THE ROLE OF NEARNESS IN CONVENIENT TOPOLOGY

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Abstract. In Convenient Topology semiuniform convergences spaces including filter spaces, uniform spaces and symmetric topological spaces as well as their generalizations are studied in detail. Higher separation axioms, paracompactness and dimension theory profit from the better behaviour of subspaces of semiuniform convergence spaces which results from their relation to nearness spaces. This has been demonstrated by the author in an earlier paper [6]. In the present paper, subspaces of compact symmetric topological spaces (resp. compact Hausdorff spaces) are characterized axiomatically where the Herrlich completion of nearness spaces (resp. Hausdorff completion of uniform spaces) is needed.

0. Introduction

Convenient Topology consists essentially in the study of semiuniform convergence spaces and their invariants, i.e. properties of semiuniform convergence spaces which are preserved by isomorphisms. The construct **SUConv** of semiuniform convergence spaces (and uniformly continuous maps) is a strong topological universe, i.e. a topological construct which is a quasitopos in the sense of M.J. Penon [3] with the additional property that products of quotients are quotients. Furthermore, in **SUConv** convergence structures and uniform convergence structures are available such as topological structures and uniform structures, and initial and final structures have an easy description! Via the subconstruct **Fil** of filter spaces, which form the link between convergence structures and uniform convergence structures, **SUConv** is related to the construct **Mer** of merotopic spaces and via the subconstruct **SubTop** of subtopological spaces to the construct **Near** of nearness spaces, namely in both cases by means of bicoreflective embedding. Topological spaces behave badly with respect to the formation of subspaces as the following example shows: Though there is a difference of a topological nature between the removal of a point and the removal of the closed unit interval [0,1] from the usual topological space $\mathbb{R}$ of real numbers, the obtained topological spaces are not distinguishable, i.e. they are homeomorphic. Much better results are obtained by forming subspaces of (symmetric) topological spaces in **SUConv** (or **Fil**). Then in the above example non–isomorphic spaces result, and subspaces of normal (resp. paracompact) spaces are normal (resp. paracompact); even dimension theory (including cohomological dimension theory) profits from this better behaviour of subspaces (cf. [6]). But the decisive step for obtaining these results is the above mentioned relation to nearness spaces.

In this paper the question how the subspaces formed in **Fil** (resp. **SUConv**) of compact symmetric topological spaces regarded as filter spaces (resp. semiuniform convergence

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*Dedicated to Professors Eraldo Giuli and Mirek Hušek on their 60th birthday spaces), called subcompact spaces, can be characterized axiomatically and the corresponding question for compact Hausdorff spaces are solved. For the proof of the characterization of subcompact spaces the Herrlich completion of nearness spaces is used, whereas for the proof of the corresponding characterization of the so–called sub–(compact Hausdorff) spaces the Hausdorff completion of uniform spaces due to A. Weil [9] suffices, since the construct of sub–(compact Hausdorff) spaces is conretely isomorphic to the construct **SepProx** of separated proximity spaces (separated proximity spaces are identified with totally bounded uniform spaces).

The terminology of this article corresponds to [1] and [4].

Convention. Subconstructs are always assumed to be full and isomorphism–closed.

1. Some basic definitions and results

1.1 Definition. 1) A *filter space* is pair (X, γ), where X is a set and γ a subset of the set $F(X)$ of all filters on X such that the following are satisfied:

(1) $\dot{x} \in \gamma$ for each $x \in X$, where $\dot{x} = \{A \subset X : x \in A\}$,

(2) $\mathcal{G} \in \gamma$ wheneverr $\mathcal{F} \in \gamma$ and $\mathcal{F} \subset \mathcal{G}$.

If (X, γ) is a filter space, then the elements of γ are called *Cauchy filters.*

2) A map $f : (X, \gamma) \to (X', \gamma')$ between filter spaces is called *Cauchy continuous* provided that $f(\mathcal{F}) \in \gamma'$ for each $\mathcal{F} \in \gamma$.

1.2 Remarks. 1) The construct **Fil** of filter spaces (and Cauchy continuous maps) can be embedded bireflectively and bicoreflectively into the construct **SUConv** of semiuniform convergence spaces (and uniformly continuous maps) as well as bicoreflectively into the construct **Mer** of merotopic spaces (and uniformly continuous maps) [cf. [5] and [7]].

