ON PRIME IDEALS OF GROUPOIDS-ORDERED GROUPOIDS

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Abstract. In a groupoid (resp. ordered groupoid) G, the non-empty intersection of the elements of a chain of prime ideals, is a prime ideal of G. As a consequence, each prime ideal of a groupoid (resp. ordered groupoid) G containing a non-empty subset K of G, contains a prime ideal P^* of S having the property: If T is a prime ideal of G such that $K \subseteq T \subseteq P^*$, then $T = P^*$. As a result, in a groupoid (resp. ordered groupoid) G with zero, each prime ideal of G contains a minimal prime ideal of G. Some further results on prime ideals of groupoids (resp. ordered groupoids) are also given.

If $(G, ., \leq)$ is an ordered groupoid, a non-empty subset I of G is called an ideal of G if 1) $GI \subseteq I$ and $IG \subseteq I$ and 2) $a \in I$, $G \ni b \leq a$ implies $b \in I$ [2]. If G is a groupoid, an ideal of G is a non-empty subset I of G such that $GI \subseteq I$ and $IG \subseteq I$. An ideal I of a groupoid (resp. ordered groupoid) G is called prime if $a, b \in G$ such that $ab \in I$ implies $a \in I$ or $b \in I$. Equivalent Definition: $A, B \subseteq G$ such that $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ [2]. If $(G, ., \leq)$ is an ordered groupoid, a zero of G is an element 0 of G such that $0x = x0 = 0$ and $0 \leq x$ for every $x \in G$ [1]. If G is a groupoid, a zero of G is an element 0 of G such that $0x = x0 = 0$ for every $x \in G$.

1. Each prime ideal of G contains a minimal prime ideal of G

Proposition 1. *Let G be a groupoid (resp. ordered groupoid) and $\mathcal{B}$ a chain (under set inclusion) of prime ideals of G. If the intersection $\cap\{B \mid B \in \mathcal{B}\}$ is non-empty, then it is a prime ideal of G.*

Proof. Since $\cap\{B \mid B \in \mathcal{B}\} \neq \emptyset$, the set $\cap\{B \mid B \in \mathcal{B}\}$ is an ideal of G. Let $a, b \in G$, $ab \in \cap\{B \mid B \in \mathcal{B}\}$, $a \notin \cap\{B \mid B \in \mathcal{B}\}$ and $b \notin \cap\{B \mid B \in \mathcal{B}\}$. Let $B_1, B_2 \in \mathcal{B}$ such that $a \notin B_1$ and $b \notin B_2$. Since $ab \in B_1$, $a \notin B_1$ and B_1 is prime, we have $b \in B_1$. Since $ab \in B_2$, $b \notin B_2$ and B_2 is prime, we have $a \in B_2$. Since $b \in B_1$ and $b \notin B_2$, we have $B_1 \not\subseteq B_2$. Then, since $\mathcal{B}$ is a chain, we have $B_2 \subseteq B_1$. Then $a \in B_1$. Impossible.

Proposition 2. *Let G be a groupoid (resp. ordered groupoid), $\emptyset \neq K \subseteq G$ and P a prime ideal of G such that $K \subseteq P$. Then, there exists a prime ideal P^* of G having the properties:*
1) *$P^* \subseteq P$.*
2) *For each prime ideal T of G such that $K \subseteq T \subseteq P^*$, we have $T = P^*$.*

Proof. Let $\mathcal{A} := \{A \mid A$ prime ideal of G, $K \subseteq A \subseteq P\}$.
Since $P \in \mathcal{A}$, we have $\mathcal{A} \neq \emptyset$.
Then, the set $\mathcal{A}$ with the relation "$\preceq$" on $\mathcal{A}$ defined by:

$$\preceq := \{(A, B) \in \mathcal{A} \times \mathcal{A} \mid B \subseteq A\}$$

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is an ordered set.
Let $(\mathcal{B}, \subseteq)$ be a chain in $\mathcal{A}$. The set $\cap\{B \mid B \in \mathcal{B}\}$ is an upper bound of $\mathcal{B}$ in $\mathcal{A}$. In fact:
The set $\cap\{B \mid B \in \mathcal{B}\}$ is a prime ideal of G. Indeed:
B is a prime ideal of G for every $B \in \mathcal{A} \supseteq \mathcal{B}$. Since $K \subseteq B$ for every $B \in \mathcal{A} \supseteq \mathcal{B}$, we have $\emptyset \neq K \subseteq \cap\{B \mid B \in \mathcal{B}\}$, then $\cap\{B \mid B \in \mathcal{B}\} \neq \emptyset$.
The set $\mathcal{B}$ is a chain of prime ideals of G and $\cap\{B \mid B \in \mathcal{B} \neq \emptyset$. By Proposition 1, the set $\cap\{B \mid B \in \mathcal{B}\}$ is a prime ideal of G.
Moreover, $K \subseteq \cap\{B \mid B \in \mathcal{B}\} \subseteq P$. Indeed: Since $K \subseteq B \subseteq P$ for every $B \in \mathcal{A} \supseteq \mathcal{B}$, we have $K \subseteq \cap\{B \mid B \in \mathcal{B}\} \subseteq P$.

By Zorn's Lemma, the set $\mathcal{A}$ has a maximal element, say P^*. For the set P^*, we have the following:

1) $P^* \subseteq P$ (since $P^* \in \mathcal{A}$).
2) Let T be a prime ideal of G such that $K \subseteq T \subseteq P^*$ ($\Rightarrow\ T = P^*$?)

Since $K \subseteq T \subseteq P^*$, we have $K \subseteq T \subseteq P$ (by 1)). Then, since T is a prime ideal of G, we have $T \in \mathcal{A}$. Since $T, P^* \in \mathcal{A}$, $T \subseteq P^*$, we have $P^* \preceq T$. Since $P^* \preceq T \in \mathcal{A}$ and P^* is a maximal in $\mathcal{A}$, we have $P^* = T$. □

Let G be a groupoid (resp. ordered groupoid). A prime ideal P of G is called a minimal prime ideal of G if

For every prime ideal T of G such that $T \subseteq P$, we have $T = P$.

Proposition 3. *Let G be a groupoid (resp. ordered groupoid) with zero and P a prime ideal of G. Then there exists a minimal prime ideal P^* of G such that $P^* \subseteq P$.*

Proof. The set P is a prime ideal of G and $\{0\} \subseteq P$. By Proposition 2, there exists a prime ideal P^* of G having the properties:

1) $P^* \subseteq P$.
2) For each prime ideal T of G such that $\{0\} \subseteq T \subseteq P^*$, we have $T = P^*$.

The set P^* is a minimal prime ideal of G. In fact: Let T be a prime ideal of G such that $T \subseteq P^*$. Since T is an ideal of G, we have $\{0\} \subseteq T$. Then $\{0\} \subseteq T \subseteq P^*$. Then, by 2), $T = P^*$.

2. Some further remarks on prime ideals

Let G be a groupoid (resp. ordered groupoid) with 0. We say that G does not contain divisors of zero if

$a, b \in G, \quad ab = 0$ implies $a = 0$ or $b = 0$.

Remark 1. A groupoid (resp. ordered groupoid) G does not contain divisors of zero if and only if the set $\{0\}$ is a prime ideal of G.

Lemma 1. *Let G be a groupoid (resp. ordered groupoid) and $\{I_i \mid i \in I\}$ a (non-empty) family of ideals of G. Then the set $\bigcup_{i \in I} I_i$ is an ideal of G.* □

When we speak about a family, we always consider that it is non-empty.

Proposition 4. *Let G be a groupoid (resp. ordered groupoid) and $\{P_i \mid i \in I\}$ a family of prime ideals of G. Then the set $\bigcup_{i \in I} P_i$ is a prime ideal of G.*

Proof. By Lemma 1, the set $\bigcup_{i\in I} P_i$ is an ideal of G. Let $a, b \in G$, $ab \in \bigcup_{i\in I} P_i$. Let $j \in I$ such that $ab \in P_j$. Since P_j is prime, we have $a \in P_j \subseteq \bigcup_{i\in I} P_i$ or $b \in P_j \subseteq \bigcup_{i\in I} P_i$.

