SECTIONAL REPRESENTATIONS OF GELFAND-MAZUR ALGEBRAS *

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Abstract. It is shown that if a topologically semisimple algebra A, which has at least one closed maximal left (or right) ideal, is a locally pseudoconvex Waelbroeck algebra, a locally A-pseudoconvex algebra, a locally pseudoconvex Fréchet algebra, an exponentially galbed algebra with bounded elements, or a Gelfand-Mazur algebra, for which there exists at least one closed 2-sided ideal, which is maximal as left (or right) ideal, then A is representable as a subalgebra of a *section algebra.*

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers. A linear topological space $(\mathcal{A}, \tau)$ over $\mathbb{C}$ is called a *topological algebra over* $\mathbb{C}$ (shortly, a *topological algebra*) if there has been defined an associative separately continuous multiplication such that $\mathcal{A}$ is an algebra. It means that for each neighbourhood $\mathcal{O}$ of zero and each $a \in \mathcal{A}$ there exists a neighbourhood $\mathcal{U}$ of zero such that $a\mathcal{U} \subset \mathcal{O}$ and $\mathcal{U}a \subset \mathcal{O}$.

We say that $\mathcal{A}$ is a *locally pseudoconvex algebra* if $\mathcal{A}$ has a base of neighbourhoods of zero consisting of balanced pseudoconvex sets. The set U is called *pseudoconvex* if $U+U \subset \nu U$ for some $\nu > 0$. A locally pseudoconvex algebra $\mathcal{A}$ is called *a locally absorbingly pseudoconvex algebra* (shortly, *a locally A-pseudoconvex algebra*) if $\mathcal{A}$ has a base $\mathcal{B}$ of neighbourhoods of zero consisting of balanced pseudoconvex sets which satisfies the following condition: *for each $\mathcal{U} \in \mathcal{B}$ and for each $a \in \mathcal{A}$ there exists $\nu = \nu(a, \mathcal{U}) > 0$ such that $a\mathcal{U}, \mathcal{U}a \subset \nu\mathcal{U}$.* We also say that a topological algebra $\mathcal{A}$ is *an exponentially galbed algebra* if for each neighbourhood $\mathcal{O}$ of zero of $\mathcal{A}$ there exists another neighbourhood $\mathcal{U}$ of zero such that

$$\{\sum_{k=0}^{n} \frac{a_k}{2^k} : a_0, ..., a_n \in \mathcal{U}\} \subset \mathcal{O}$$

for each $n \in \mathbb{N}$ (see [1], p. 65) and *a Frèchet algebra* if $\mathcal{A}$ is complete and metrizable.

Let $\mathcal{A}$ be a topological algebra with unit $e_{\mathcal{A}}$, $m(\mathcal{A})$ denote the set of all closed two-sided ideals of $\mathcal{A}$ which are maximal as left (right) ideals. In case when the quotient algebra $\mathcal{A}/\mathcal{M}$ (in the quotient topology) is topologically isomorphic with $\mathbb{C}$ for each $\mathcal{M} \in m(\mathcal{A})$, then $\mathcal{A}$ is called *a Gelfand-Mazur* algebra. The term Gelfand-Mazur algebra is initially applied, independently of each other, by Mati Abel ([2], [3]) and A. Mallios [7]. Since then this terminology has been extensively employed.

An element a of a topological algebra $\mathcal{A}$ is called to be *bounded* in $\mathcal{A}$ if there exists a number $\lambda \in \mathbb{C}\backslash\{0\}$ such that the set $\{(\frac{a}{\lambda})^n : n \in \mathbb{N}\}$ is bounded in $\mathcal{A}$ and $\mathcal{A}$ is called to be a *topological algebra with bounded elements* if every element of $\mathcal{A}$ is bounded in $\mathcal{A}$.

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An element $a \in \mathcal{A}$ is called quasi-invertible in $\mathcal{A}$ if there exists $b \in \mathcal{A}$ such that $a+b-ab = a+b-ba = \theta_{\mathcal{A}}$ (here $\theta_{\mathcal{A}}$ is the zero element of $\mathcal{A}$). A topological algebra $\mathcal{A}$ is *a Q-algebra* if the set of all quasi- invertible elements $Q\mathrm{inv}\mathcal{A}$ is open in $\mathcal{A}$. It is known that a topological algebra $\mathcal{A}$ with unit $e_{\mathcal{A}}$ is Q-algebra if and only if the set of its invertible elements $\mathrm{Inv}\mathcal{A}$ is open in $\mathcal{A}$ (since $Q\mathrm{inv}\mathcal{A} = e_{\mathcal{A}} - \mathrm{Inv}\mathcal{A}$). A topological algebra with unit is called *a Waelbroeck algebra* if it is both a Q-algebra and a topological algebra with continuous inversion (see [7], p. 54).

