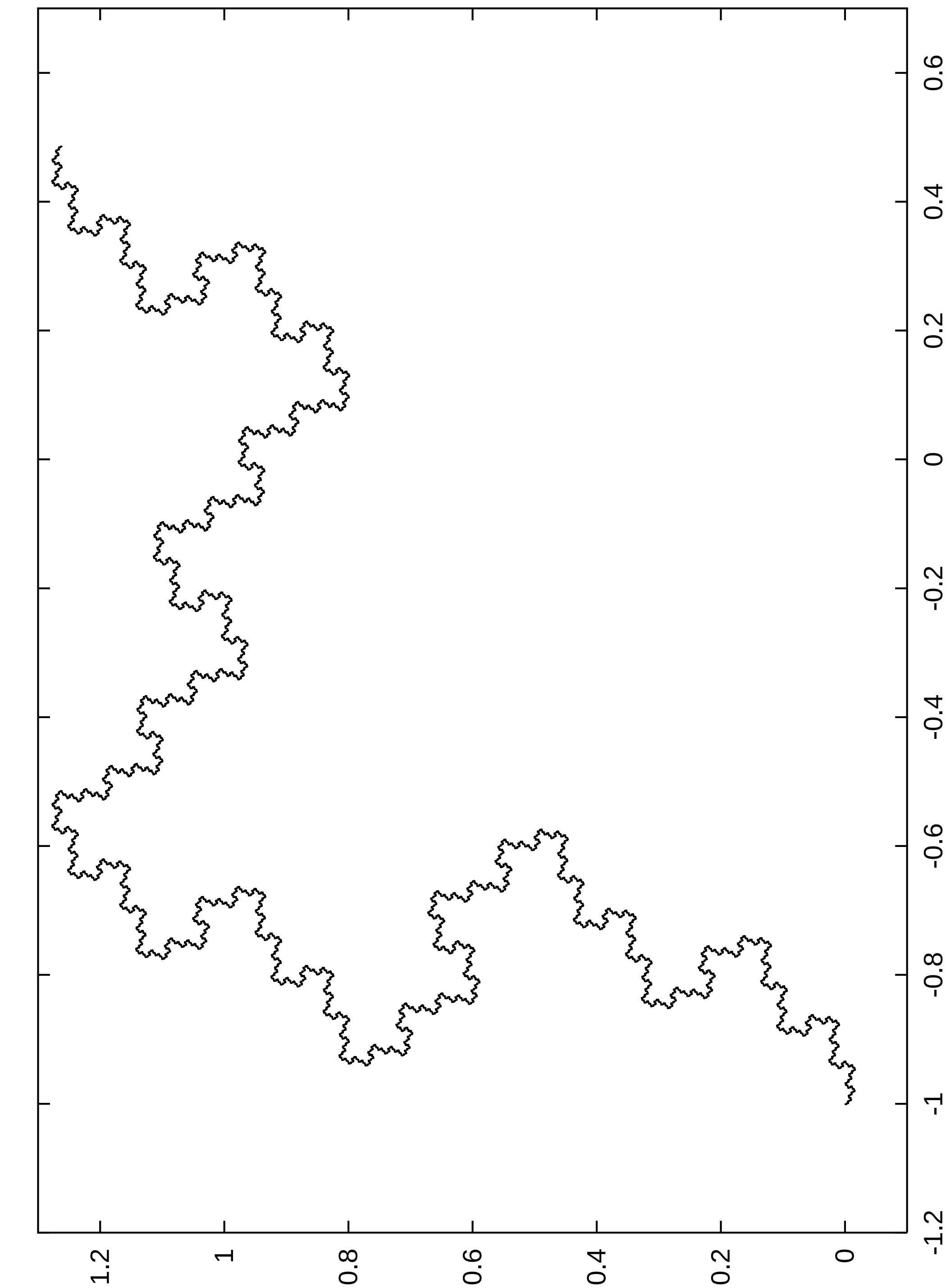

Figure 3: E

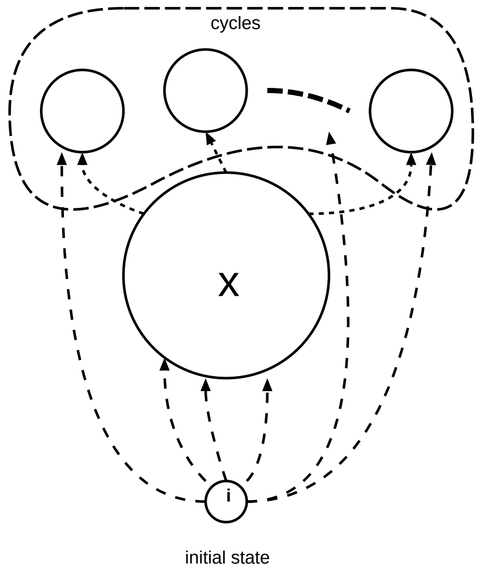

Figure 4: decomposition to strongly connected components

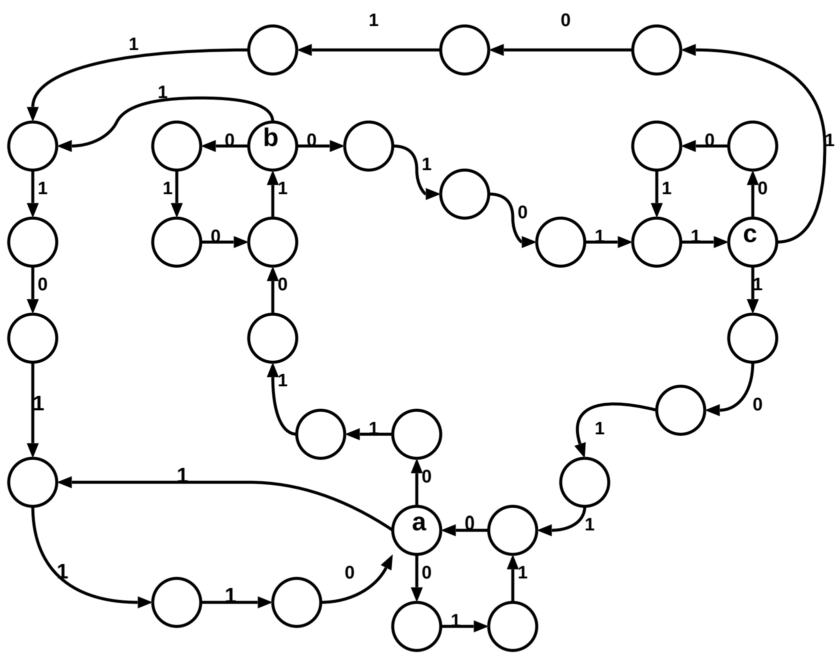

Figure 5: X

$$\begin{aligned}
f_1(x) &= \alpha^4 x + \alpha^2 + \alpha + 1\\
f_2(x) &= \alpha^4 x + \alpha^2 + \alpha\\
f_3(x) &= \alpha^5 x + \alpha^4 + \alpha^2 + \alpha\\
g_1(x) &= \alpha^7 x + \alpha^5 + \alpha^4 + \alpha^3 + \alpha + 1\\
g_2(x) &= \alpha^4 x + \alpha^3 + \alpha\\