2) If $(X, \mathcal{J}_X) \in |\mathbf{SUConv}|$, then its bicoreflective **Fil**–modification is $(X, \gamma_{\mathcal{J}_X})$ with $\gamma_{\mathcal{J}_X} = \{\mathcal{F} \in F(X) : \mathcal{F} \times \mathcal{F} \in \mathcal{J}_X\}$; it is called the *underlying filter space* of $(X, \mathcal{J}_X)$.

3) According to 1) there is an alternative description of filter spaces in the realm of merotopic spaces. In particular, if (X, γ) is a filter space, then (X, μ_γ) is the corresponding merotopic space, where the set μ_γ of uniform covers is given by $\{\mathcal{A} \subset \mathcal{P}(X)$: for each $\mathcal{F} \in \gamma$, there is some $A \in \mathcal{A}$ with $A \in \mathcal{F}\}$.

4) There is also an alternative description of symmetric topological spaces in the realm of filter spaces (cf. [8]). If $(X, \mathcal{X})$ is a symmetric topological space, then $(X, \gamma_{\mathcal{X}})$ is its corresponding filter space, where $\gamma_{\mathcal{X}} = \{\mathcal{F} \in F(X)$: $\mathcal{F}$ converges to some $x \in X$ w.r.t. the topology $\mathcal{X}$ on $X\}$.

5) In order to characterize subspaces of symmetric topological spaces in **SUConv** it suffices to characterize subspaces of symmetric topological spaces in **Fil** (cf. 1) and 4)).

1.3 Theorem (cf. [8]). A filter space (X, γ) is subtopological, i.e. a subspace in **Fil** of some symmetric topological space, iff each $\mathcal{F} \in \gamma$ contains some $\mathcal{G} \in \gamma$ with a γ–open base $\mathcal{B}$, i.e. each $B \in \mathcal{B}$ is γ–open, where a subset O of X is γ–open iff for each $x \in X$ and each $\mathcal{F} \in F(X)$ with $\mathcal{F} \cap \dot{x} \in \gamma, O \in \mathcal{F}$.

1.4 Theorem [Bentley] (cf. e.g. [2;3.1.9]). A filter space (X, γ) is subtopological iff (X, μ_γ) is a nearness space.

2. Subcompact spaces

2.1 Definiton A filter space (X, γ) is called *m-contigual* provided that (X, μ_γ) is contigual, i.e. each $\mathcal{A} \in \mu_\gamma$ is refined by some finite $\mathcal{B} \in \mu_\gamma$.

2.2 Proposition. A filter space (X, γ) is m–contigual iff the following is satisfied: If from each $\mathcal{F} \in \gamma$ some $F_{\mathcal{F}}$ is chosen, then there are finitely many $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \gamma$ such that for each $\mathcal{G} \in \gamma$ there is some $i \in \{1, \ldots, n\}$ with $F_{\mathcal{F}_i} \in \mathcal{G}$.

2.3 Remark. If (X, μ) is a contigual nearness space, then its Herrlich completion (= canonical completion) (X^*, μ^*) is a compact topological nearness space, i.e. a compact symmetric topological space regarded as a nearness space (cf. e.g. [4; 6.2.11] and note that if $(X, \mathcal{X})$ is a symmetric topological space, then $(X, \mu_{\mathcal{X}})$ is a topological nearness space, where $\mu_{\mathcal{X}}$ consists of all covers of X which are refined by some open cover of X).

2.4 Theorem. Let (X, γ) be a filter space. Then the following are equivalent:
(1) (X, γ) is subcompact, i.e. a subspace (in **Fil**) of a compact symmetric topological space (regarded as a filter space),
(2) (X, γ) is a dense subspace (in **Fil**) of a compact symmetric topological space (regarded as a filter space),
(3) (X, γ) is subtoplogical and m–contigual.