Let $\mathcal{M}$ be a maximal regular left (right) ideal of an algebra $\mathcal{A}$ and let $\mathcal{P} = \{a \in \mathcal{A} : a\mathcal{A} \subset \mathcal{M}\}$ ($\mathcal{P} = \{a \in \mathcal{A} : \mathcal{A}a \subset \mathcal{M}\}$ respectively). Then we say that $\mathcal{P}$ is *a primitive ideal of $\mathcal{A}$ (with respect to $\mathcal{M}$)*. Let now $\mathcal{A}$ be a topological algebra with unit and let $\mathcal{Z}(\mathcal{A}) = \{z \in \mathcal{A} : za = az$ for each $a \in \mathcal{A}\}$ be the *center* of $\mathcal{A}$. An ideal $\mathcal{M} \in m(\mathcal{Z}(\mathcal{A}))$ is called *an extendible ideal in $\mathcal{A}$* if

$$\mathcal{I}(\mathcal{M}) = cl_{\mathcal{A}}\{\sum_{k=1}^{n} \alpha_k m_k : \ n \in \mathrm{N}, \ \alpha_1, \alpha_2, ..., \alpha_n \in \mathcal{A}, \ m_1, m_2, ..., m_n \in \mathcal{M}\} \neq \mathcal{A}.$$

(Here $cl_{\mathcal{A}}(\mathcal{U})$ stands for the closure of the set $\mathcal{U}$ in the topology of $\mathcal{A}$.) We know (see [8], p. 169) that for every two-sided ideal $\mathcal{I}$ of $\mathcal{A}$ for which $cl_{\mathcal{A}}(\mathcal{I}) \neq \mathcal{A}$ the set $cl_{\mathcal{A}}(\mathcal{I})$ is a two-sided ideal of $\mathcal{A}$. So $\mathcal{I}(\mathcal{M})$ is also a two-sided ideal in $\mathcal{A}$. Let $m_e(\mathcal{Z}(\mathcal{A})) = \{\mathcal{M} \in m(\mathcal{Z}(\mathcal{A})) : \mathcal{M}$ is an extendible ideal in $\mathcal{A}\}$.

Let $\mathcal{A}$ be again a topological algebra. The set $\mathcal{R} = \bigcap\{\mathcal{M} : \mathcal{M}$ is a closed maximal regular left ideal of $\mathcal{A}\} = \bigcap\{\mathcal{M} : \mathcal{M}$ is a closed maximal regular right ideal of $\mathcal{A}\}$ is called *the topological radical of the algebra $\mathcal{A}$*. The topological algebra $\mathcal{A}$ is said to be *topologically semisimple* if its topological radical $\mathcal{R} = \{\theta_{\mathcal{A}}\}$.

Let now B and X be topological spaces and $\pi : B \to X$ a continuous and open surjection. Then the complex (B, π, X) is called *a fiber bundle*. The mapping $f : X \to B$ is said to be *a section* of the fiber bundle (B, π, X) (shortly, a *section of* π) if and only if $\pi f(x) = x$ for every $x \in X$. Let (B, π, X) be a fiber bundle for which the fibers

$$B_x = \{b \in B : \pi(b) = x\}$$

are topological algebras, for every $x \in X$. Then the set of all continuous sections of π is denoted by $\Gamma(\pi)$. Defining algebraic operations in $\Gamma(\pi)$ point-wise and topology by giving the subbase of $f_0 \in \Gamma(\pi)$ by

$$B(f_0) = \{U_O(f_0) : O \in B(\theta_P)\},$$

where $B(\theta_P)$ is a base of neighbourhoods of zero of the algebra

$$P = \prod_{x \in X} B_x$$

in the product topology and $U_O(f_0) = \{f \in \Gamma(\pi) : ((f - f_0)(x))_{x \in X} \in O\}$, we see that $\Gamma(\pi)$ is a topological algebra which is called *a section algebra*.

Let now f be *a representation of a topological algebra $\mathcal{A}$ in another topological algebra* $\mathcal{B}$, that is f is a continuous homomorphism from $\mathcal{A}$ into $\mathcal{B}$. In case when $\mathcal{B}$ is a section algebra, then f is called a *sectional representation of* $\mathcal{A}$. The aim of this paper is to generalize sectional representations of Banach algebras with unit given in [4] to the case of topological algebras and find the possible general conditions for a topological algebra with unit to be representable as a subalgebra of a section algebra.

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2. Properties of the quotient algebra and its center

To describe properties of a quotient algebra and its center, we need the following results.

Theorem 1. *Let $\mathcal{A}$ be a topological algebra and $\mathcal{I}$ a two-sided ideal of $\mathcal{A}$. If $\mathcal{A}$ is a Gelfand-Mazur algebra for which $m(\mathcal{A}) \neq \emptyset$, then there exists such a topology τ on A that $\mathcal{A}/\mathcal{I}$ in the quotient topology (defined by τ) and $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ in the subspace topology are exponentially galbed algebras with bounded elements.*

Proof. It is known (see [2], p. 123, Theorem 2) that any topological algebra for which $m(\mathcal{A}) \neq \emptyset$ is a Gelfand-Mazur algebra iff there exists a topology τ on $\mathcal{A}$ such that $(\mathcal{A}, \tau)$ is an exponentially galbed algebra with bounded elements.

Let $\tau_{\mathcal{I}}$ be the quotient topology on $\mathcal{A}/\mathcal{I}$ and $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ the canonical homomorphism. Then $\tau_Z = \{T' \bigcap \mathcal{Z}(\mathcal{A}/\mathcal{I}) : \ T' \in \tau_{\mathcal{I}}\}$ is the topology on $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ generated by $\tau_{\mathcal{I}}$.

