STRUCTURE OF RINGS WITH A CONDITION ON ZERO DIVISORS

HAZAR ABU-KHUZAM, HOWARD E. BELL, AND ADIL YAQUB

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ABSTRACT. Let R be a ring such that every zero divisor is nilpotent. We call such a ring a D-ring. We give the structure of periodic D-rings, weakly periodic D-rings, Artinian D-rings, semiperfect D-rings, von Neumann regular D-rings, D-rings satisfying certain polynomial identities, and semiprime D-rings. We also include some indecomposability results.

1. INTRODUCTION

Throughout, R will represent an associative ring, N the set of nilpotent elements of R, J the Jacobson radical, and $C(R)$ the commutator ideal of R. For each integer $n > 1$, we set $E_n = \{x \in R | x^n = x\}$. An element x in R is called *potent* if $x \in \cup_{n=2}^{\infty} E_n$. A ring R is called *periodic* if for every x in R, $x^m = x^n$ for some distinct positive integers $m = m(x), n = n(x)$. By a theorem of Chacron(see [4; Theorem 1]) R is periodic if and only if for each $x \in R$, there exists a positive integer $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer coefficients such that $x^k = x^{k+1} f(x)$. A ring R is called *weakly periodic* if every element of R is expressible as a sum of a nilpotent element and a potent element of R. Recall that a ring R is *local* [2; page 170] if and only if R/J is a division ring. We study rings in which every zero divisor (left or right) is nilpotent. We call such a ring a *D-ring.* Clearly, every nil ring is a D-ring; every domain is a D-ring; and the ring of integers (mod p^k), p prime, is a D-ring. A less trivial example is Example 1.1 of [11].

2. STRUCTURAL RESULTS FOR VARIOUS CLASSES OF D-RINGS

We start by stating the following lemmas.

Lemma 1. *Let R be D-ring. Then aR is a nil right ideal for all $a \in N$.*

Lemma 1 follows at once, since $a^k = 0$, $a^{k-1} \neq 0$ implies $a^{k-1}(ax) = 0$, and thus $ax \in N$.

Lemma 2. *Let R be a D-ring. If e is an idempotent element of R, then $e = 0$ or $e = 1$.*

Proof. Suppose $e^2 = e \neq 0$, and $x \in R$. Then $e(ex - x) = 0$ and hence $ex - x = 0$; otherwise, e will be nilpotent (R is a D-ring) forcing $e = 0$. Similarly, $xe - x = 0$ for all x in R, and thus $e = 1$.

Theorem 1. *Let R be a D-ring such that N is an ideal of R. Then, either $R = N$ or R/N is a domain.*

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Proof. Suppose that $R \neq N$. Let $\bar{x} = x + N$ and $\bar{y} = y + N$ be two elements in R/N such that $\bar{x}\bar{y} = \bar{0}$. Then $xy \in N$. This implies that $(xy)^m = 0$ and $(xy)^{m-1} \neq 0$ for some positive integer m. Hence $(xy)^{m-1}(xy) = 0$. This implies that

$$y \text{ is a zero divisor or } (xy)^{m-1}x = 0;$$

therefore, y is a zero divisor or x is a zero divisor, since $(xy)^{m-1} \neq 0$. Hence $y \in N$ or $x \in N$, so $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$, and thus R/N is a domain.

Corollary 1. *Let R be a D-ring with N commutative. Then either $R = N$, or N is an ideal and R/N is a domain.*

Proof. If N is commutative, N is an additive subgroup of R, hence an ideal by Lemma 1.

Theorem 2. *If R is a periodic D-ring, then R is either nil or local. Further, if R has an identity element, then N is an ideal and R/N is a field.*

Proof. Since R is periodic, for each $x \in R$, there exists a positive integer $k = k(x)$ such that x^k is idempotent [4]. Using Lemma 2, $x^k = 0$ or $x^k = 1$, and hence x is either nilpotent or invertible. Therefore, R is nil or local. If R has an identity element, then R is local, and hence N is an ideal. Thus, R/N is a periodic division ring, and hence R/N is a field.