Proof. (3) $\Rightarrow$ (2). By 1.4, (X, μ_γ) is a contigual nearness space and hence, by 2.3., (X^*, μ^*_γ) is a compact topological nearness space. Since (X^*, μ^*_γ) is topological, $\mu^*_\gamma = \mu_{\mathcal{X}_{\mu^*_\gamma}}$ with the symmetric topology $\mathcal{X}_{\mu^*_\gamma} = \{O \subset X : O = int_{\mu^*_\gamma} O\}$. Put $\mathcal{X}^* = \mathcal{X}_{\mu^*_\gamma}$. Hence, $\mu_{\gamma_{\mathcal{X}^*}} = \mu_{\mathcal{X}^*} = \mu^*_\gamma$, i.e. (X^*, μ^*_γ) is the corresponding merotopic space of the filter space $(X^*, \gamma_{\mathcal{X}^*})$. Consequently, since (X, μ_γ) is a subspace in the construct **Near** of nearness spaces (and uniformly continuous maps) of $(X^*, \mu^*_\gamma), (X, \gamma)$ is a subspace in **Fil** of $(X^*, \gamma_{\mathcal{X}^*})$ [note: $\gamma_{\mu_\gamma} = \gamma$ for each **Fil**–structure γ on a set X, where γ_{μ_γ} denotes the set of all Cauchy filters (cf. [4;3.2.3.8] for their definition) in (X, μ_γ), where $(X^*, \gamma_{\mathcal{X}^*})$ is a compact symmetric topological space regarded as a filter space. Furthermore, X is dense in $(X^*, \mathcal{X}^*)$, i.e. in $(X^*, \gamma_{\mathcal{X}^*})$.
(2) $\Rightarrow$ (1). This implication is obvious.
(1) $\Rightarrow$ (3). Let (X, γ) be a subspace in **Fil** of a compact topological filter space (X', γ'), i.e. there is a compact symmetric topological space $(X', \mathcal{X}')$ such that $\gamma' = \gamma_{\mathcal{X}'}$. Obviously, (X, γ) is subtopological. Furthermore, since $\mu_{\mathcal{X}'} = \mu_{\gamma_{\mathcal{X}'}} = \mu_{\gamma'}, (X', \mu_{\gamma'})$ is a contigual nearness space and (X, μ_γ) is a subspace of it in **Mer**. Thus, by [4;3.1.3.3.], (X, μ_γ) is a contigual nearness space (note that subspaces of nearness spaces are formed in **Near** as in **Mer**). Hence, by definition, (X, γ) is m–contigual.

2.5 Remark. It is easliy checked that *subcompact filter spaces form a bireflective subconstruct* of **Fil**, i.e. they are closed under formation subspaces and products and contain all indiscrete **Fil**–objects.

3. Sub–(compact Hausdorff) spaces

3.1 Proposisition. Let (X, γ) be a subtopological filter space and (X, μ_γ) its corresponding nearness space. Then the topology $\mathcal{X}_\gamma = \{O \subset X : O \text{ is } \gamma\text{–open}\}$ coincides with the topology $\mathcal{X}_{\mu_\gamma} = \{O \subset X : \text{int}_{\mu_\gamma} O = O\}$.

3.2 Corollary. Let (X, γ) be a subtopological filter space. Then (X, γ) is T_1 (i.e. $(X, \mathcal{X}_\gamma)$ is T_1) iff (X, μ_γ) is T_1 (i.e. $(X, \mathcal{X}_{\mu_\gamma})$ is T_1).

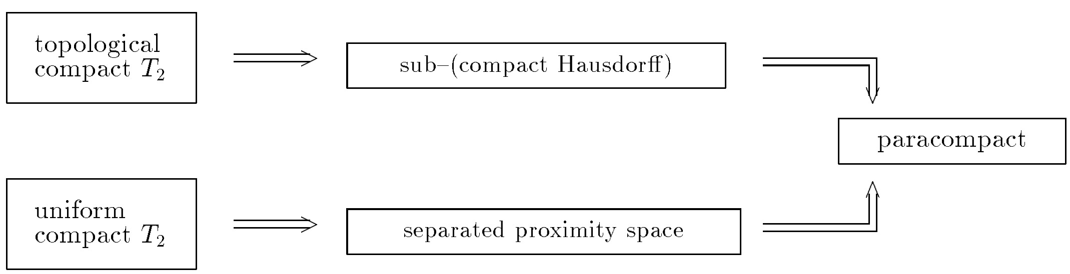
3.3 Definition. A filter space (X, γ) is called *m-proximal* provided that (X, μ_γ) is a proximity space (= totally bounded uniform space).