Let $\mathcal{O}'$ be a neighbourhood of zero in $(\mathcal{A}/\mathcal{I}, \tau_{\mathcal{I}})$. Since π is a continuous open mapping (see [6], p 104), then $\mathcal{O} = \pi^{-1}(\mathcal{O}')$ is a neighbourhood of zero in $(\mathcal{A}, \tau)$. Now we can find a neighbourhood of zero $\mathcal{U}$ in $(\mathcal{A}, \tau)$ such that

$$\{\sum_{k=0}^{n} \frac{a_k}{2^k} : a_0, ..., a_n \in \mathcal{U}\} \subset \mathcal{O}$$

for every $n \in \mathbb{N}$. Taking $\mathcal{V}' = \pi(\mathcal{U})$, we can see that

$$\{\sum_{k=0}^{n} \frac{x_k}{2^k} : x_0, x_1, ..., x_n \in \mathcal{V}'\} \subset \mathcal{O}'$$

for every $n \in \mathbb{N}$ which implies that $\mathcal{A}/\mathcal{I}$ is exponentially galbed.

Every neighbourhood of zero in $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ is representable in the form $\mathcal{O}'' = \mathcal{O}' \bigcap \mathcal{Z}(\mathcal{A}/\mathcal{I})$ where $\mathcal{O}'$ is a neighbourhood of zero in $\mathcal{A}/\mathcal{I}$. As above we can find the neighborhood $\mathcal{V}'$ of zero. Taking now $\mathcal{V}'' = \mathcal{V}' \bigcap \mathcal{Z}(\mathcal{A}/\mathcal{I})$, we can see that

$$\{\sum_{k=0}^{n} \frac{y_k}{2^k} : y_0, y_1, ..., y_n \in \mathcal{V}''\} \subset \mathcal{O}''$$

for every $n \in \mathbb{N}$ which implies that $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ is also exponentially galbed.

It is easy to see that the elements of $\mathcal{A}/\mathcal{I}$ and $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ are bounded.

Corollary 1. *Let $\mathcal{A}$ be a topological algebra and $\mathcal{I}$ a two-sided ideal of $\mathcal{A}$. If $\mathcal{A}$ is a Gelfand-Mazur algebra for which $m(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}/\mathcal{I}$ is a Gelfand-Mazur algebra. If hereby $m(\mathcal{A}/\mathcal{I}) \neq \emptyset$ then $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ is a Gelfand-Mazur algebra too.*

Lemma 1. *Let $\mathcal{A}$ be a locally pseudoconvex (a locally A-pseudoconvex) algebra and $\mathcal{I}$ a two-sided ideal of $\mathcal{A}$. Then $\mathcal{A}/\mathcal{I}$ and $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ are also locally pseudoconvex (locally A-pseudoconvex) algebras.*

Proof. Since $\mathcal{A}$ is a locally pseudoconvex algebra, we can find a base $\mathcal{B}$ of neighbourhoods of zero of $\mathcal{A}$ consisting of balanced pseudoconvex neighbourhoods of zero. It is easy to see that $\mathcal{B}' = \pi(\mathcal{B})$ and

$$\mathcal{B}'' = \{\mathcal{U}' \bigcap \mathcal{Z}(\mathcal{A}/\mathcal{I}) : \ \mathcal{U}' \in \mathcal{B}'\}$$

are suitable bases of neighbourhoods of zero for $\mathcal{A}/\mathcal{I}$ and $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ respectively.

Lemma 2. *Let $\mathcal{A}$ be a Fréchet (a unital locally pseudoconvex Fréchet) algebra and $\mathcal{I}$ a closed two-sided ideal. Then $\mathcal{A}/\mathcal{I}$ and $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ are also Fréchet (unital locally pseudoconvex Fréchet) algebras.*

Proof. According to our hypothesis (see also [6], p. 138, Theorem 2), $\mathcal{A}/\mathcal{I}$ is a Fréchet algebra. Since $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ is a closed linear subspace of $\mathcal{A}/\mathcal{I}$, then $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ is also complete and metrizable, that is, a Fréchet algebra too.

Using Lemmas 1 and 3, we can get the following.

Lemma 3. *Let $\mathcal{A}$ be a (locally pseudoconvex) Waelbroeck algebra with unit and $\mathcal{I}$ a closed two-sided ideal. Then $\mathcal{A}/\mathcal{I}$ and $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ are also (locally pseudoconvex) Waelbroeck algebras with unit.*

Proof. It is shown in [11] that $\mathcal{A}/\mathcal{I}$ is a Waelbroeck algebra with unit. This implies that $\mathrm{Inv}\mathcal{A}/\mathcal{I}$ is an open set in $\mathcal{A}/\mathcal{I}$. Since $\mathrm{Inv}\mathcal{Z}(\mathcal{A}/\mathcal{I}) = = \mathrm{Inv}\mathcal{A}/\mathcal{I} \bigcap \mathcal{Z}(\mathcal{A}/\mathcal{I})$, then $\mathrm{Inv}\mathcal{Z}(\mathcal{A}/\mathcal{I})$ is an open set in $\mathcal{Z}(\mathcal{A})$. We can see that the inversion is continuous in $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ (because $\mathrm{Inv}\mathcal{Z}(\mathcal{A}/\mathcal{I})$ is a subset of $\mathrm{Inv}\mathcal{A}/\mathcal{I}$). Hence, $\mathcal{Z}(\mathcal{A}/\mathcal{I})$ is a Waelbroeck algebra too.