Theorem 3. *Let R be an Artinian D-ring such that $R \neq N$. Then R has an identity and R is a local ring. In fact, $N = J$ and R/N is a division ring.*

Proof. Let a be any element of N. Then by Lemma 1, aR is a nil right ideal of R. This implies that $aR \subseteq J$ for all $a \in N$, hence $a \in J$ and thus $N \subseteq J$. Also, R being Artinian implies that J is a nilpotent ideal and hence $J \subseteq N$. It follows that $N = J$ is an ideal, and hence by Theorem 1, R/N is a domain. Thus, R/N is an Artinian domain which can be easily shown to be a division ring. Since $N = J$, we see that R is a local ring. Let $\bar{e} = e + N$ be the identity element of R/N. Then $e - e^2 \in N$ and hence there exists a positive integer k such that $e^k = e^{k+1}p(e)$ for some polynomial $p(\lambda) \in \mathbb{Z}[\lambda]$. From this equation, it is easy to show that $e^k(p(e))^k$ is a nonzero idempotent; hence $1 \in R$, by Lemma 2.

Now, we consider semiperfect D-rings. Recall that a ring R is *semiperfect* [2] if and only if R/J is semisimple (Artinian) and idempotents lift modulo J.

Theorem 4. *A semiperfect D-ring R such that $R \neq J$ must be local.*

Proof. Let R be a semiperfect D-ring such that $R \neq J$. Then R/J is semisimple (Artinian) with more than one element, and hence it is isomorphic to a finite direct product $R_1 \times R_2 \times \cdots \times R_n$ where each R_i is a complete $t_i \times t_i$ matrix ring over a division ring D_i. By Lemma 2, the only idempotents of R are 0 and 1. Since R is semiperfect, the idempotents of R/J lift to idempotents in R. Hence, $\bar{0}$ and $\bar{1}$ are the only idempotents of R/J. If $n > 1$, then the element $(0, \ldots, 0, 1_j, 0, \ldots, 0)$, where 1_j is the identity of R_j, is an idempotent of R/J other that $\bar{0}$ and $\bar{1}$; so $n = 1$ and $R/J \cong R_1$, the complete $t_1 \times t_1$ matrix ring over a division ring D_1. Now, if $t_1 > 1$, then E_{11} is an idempotent of R/J other than $\bar{0}$ and $\bar{1}$; therefore $t_1 = 1$ and $R/J \cong D_1$. Hence, R is local.

Remark. Note that J need not be equal to N in Theorem 4, as a consideration of this local (and hence semiperfect) ring shows:

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} | b \notin p\mathbb{Z},\ (a,b) = 1,\ p \text{ prime} \right\}.$$

Here, $\mathbb{Z}(p)$ is a D-ring, since it is a domain, and $J \neq N$ [2; p. 174].

Recall that a ring R is *regular* (von Neumann) if for each a in R, there exists an element x in R such that $axa = a$. A ring is said to be $\pi - regular$ if for each a in R, there exists a positive integer n, and an element x in R such that $a^n x a^n = a^n$. A ring R is called *semiprimitive* if $J = \{0\}$.

Theorem 5. *Let R be a D-ring with $R \neq N$.*

(i) *If R has no nonzero nil ideals, then R is prime.*

(ii) *If R has no nonzero nil right ideals, then R is a domain.*

(iii) *If R is semiprimitive, then R is a domain.*

(iv) *If R is regular, then R is a division ring.*

(v) *If R is $\pi-$regular, then N is an ideal and R/N is a division ring.*

Proof.

(i) Let K be any nonzero right ideal. Then the right annihilator $A_r(K)$ is a nil ideal and hence $A_r(K) = \{0\}$. This implies that R is a prime ring [9; p. 44].