g_3(x) &= \alpha^5 x + \alpha^4 + \alpha^3 + \alpha\\
h_1(x) &= \alpha^{10} x + \alpha^8 + \alpha^7 + \alpha^6 + \alpha^4 + \alpha^3 + \alpha^2 + 1\\
h_2(x) &= \alpha^5 x + \alpha^3 + \alpha^2 + 1\\
h_3(x) &= \alpha^4 x + \alpha^3 + \alpha^2
\end{aligned}$$

The Mauldin-Williams graph G for this system is shown in Figure 6. The dimension s

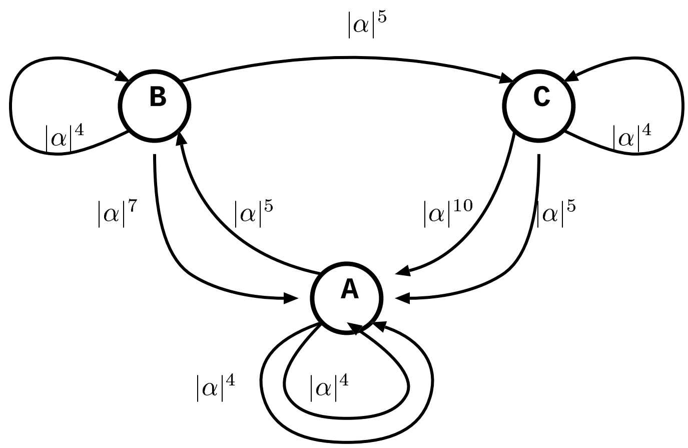

Figure 6: Mauldin-Williams graph for E

associated to G is computed as follows [5, 9]:

$$\det \begin{pmatrix} 2|\alpha|^{4s}-1 & |\alpha|^{5s} & 0 \\ |\alpha|^{7s} & |\alpha|^{4s}-1 & |\alpha|^{5s} \\ |\alpha|^{5s}+|\alpha|^{10s} & 0 & |\alpha|^{4s}-1 \end{pmatrix} = 0. \tag{2}$$

$$|\alpha|^{20s} - |\alpha|^{16s} + |\alpha|^{15s} + 3|\alpha|^{12s} - 5|\alpha|^{8s} + 4|\alpha|^{4s} - 1 = 0.$$

Thus $|\alpha|^s$ can take the following four real values,

$$\begin{aligned}
&-1,\\
&-0.970402824857290896445 6\cdots,\\
&-0.826523355356233482090 3\cdots,\\
&0.792543498375732550062 71\cdots.
\end{aligned}$$

Since $s > 0$, we obtain $|\alpha|^s = 0.7925434983757325500 62\cdots$ and $s = 1.15931959819470279575 \cdots$. By using the result of [5], this value is the upper bound of the Hausdorff dimension of A, B, C .

Theorem 2 *E has the Hausdorff dimension smaller than or equal to* $1.15931959819470279575\cdots$.

From numerical experiments, the upper bound in the theorem above seems to be exactly equal to the Hausdorff dimension of E.

Conjecture 1 *Each of the sets, $f_1(A)\cap f_2(A)$, $f_2(A)\cap f_3(B)$, $f_3(B)\cap f_1(A)$, $g_1(A)\cap g_2(B)$, $g_2(B)\cap g_3(C)$, $g_3(C)\cap g_1(A)$, $h_1(A)\cap h_2(A)$, $h_2(A)\cap h_3(C)$, $h_3(C)\cap h_1(A)$ consits of only one point.*

A, B and C has the same Hausdorff dimension s, and each of the s-dimensional Hausdorff measure of A, B, C is positive and finite. (See the first half of the proof of Corollary 3.5 in [6].) Regarding this conjecture to be true, we have

$$\begin{aligned} \mathcal{H}^s(A) &= \mathcal{H}^s(f_1(A)) + \mathcal{H}^s(f_2(A)) + \mathcal{H}^s(f_3(B)) \\ &= |\alpha|^{4s}\mathcal{H}^s(A) + |\alpha|^{4s}\mathcal{H}^s(A) + |\alpha|^{5s}\mathcal{H}^s(B). \end{aligned}$$

where $\mathcal{H}^s(A)$ denotes the s-dimensional Hausdorff measure of A. In the same way, we obtain

$$\begin{pmatrix} \mathcal{H}^s(A) \\ \mathcal{H}^s(B) \\ \mathcal{H}^s(C) \end{pmatrix} = \begin{pmatrix} 2|\alpha|^{4s} & |\alpha|^{5s} & 0 \\ |\alpha|^{7s} & |\alpha|^{4s} & |\alpha|^{5s} \\ |\alpha|^{5s} + |\alpha|^{10s} & 0 & |\alpha|^{4s} \end{pmatrix} \begin{pmatrix} \mathcal{H}^s(A) \\ \mathcal{H}^s(B) \\ \mathcal{H}^s(C) \end{pmatrix}$$

and we have (2).

Figure 7 shows the points $\{z : (z, w) \in T(0) \cap T \cap T',\ T, T' \in \mathcal{T}\}$,which seems to have the same dimension as that of $T(0) \cap T(1) \cap T(11)$.

3 Appendix

The transision function of the automaton M are shown below. The notation

```
m
= 0 => n
= 0 => l
= 1 => k
```

means that there are edges from the state m, one to the state n labeled by 0, one to l labeled by 0, and one to k labeled by 1.

1
= 0 => 2
= 0 => 106
= 0 => 109
= 0 => 111
= 1 => 114
2
= 0 => 3
= 0 => 99
= 0 => 100
= 1 => 103
= 1 => 104
3
= 1 => 4
4
= 1 => 5
5
= 0 => 6
6
= 0 => 3
= 0 => 7
= 0 => 80
= 1 => 93
= 1 => 96

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= 0 => 112

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