Using [2], p. 120-122 (Theorem 1 and Corollary 1), see also [3], the following result can be proved:

Lemma 4. *Let $\mathcal{A}$ be a topological division algebra for which at least one of the following statements holds:*

a) *there exists a topology τ on $\mathcal{A}$ such that $(\mathcal{A}, \tau)$ is a locally pseudoconvex Hausdorff algebra with continuous inversion;*

b) *$\mathcal{A}$ is a locally A-pseudoconvex Hausdorff algebra;*

c) *$\mathcal{A}$ is a locally pseudoconvex Fréchet algebra;*

d) *there exists a topology τ on $\mathcal{A}$ such that $(\mathcal{A}, \tau)$ is an exponentially galbed Hausdorff algebra with bounded elements.*

Then $\mathcal{A}$ is topologically isomorphic to $\mathbb{C}$.

Theorem 3. *Let $\mathcal{A}$ be a topological algebra with unit, $\mathcal{P}$ a closed primitive ideal of it and let one of the following statements be true:*

a) *$\mathcal{A}$ is a locally pseudoconvex Waelbroeck algebra;*

b) *$\mathcal{A}$ is a locally A-pseudoconvex algebra;*

c) *$\mathcal{A}$ is a locally pseudoconvex Fréchet algebra;*

d) *$\mathcal{A}$ is an exponentially galbed algebra with bounded elements;*

e) *$\mathcal{A}$ is a Gelfand-Mazur algebra for which $m(\mathcal{A}) \neq \emptyset$.*

Then $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ is topologically isomorphic to $\mathbb{C}$.

Proof. Since $\mathcal{P}$ is a primitive ideal of $\mathcal{A}$ then $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ is a field (see [5], p. 136 (Proposition 9) and [9], p. 61 (Corollary 2.4.5)). Thus $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ is a division Hausdorff algebra.

In case a) we obtain from Lemma 3 that $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ is a locally pseudoconvex Waelbroeck algebra. Since every Waelbroeck algebra is an algebra with continuous inversion, then $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ satisfies condition a) of Lemma 4.In case b) we obtain from Lemma 1 that $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ is a locally A-pseudoconvex algebra and thus satisfies the condition b) of Lemma 4. In case c) we obtain from Lemma 2 that $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ satisfies the condition c) of Lemma 4. In cases d) and e) we obtain from Theorem 1 that $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ has a topology in which $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ is an exponentially galbed algebra with bounded elements and thus satisfies condition d) of Lemma 4.

Corollary 2. *Let $\mathcal{A}$ be a topological algebra with unit and $\mathcal{M}$ be a closed maximal left (right) ideal of $\mathcal{A}$. If at least one of the statements* a) - e) *of the Theorem* 3 *is true, then the following statements are also true:*

1) *every $b \in \mathcal{Z}(\mathcal{A})$ defines such $\lambda \in \mathbb{C}$ that $b - \lambda e_{\mathcal{A}} \in \mathcal{M}$;*

2) $\mathcal{M} \bigcap \mathcal{Z}(\mathcal{A}) \in m_e(\mathcal{Z}(\mathcal{A}))$.

Proof. 1) Let $b \in \mathcal{Z}(\mathcal{A})$, $\mathcal{P}$ be a primitive ideal in $\mathcal{A}$ with respect to $\mathcal{M}$ and $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{P}$ the canonical homomorphism. Then $\mathcal{Z}(\mathcal{A}/\mathcal{P})$ is topologically isomorphic to $\mathbb{C}$, according Theorem 3. We will denote this isomorphism by μ. Since $\pi(b) \in \mathcal{Z}(\mathcal{A}/\mathcal{P})$, we can find $\lambda \in \mathbb{C}$ such that $\mu(\pi(b)) = \lambda = \mu(\pi(\lambda e_{\mathcal{A}}))$. Therefore $\pi(b) = \pi(\lambda e_{\mathcal{A}})$ which implies that $b - \lambda e_{\mathcal{A}} \in \mathcal{P} \subseteq \mathcal{M}$.