(ii) Let $a, b \in N$. Then aR, bR are nil right ideals, by Lemma 1. Hence $aR = bR = \{0\}$, by hypothesis, so $a^2 = 0 = ab = ba = b^2$ for all $a, b \in N$. Therefore, $(a - b)^2 = 0$, and $a - b \in N$. Moreover, for all a in N, $aR \subseteq N$ (Lemma 1), and hence N is a nil right ideal of R. Thus, $N = \{0\}$ by hypothesis, and hence R is a domain by Theorem 1.(Recall that, by hypothesis, $R \neq N$.)

(iii) This is an immediate consequence of (ii) (since $J = \{0\}$ and any nil right ideal of R is contained in J).

(iv) Let a be a nonzero element of R. Since R is regular, there exists an element $x \in R$ such that $axa = a$. This implies that $(ax)^2 = ax$, and thus ax is an idempotent element in R; hence, by Lemma 2, $ax = 0$ or $ax = 1$. If $ax = 0$, then $a = axa = 0$. This is not true since $a \neq 0$, so $ax = 1$. Similarly, $xa = 1$. Hence, a is invertible and thus R is a division ring.

(v) Suppose that R is π-regular, and let $a \in R$. Then, $a^n x a^n = a^n$ for some $x \in R$ and some positive integer n. This implies that $(a^n x)^2 = a^n x$, so by Lemma 2, $a^n x = 0$ or $a^n x = 1$. Thus, $a^n = a^n x a^n = 0$ or $a^n x = 1$, so $a \in N$ or a has a right inverse. Similarly, $a \in N$ or a has a left inverse; hence for all $a \in R$, either $a \in N$ or a is invertible, which readily implies that N is an ideal and R/N is a division ring.

3. D-RINGS AND COMMUTATIVITY CONDITIONS

In this section we study the structure of periodic D-rings, weakly periodic D-rings, semiprime D-rings, and D-rings satisfying certain polynomial identities. We will begin with the following lemmas which were proved in [7].

Lemma 3. *Let R be a weakly periodic ring. Then the Jacobson radical J of R is nil. If, furthermore, $xR \subseteq N$ for all $x \in N$, then $N = J$ and R is periodic.*

Lemma 4. *If R is a weakly periodic division ring, then R is a field.*

Theorem 6. *If R is a periodic D-ring, then $C(R)$ is nil.*

Proof. If R is nil, there is nothing to prove. Suppose $R \neq N$, and let $x \in R \backslash N$. Then $x^n = x^m$ for some integers $n > m \geq 1$. It is readily verified that $x^{m(n-m)}$ is a nonzero idempotent, and hence by Lemma 2, $1 \in R$. By Theorem 2, R is local, N is an ideal, and R/N is a field. Thus, $C(R)$ is nil.

Theorem 7. *If R is a weakly periodic D-ring, then R is periodic.*

Proof. Since R is a D-ring, aR is a nil right ideal for all $a \in N$, by Lemma 1. By Lemma 3, $N = J$ and R is periodic.

Theorem 8. *Let R be a semiprime D-ring with commuting nilpotent elements, and suppose $R \neq N$. Then R is a domain.*

Proof. Let $a \in N$ with $a^n = 0$. By Lemma 1, $aR \subseteq N$. Moreover, since N is commutative, it follows that $(aR)^n = \{0\}$. Thus $aR = \{0\}$, since R is semiprime; hence $a = 0$ and R has no zero divisors.

Remark. The hypothesis that R is a D-ring in Theorem 8 cannot be dropped, as a consideration of the ring of integers mod 6 shows.

Theorem 9. *Let R be a semiprime D-ring with $R \neq N$. If R satisfies a polynomial identity, then R is a domain.*

Proof. Let a be any nilpotent element of R. Then, by Lemma 1, aR is nil right ideal of R. Suppose $aR \neq \{0\}$. Since, by hypothesis, R satisfies a polynomial identity, aR is a nonzero nil right ideal satisfying the same polynomial identity. Hence, by Lemma 2.1.1 of [8], R has a nonzero nilpotent ideal, contradicting the fact that R is semiprime. Thus $aR = \{0\}$ and hence $a = 0$. Therefore, $N = \{0\}$ and hence R is a domain.