3.4 Theorem. Let (X, γ) be a filter space. Then the following are equivalent:
(1) (X, γ) is sub–(compact Hausdorff), i.e. a subspace (in **Fil**) of a compact Hausdorff space

(regarded as a filter space),
(2) (X,γ) is a dense subspace (in **Fil**) of a compact Hausdorff space (regarded as a filter space),
(3) (X,γ) is m-proximal and T_1.

Proof. (3) $\Rightarrow$ (2). Since (X,γ) is T_1 and m–proximal, (X,μ_γ) is a separated proximity space (cf. 3.2. and note that, since (X,μ_γ) is a proximity space, it is a nearness space and thus, by 1.4., (X,γ) is subtopological). Consequently, the Hausdorff completion (X^*,μ_γ^*) of (X,μ_γ) is compact (i.e. $(X^*,\mathcal{X}^*_{\mu_\gamma})$ is compact) and contains (X,μ_γ) as a dense subspace. Put $\mathcal{X}_{\mu_\gamma^*}=\mathcal{X}^*$. It is easily checked that $\mu_{\mathcal{X}^*}=\mu_{\gamma_{\mathcal{X}^*}}$. Since furthermore a compact Hausdorff space is uniquely uniformizable, $\mu_\gamma^*=\mu_{\mathcal{X}^*}=\mu_{\gamma_{\mathcal{X}^*}}$. Additionally, (X,γ) is a dense subspace of $(X^*,\gamma_{\mathcal{X}^*})$.
(2) $\Rightarrow$ (1) is obvoius.
(1) $\Rightarrow$ (3). Let (X,γ) be a subspace (in **Fil**) of (X',γ') with $\gamma'=\gamma_{\mathcal{X}'}$, where $(X',\mathcal{X}')$ is a compact Hausdorff space. Then (X,γ) is T_1 as a subspace of a T_1–space. Furthermore, (X,γ) is a subspace of $(X',\mu_{\gamma'})=(X,\mu_{\mathcal{X}'})$ in **Mer**. Since $(X',\mu_{\mathcal{X}'})$ is contigual and uniform, it follows that (X,μ_γ) is contigual and uniform ('contigual' and 'uniform' are hereditary properties!), in other words: a proximity space, i.e. (X,γ) is m–proximal.

3.6 Remarks. 1) It is easily verified that *sub-(compact Hausdorff) spaces form an epireflective subconstruct of* **Fil**, i.e. they are closed under formation of subspaces and products (in **Fil**).
2) Sub–(compact Hausdorff) spaces can also be described in the realm of semiuniform convergence spaces since they are filter spaces (cf. 1.2.1)). Using the definition of paracompactness for semiuniform convergence spaces introduced in [6], the following implication scheme in **SUConv** is obvious:



3) The subconstructs (of **SUConv**) $\mathbf{Sub_{SUConv}CompH}$ of sub–(compact Hausdorff) spaces and **SepProx** of separated proximity spaces are concretely isomorphic, but $|\mathbf{Sub_{SUConv}CompH}|\cap|\mathbf{SepProx}|$ does not contain a space with more than one point.

References

[1] Adámek, J., H. Herrlich, and G.E. Strecker: 1990, *Abstract and Concrete Categories*, Wiley Intersci. Publ., New York.

[2] Herrlich, H.: 1988, *Topologie II: Uniforme Räume*, Heldermann, Berlin.

[3] Penon, M.J.: 1973, 'Quasitopos', *C.R. Acad. Sci. Paris*, Sér. A. **276**, 237–240.

[4] Preuß, G.: 1988, *Theory of Topological Structures*, Reidel, Dordrecht.

[5] Preuß, G.: 1995, 'Semiuniform convergence spaces', *Math. Japonica* **41**, 465–491.

[6] Preuß, G.: 1999, 'Higher separation axioms, paracompactness and dimension for semiuniform convergence spaces', *Scientiae Mathemticae* **2**, 321–335.

[7] Preuß, G.: 2000, 'Some algebraically topological aspects in the realm of Convenient Topology', *Quest. Math.* (to appear).

[8] Preuß, G.: 2000, 'Characterization of subspaces of important types of convergence spaces', to appear.

[9] Weil, A.: 1937, *Sur les espaces à structures uniformes et sur la topologie générale*, Hermann, Paris.

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