2) Let $\mathcal{M}_{\mathcal{Z}} = \mathcal{M} \bigcap \mathcal{Z}(\mathcal{A})$. Then $\mathcal{M}_{\mathcal{Z}}$ is a closed linear subspace of $\mathcal{A}$ because $\mathcal{M}$ and $\mathcal{Z}(\mathcal{A})$ were closed subsets of $\mathcal{A}$. Since $e_{\mathcal{A}} \notin \mathcal{M}$ then $\mathcal{M}_{\mathcal{Z}} \neq \mathcal{Z}(\mathcal{A})$. Let now $z \in \mathcal{M}_{\mathcal{Z}}$. Then $z \in \mathcal{M}$ which implies $za \in \mathcal{M}$ for every $a \in \mathcal{A}$. Hence $za \in \mathcal{M}_{\mathcal{Z}}$ for every $a \in \mathcal{Z}(\mathcal{A})$ which implies that $\mathcal{M}_{\mathcal{Z}}$ is a closed ideal of $\mathcal{Z}(\mathcal{A})$. Let now $\mathcal{I}$ be an ideal of $\mathcal{Z}(\mathcal{A})$ such that $\mathcal{M}_{\mathcal{Z}} \subset \mathcal{I}$. If $\mathcal{I} \neq \mathcal{M}_{\mathcal{Z}}$ then there exists $b \in \mathcal{I} \backslash \mathcal{M}_{\mathcal{Z}}$ and according to 1) there exists $\lambda \in \mathbb{C}$ such that $b - \lambda e_{\mathcal{A}} \in \mathcal{M}_{\mathcal{Z}}$. Since $b \notin \mathcal{M}_{\mathcal{Z}}$ we see that $\lambda \neq 0$ and therefore there exists λ^{-1}. Since $b - \lambda e_{\mathcal{A}} \in \mathcal{M}_{\mathcal{Z}} \subset \mathcal{I}$ we have $e_{\mathcal{A}} = \lambda^{-1}[b - (b - \lambda e_{\mathcal{A}})] \in \mathcal{I}$ and therefore $\mathcal{I} = \mathcal{Z}(\mathcal{A})$. So $\mathcal{M}_{\mathcal{Z}} \in m(\mathcal{Z}(\mathcal{A}))$. Since $\mathcal{M}_{\mathcal{Z}} \subset \mathcal{M}$ and $\mathcal{M}$ is a closed subset, we have $\mathcal{I}(\mathcal{M}_{\mathcal{Z}}) \subseteq \mathcal{M} \neq \mathcal{A}$. Hence $\mathcal{M}_{\mathcal{Z}} \in m_e(\mathcal{Z}(\mathcal{A}))$.

3. Sectional representations

Let $\mathcal{A}$ be a topological algebra with unit for which at least one of the conditions a) - e) of Theorem 3 is true and which has at least one closed left (right) maximal ideal. Then $\mathcal{Z}(\mathcal{A})$ is a (commutative) Gelfand-Mazur algebra by Theorem 1 and Lemmas 1, 2 and 3 (see also [2], p. 125-126, Corollaries 2 and 3) and $m_e(\mathcal{Z}(\mathcal{A})) \neq \emptyset$, by Corollary 2.

For every $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$ let $\mathcal{A}_{\mathcal{M}} = \mathcal{A}/\mathcal{I}(\mathcal{M})$ and $\kappa_{\mathcal{M}} : \mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ the canonical homomorphism, $a\hat{}(\mathcal{M}) = \kappa_{\mathcal{M}}(a)$ for every $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$ and

$$\mathcal{B} = \bigcup_{\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))} \mathcal{A}_{\mathcal{M}}.$$

Then $a\hat{}$ maps $m_e(\mathcal{Z}(\mathcal{A}))$ into $\mathcal{B}$.

Let now $\pi : \mathcal{B} \to m_e(\mathcal{Z}(\mathcal{A}))$ be such mapping, which will assign to every $b \in \mathcal{B}$ such ideal $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$ that $b \in \mathcal{A}_{\mathcal{M}}$ i.e. $b = \kappa_{\mathcal{M}}(a)$ for some $a \in \mathcal{A}$. It is easy to see that π is well defined. Indeed, if $b = \kappa_{\mathcal{M}_1}(a_1)$ (i.e. $\pi(b) = \mathcal{M}_1$) and $b = \kappa_{\mathcal{M}_2}(a_2)$ (i.e. $\pi(b) = \mathcal{M}_2$), then $a_1 + \mathcal{I}(\mathcal{M}_1) = b = a_2 + \mathcal{I}(\mathcal{M}_2)$. Since $a_1 = a_1 + \theta_{\mathcal{A}} \in a_1 + \mathcal{I}(\mathcal{M}_1) = a_2 + \mathcal{I}(\mathcal{M}_2)$, we can find $d_2 \in \mathcal{I}(\mathcal{M}_2)$ such that $a_1 = a_2 + d_2$ or $a_1 - a_2 = d_2 \in \mathcal{I}(\mathcal{M}_2)$.

Let $c_1 \in \mathcal{I}(\mathcal{M}_1)$ be an arbitrary element. Then

$$a_1 + c_1 \in a_1 + \mathcal{I}(\mathcal{M}_1) = a_2 + \mathcal{I}(\mathcal{M}_2)$$

and we can find $c_2 \in \mathcal{I}(\mathcal{M}_2)$ such that $a_1 + c_1 = a_2 + c_2$. Since $a_1 - a_2 \in \mathcal{I}(\mathcal{M}_2)$ and $c_2 \in \mathcal{I}(\mathcal{M}_2)$ then $c_1 = c_2 - (a_1 - a_2) \in \mathcal{I}(\mathcal{M}_2)$, so that $\mathcal{I}(\mathcal{M}_1) \subset \mathcal{I}(\mathcal{M}_2)$. Analogously we get that $\mathcal{I}(\mathcal{M}_2) \subset \mathcal{I}(\mathcal{M}_1)$. Hence $\mathcal{I}(\mathcal{M}_1) = \mathcal{I}(\mathcal{M}_2)$ and

$$\mathcal{M}_1 = \mathcal{I}(\mathcal{M}_1) \bigcap \mathcal{Z}(\mathcal{A}) = \mathcal{I}(\mathcal{M}_2) \bigcap \mathcal{Z}(\mathcal{A}) = \mathcal{M}_2.$$

The last result proves that π is well defined and $\mathcal{A}_{\mathcal{M}_1} \cap \mathcal{A}_{\mathcal{M}_2} \neq \emptyset$ if and only if $\mathcal{M}_1 = \mathcal{M}_2$.