A consequence of Theorem 9 is the following:

Theorem 10. *Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in n noncommuting indeterminates with relatively prime integer coefficients, such that for each prime p the indentity $f = 0$ is not satisfied by the ring of 2×2 matrices over $GF(p)$. Then every semiprime D-ring R in which $R \neq N$ and which satisfies the identity $f = 0$ is a commutative domain.*

Proof. That R is a domain follows from the previous theorem. That it is commutative follows by a theorem of Kezlan [10].

Let $[x_1, x_2]_1 = [x_1, x_2]$ denote $x_1x_2 - x_2x_1$, and for $k > 1$, let $[x_1, x_2, \ldots, x_{k+1}] = [[x_1, \ldots, x_k], x_{k+1}]$. For $x_1 = x$ and $x_2 = x_3 = \cdots = x_{k+1} = y$, denote the extended commutator $[x, y, \ldots, y]$ by $[x, y]_k$. Next, we consider D-rings with a certain variable identity.

Theorem 11. *Let R be a semiprime D-ring such that $R \neq N$. If for each x, y in R, there exist positive integers $m = m(x, y) \leq S$, and $n = n(x, y) \leq T$, where S and T are fixed positive integers, such that $[x^m, y^n]_k = 0$, $k \geq 1$ fixed, then R is a domain.*

Proof. Clearly R satisfies the polynomial identity

$$[x, y]_k[x, y^2]_k \cdots [x, y^T]_k[x^2, y]_k \cdots [x^2, y^T]_k \cdots [x^S, y]_k \cdots [x^S, y^T]_k = 0.$$

The theorem now follows from Theorem 9.

3. Indecomposability considerations

The following theorem is immediate from the definition of D-ring and known results on direct-product decompositions of rings R in which $(R, +)$ is a torsion group.

Theorem 12. *Let R be an arbitrary D-ring and T its ideal of torsion elements. Then R is either nil or indecomposable. Moreover, either T is a nil ideal or $(T,+)$ is a p-group for some prime p.*

This theorem, together with known results on direct-sum decomposition, provides structural dichotomy theorems for certain classes of D-rings. Our final two theorems provide a sample of theorems of this type.

In [5] a ring R is defined to be *quasi-Boolean* if for each $x \in R$ there exists an integer $n = n(x) \geq 1$ for which $x^n = x^{n+1}$; and it is proved that a quasi-Boolean ring is a direct sum of a Boolean ring and a nil ring if and only if it contains no subring isomorphic to Q_2 or ${Q_2}'$, where Q_2 (resp. ${Q_2}'$) denotes the ring of 2×2 matrices over $GF(2)$ with second row (resp. second column) zero. This result, together with Theorem 12, yields

Theorem 13. *A quasi-Boolean D-ring is either $GF(2)$ or a nil ring.*

Proof. Let R be any quasi-Boolean D-ring. Clearly, Q_2 and Q_2' are not D-rings, hence R is a direct sum of a Boolean ring and a nil ring; and by Theorem 12, R is either Boolean or nil. But by Lemma 2, the only Boolean D-ring is $GF(2)$.

Theorem 14. *Let R be a D-ring such that for each $x, y \in R$ there exists a polynomial $p(X,Y)$ in two noncommuting indeterminates, with integer coefficients, for which*

$$(*) \qquad xy = (xy)^2 p(x,y).$$

Then R is either a zero ring or a periodic field.

Proof. Theorem 1 of [6] states that any ring R satisfying $(*)$ is a direct sum of a J-ring (i.e. a ring in which every element is potent) and a zero ring. In view of Theorem 12, a D-ring with $(*)$ must be either a J-ring or a zero ring. By Lemma 2, D-rings which are also J-rings must be periodic division rings; and J-rings are commutative by Jacobson's famous "$a^n = a$ Theorem."

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American University of Beirut, Beirut, Lebanon
Brock University, St. Catharines, Ontario, Canada L2S 3A1
University of California, Santa Barbara, CA 93106