The set $m_e(\mathcal{Z}(\mathcal{A}))$ is endowed with the topology τ, a subbase of neighbourhoods of $\mathcal{M}_0 \in m_e(\mathcal{Z}(\mathcal{A}))$ of which consists of sets

$$O(\mathcal{M}_0) = \{\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A})) \ : \mid (\varphi_{\mathcal{M}} - \varphi_{\mathcal{M}_0})(z) \mid < \ \epsilon\}$$

where $\epsilon > 0$ and $z \in \mathcal{Z}(\mathcal{A})$ vary, while $\varphi_{\mathcal{M}}$ denotes such nontrivial homomorphism $\mathcal{Z}(\mathcal{A}) \to \mathbb{C}$ for which $\ker\varphi_{\mathcal{M}} = \mathcal{M}$. On the algebras $\mathcal{A}_{\mathcal{M}}$ we shall consider quotient topologies $\tau_{\mathcal{M}}$ and on $\mathcal{B}$ the topology $\tau_{\mathcal{B}} = \{\pi^{-1}(\mathcal{U}) : \mathcal{U} \in \tau\}$. Then $(\mathcal{B}, \pi, m_e(\mathcal{Z}(\mathcal{A})))$ is a fiber bundle and $a\hat{} \in \Gamma(\pi)$ for every $a \in \mathcal{A}$.

Next we define a mapping $A : \mathcal{A} \to \Gamma(\pi)$, such that $\mathrm{A}(a) = a\hat{}$ for every $a \in \mathcal{A}$. It is easy to see that A is a continuous homomorphism. Hence, A is *a sectional representation of the topological algebra* $\mathcal{A}$.

Lemma 5. *Let $\mathcal{A}$ be a topological algebra with unit, which satisfies at least one of the conditions* a) - e) *of the Theorem* 3 *and $\mathcal{I}$ be a closed maximal left (right) ideal of $\mathcal{A}$. Then we can find such $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$ that $\kappa_{\mathcal{M}}(\mathcal{I}) = \{x\,\hat{}(\mathcal{M}) : x \in \mathcal{I}\}$ is a left (right) ideal of $\mathcal{A}_{\mathcal{M}}$.*

Proof. See [4], p. 197-198 (Theorem 2.8 (iii)).

Theorem 4. *Let $\mathcal{A}$ be a topological algebra with unit, which satisfies at least one of the conditions* a) - e) *of the Theorem* 3 *and $\mathcal{I}$ a closed maximal left (right) ideal of $\mathcal{A}$. Then $\mathcal{I} = \kappa_{\mathcal{M}}{}^{-1}(\mathcal{J}) = \{a \in \mathcal{A} : a\,\hat{}\,(\mathcal{M}) \in \mathcal{J}\}$ for some $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$ and for some closed maximal left (right) ideal $\mathcal{J}$ of $\mathcal{A}_{\mathcal{M}}$.*

Proof. Let $\mathcal{I}$ be a closed maximal left (right) ideal of $\mathcal{A}$.

According to Lemma 5, we can find $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$ such that $\mathcal{J} = \{a\,\hat{}(\mathcal{M}), a \in \mathcal{I}\}$ is a left (right) ideal of $\mathcal{A}_{\mathcal{M}}$. Next we can find a maximal left (right) ideal $\mathcal{I}_{\mathcal{M}}$ of $\mathcal{A}_{\mathcal{M}}$ such that $\mathcal{J} \subset \mathcal{I}_{\mathcal{M}}$. If there exists $j \in \mathcal{I}_{\mathcal{M}} \backslash \mathcal{J}$, then it is possible to find $b \in \mathcal{A} \backslash \mathcal{I}$ such that $b\,\hat{}(\mathcal{M}) = j$. Hence $\kappa_{\mathcal{M}}{}^{-1}(\mathcal{I}_{\mathcal{M}}) \supseteq \{b\} \bigcup \mathcal{I} \supset \mathcal{I}$. Let $c \in \mathcal{A}$ and $d \in \kappa_{\mathcal{M}}{}^{-1}(\mathcal{I}_{\mathcal{M}})$. Then $\kappa_{\mathcal{M}}(cd) = \kappa_{\mathcal{M}}(c)\kappa_{\mathcal{M}}(d) \in \mathcal{I}_{\mathcal{M}}$ which implies that $cd \in \kappa_{\mathcal{M}}{}^{-1}(\mathcal{I}_{\mathcal{M}})$ ($dc \in \kappa_{\mathcal{M}}{}^{-1}(\mathcal{I}_{\mathcal{M}})$ for a right ideal). Analogously, we get $c + d, \lambda c \in \kappa_{\mathcal{M}}{}^{-1}(\mathcal{I}_{\mathcal{M}})$. Since $\mathcal{I}$ is a maximal left (right) ideal of $\mathcal{A}$, we have $\kappa_{\mathcal{M}}{}^{-1}(\mathcal{I}_{\mathcal{M}}) = \mathcal{A}$ which implies that $a\,\hat{}(\mathcal{M}) \in \mathcal{I}_{\mathcal{M}}$ for every $a \in \mathcal{A}$. Hence $\mathcal{A}_{\mathcal{M}} = \mathcal{I}_{\mathcal{M}}$ which is not possible. So we have shown that $\mathcal{J}$ is a maximal left (right) ideal of $\mathcal{A}_{\mathcal{M}}$. According to the definition of $\mathcal{J}$, it is clear that $\mathcal{I} = \kappa_{\mathcal{M}}{}^{-1}(\mathcal{J})$.

Next we show that $\mathcal{J}$ is closed. If $cl_{\mathcal{A}_{\mathcal{M}}}(\mathcal{J}) \neq \mathcal{A}_{\mathcal{M}}$ then $cl_{\mathcal{A}_{\mathcal{M}}}(\mathcal{J})$ is also an ideal in $\mathcal{A}_{\mathcal{M}}$ (see [8], p. 169) and $\mathcal{J} = cl_{\mathcal{A}_{\mathcal{M}}}(\mathcal{J})$. Let us suppose that $cl_{\mathcal{A}_{\mathcal{M}}}(\mathcal{J}) = \mathcal{A}_{\mathcal{M}}$ and let $\mathcal{O}'$ be a neighbourhood of zero in $\mathcal{A}$. Then $\kappa_{\mathcal{M}}(\mathcal{O}') = \mathcal{O}$ is a neighbourhood of zero in $\mathcal{A}_{\mathcal{M}}$. Since $\kappa_{\mathcal{M}}(e_{\mathcal{A}}) \in cl_{\mathcal{A}_{\mathcal{M}}}(\mathcal{J})$, then there exists a family $(i_\lambda)_{\lambda\in\Lambda} \in \mathcal{I}$ such that $\kappa_{\mathcal{M}}(i_\lambda)_{\lambda\in\Lambda} \to \kappa_{\mathcal{M}}(e_{\mathcal{A}})$. Now we can find such a $\mu \in \Lambda$ that $\kappa_{\mathcal{M}}(i_\lambda - e_{\mathcal{A}}) \in \mathcal{O}$ for every $\lambda > \mu$. If $\lambda_0 > \mu$ then $i_{\lambda_0} - e_{\mathcal{A}} \in \kappa_{\mathcal{M}}{}^{-1}(\kappa_{\mathcal{M}}(\mathcal{O}')) = \mathcal{I}(\mathcal{M}) + \mathcal{O}' \subset \mathcal{I} + \mathcal{O}'$ (because $\mathcal{M} = \mathcal{I} \bigcap Z(\mathcal{A})$, see proof of Theorem 2.8 (iii) in [4]) and therefore

$$e_{\mathcal{A}} = (e_{\mathcal{A}} - i_{\lambda_0}) + i_{\lambda_0} \in \mathcal{I} + \mathcal{O}' + \mathcal{I} \subset \mathcal{I} + \mathcal{O}'.$$

Hence

$$e_{\mathcal{A}} \in \bigcap\{\mathcal{I} + \mathcal{O}' : \mathcal{O}' \text{ is a neighbourhood of a base of zero in } \mathcal{A}\} = cl_{\mathcal{A}_{\mathcal{M}}}\mathcal{I} = \mathcal{I}$$

(see [10], p. 13). Thus $\mathcal{I} = \mathcal{A}$ which is not possible. Therefore, $\mathcal{J}$ is a closed maximal left (right) ideal.

It is easy to verify that the following statement holds.

Lemma 6. *Let $\mathcal{A}$ be a topological algebra with unit, $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$ and $\mathcal{J}$ be an arbitrary closed left (right) ideal of $\mathcal{A}_{\mathcal{M}}$. Then $\kappa_{\mathcal{M}}{}^{-1}(\mathcal{J})$ is a left (right) ideal of $\mathcal{A}$.*

Lemma 7. *Let $\mathcal{A}$ be a topological algebra with unit for which at least one of the conditions* a) - e) *of Theorem* 3 *holds. Suppose that there exists a closed maximal left (right) ideal in $\mathcal{A}$. If we denote the topological radicals of the algebras $\mathcal{A}$ and $\mathcal{A}_{\mathcal{M}}$, where $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$, by $\mathcal{R}$ and $\mathcal{R}_{\mathcal{M}}$ respectively, then*

$$\mathcal{R} = \bigcap\{\kappa_{\mathcal{M}}{}^{-1}(\mathcal{R}_{\mathcal{M}}) : \mathcal{M} \in m_l(\mathcal{Z}(\mathcal{A}))\}.$$

Proof. Suppose that $x \in \bigcap\{\kappa_{\mathcal{M}}{}^{-1}(\mathcal{R}_{\mathcal{M}}) : \mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))\}$ and $\mathcal{I}$ is an arbitrary closed maximal left (right) ideal of $\mathcal{A}$. According to Theorem 4, we can find $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$ and a closed maximal left (right) ideal $\mathcal{J}$ of $\mathcal{A}_{\mathcal{M}}$ such that $\mathcal{I} = \kappa_{\mathcal{M}}{}^{-1}(\mathcal{J})$. Since $\kappa_{\mathcal{M}}(x) \in \mathcal{R}_{\mathcal{M}} \subset \mathcal{J}$ then $x \in \mathcal{I}$, for any closed maximal left (right) ideal $\mathcal{I}$ of $\mathcal{A}$. Therefore, $x \in \mathcal{R}$ such that

$$\mathcal{R} \supseteq \bigcap\{\kappa_{\mathcal{M}}{}^{-1}(\mathcal{R}_{\mathcal{M}}) : \mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))\}.$$

Suppose now that $y \in \mathcal{R}$ and $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$. If $\mathcal{J}$ is an arbitrary closed maximal left (right) ideal of $\mathcal{A}_{\mathcal{M}}$, then ${\kappa_{\mathcal{M}}}^{-1}(\mathcal{J})$ is a closed left (right) ideal of $\mathcal{A}$ by Lemma 6. It is clear that $\mathcal{I}(\mathcal{M}) \subset {\kappa_{\mathcal{M}}}^{-1}(\mathcal{J})$. Suppose that ${\kappa_{\mathcal{M}}}^{-1}(\mathcal{J}) \subset \mathcal{H}$ for a left (right) ideal $\mathcal{H}$ of $\mathcal{A}$. Then $\mathcal{J} \subset \kappa_{\mathcal{M}}(\mathcal{H})$. If $\kappa_{\mathcal{M}}(\mathcal{H}) = \mathcal{A}_{\mathcal{M}}$ then we can find such a $h \in \mathcal{H}$ that $\kappa_{\mathcal{M}}(h) = e_{\mathcal{A}_{\mathcal{M}}}$ (here $e_{\mathcal{A}_{\mathcal{M}}}$ denotes the unit element of $\mathcal{A}_{\mathcal{M}}$). But this means that $h - e_{\mathcal{A}} \in \mathcal{I}(\mathcal{M}) \subset {\kappa_{\mathcal{M}}}^{-1}(\mathcal{J}) \subset \mathcal{H}$, which implies $e_{\mathcal{A}} \in \mathcal{H}$. Hence $\mathcal{H} = \mathcal{A}$ which contradicts the assumption that $\mathcal{H}$ is an ideal of $\mathcal{A}$. Since $\mathcal{J}$ is a maximal left (right) ideal of $\mathcal{A}_{\mathcal{M}}$ then $\mathcal{J} = \kappa_{\mathcal{M}}(\mathcal{H})$, that is, $\mathcal{H} = {\kappa_{\mathcal{M}}}^{-1}(\mathcal{J})$. Hence, ${\kappa_{\mathcal{M}}}^{-1}(\mathcal{J})$ is a closed maximal left (right) ideal of $\mathcal{A}$ and $\kappa_{\mathcal{M}}(y) \in \mathcal{J}$ for every closed maximal left (right) ideal $\mathcal{J}$ of $\mathcal{A}_{\mathcal{M}}$. Therefore $\kappa_{\mathcal{M}}(y) \in \mathcal{R}_{\mathcal{M}}$ if $\mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))$, so that

$$\mathcal{R} \subseteq \bigcap \{{\kappa_{\mathcal{M}}}^{-1}(\mathcal{R}_{\mathcal{M}}) : \mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))\}$$

which completes the proof.

Corollary 3. *Let $\mathcal{A}$ be a topologically semisimple algebra, which satisfies the conditions of Lemma 7. Then the mapping A is one-to-one.*

Proof. Since $\mathcal{A}$ is topologically semisimple, its topological radical $\mathcal{R} = \{\theta_{\mathcal{A}}\}$. Hence from

$$\ker A = \bigcap \{{\kappa_{\mathcal{M}}}^{-1}(\theta_{\mathcal{M}}) : \mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))\} \subseteq$$

$$\subseteq \bigcap \{{\kappa_{\mathcal{M}}}^{-1}(\mathcal{R}_{\mathcal{M}}) : \mathcal{M} \in m_e(\mathcal{Z}(\mathcal{A}))\} = \mathcal{R} = \{\theta_{\mathcal{A}}\}$$

one obtains that A is one-to-one.

Now we formulate the main result of this paper.

Theorem 5. *Let $\mathcal{A}$ be a topologically semisimple algebra with unit, having at least one closed maximal left (right) ideal. If one of the following statements is true*

a) *$\mathcal{A}$ is a locally pseudoconvex Waelbroeck algebra;*
b) *$\mathcal{A}$ is a locally A-pseudoconvex algebra;*
c) *$\mathcal{A}$ is a locally pseudoconvex Fréchet algebra;*
d) *$\mathcal{A}$ is an exponentially galbed algebra with bounded elements;*
e) *$\mathcal{A}$ is a Gelfand-Mazur algebra for which $m(\mathcal{A}) \neq \emptyset$,*

then $\mathcal{A}$ can be considered as a subalgebra of the section algebra $\Gamma(\pi)$.

Proof. Since A is a one-to-one representation of $\mathcal{A}$ in $\Gamma(\pi)$, we can consider $\mathcal{A}$ as a subalgebras of $\Gamma(\pi)$.

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