Lemma 2. *Let G be a groupoid (resp. ordered groupoid) and $\{I_i \mid i \in I\}$ a family of ideals of G. If $\bigcap_{i\in I} I_i \neq \emptyset$, then the set $\bigcap_{i\in I} I_i$ is an ideal of G.*

Corollary 1. *Let S be a semigroup (resp. ordered semigroup). If I_i is an ideal of S for every $i = 1, 2, \cdots, n$, then the set $\bigcap_{i=1}^{n} I_i$ is an ideal of S.*

Proof. By Lemma 2, it is enough to prove that $\bigcap_{i=1}^{n} I_i \neq \emptyset$. We have $\emptyset \neq I_i \subseteq S$ for every $i = 1, 2, \cdots, n$, so $I_1 I_2 \cdots I_n \neq \emptyset$. Since I_i is an ideal of S, we have $I_1 I_2 \cdots I_n \subseteq I_i$ for every $i = 1, 2, \cdots, n$, then $I_1 I_2 \cdots I_n \subseteq \bigcap_{i=1}^{n} I_i$. Hence we have $\bigcap_{i=1}^{n} I_i \neq \emptyset$. □

As in Corollary 1, we prove the

Corollary 2. *Let G be a groupoid (resp. ordered groupoid). If I_1, I_2 are ideals of G, then the set $I_1 \cap I_2$ is an ideal of G.*

Proposition 5. *Let G be a groupoid (resp. ordered groupoid), P_1, P_2 ideals of G such that $P_1 \cap P_2$ be a prime ideal of G. Then $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.*

Proof. Let $P_1 \nsubseteq P_2$ and let $a \in P_2$. Let $b \in P_1$ and $b \notin P_2$. We have $ab \in P_2 G \subseteq P_2$, $ab \in G P_1 \subseteq P_1$. Then $ab \in P_1 \cap P_2$. Since $P_1 \cap P_2$ is prime, we have $a \in P_1 \cap P_2$ or $b \in P_1 \cap P_2$. Since $b \notin P_2$, we have $b \notin P_1 \cap P_2$. Then $a \in P_1 \cap P_2$, and $a \in P_1$.

Remark 2. Let G be a groupoid (resp. ordered groupoid), P_1 a prime ideal of G and P_2 an ideal of G such that $P_1 \subseteq P_2$. Then the set $P_1 \cap P_2$ is a prime ideal of G. □

By Proposition 5 and Remark 2, we have the following:

Proposition 6. *Let G be a groupoid (resp. ordered groupoid), P_1, P_2 prime ideals of G. The following are equivalent:*
1) *$P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.*
2) *$P_1 \cap P_2$ is a prime ideal of G.*

Proposition 7. *Let G be a groupoid (resp. ordered groupoid), $\{P_i \mid i \in I\}$ a family of prime ideals of G which is a chain. If $\bigcap_{i\in I} P_i \neq \emptyset$, then the set $\bigcap_{i\in I} P_i$ is a prime ideal of G.*

Proof. By Lemma 2, the set $\bigcap_{i\in I} P_i$ is an ideal of G. Let $a, b \in G$, $ab \in \bigcap_{i\in I} P_i$, $a \notin \bigcap_{i\in I} P_i$ and $b \notin \bigcap_{i\in I} P_i$. Let $j, k \in I$ such that $a \notin P_j$ and $b \notin P_k$.

We have $P_j \subseteq P_k$ or $P_k \subseteq P_j$.
Let $P_j \subseteq P_k$. Since $ab \in P_j$, P_j prime and $a \notin P_j$, we have $b \in P_j \subseteq P_k$.
Impossible.
The case $P_k \subseteq P_j$ is also impossible.

Proposition 8. *Let $(S, ., \leq)$ be an ordered semigroup, I an ideal of S and P a prime ideal of I. Then P is an ideal of S.*

Proof. First of all, $\emptyset \neq P \subseteq I \subseteq S$.
Let $a \in S$, $b \in P$. Since $b \in I$, we have $ab \in SI \subseteq I$, $aba \in IS \subseteq I$, $(ab)^2 = (aba)b \in IP \subseteq P$. Since P is prime, we have $ab \in P$. Similarly, $PS \subseteq P$.
Let $a \in P$, $S \ni b \leq a$. Since $S \ni b \leq a \in I$, I an ideal of S, we have $b \in I$. Since $I \ni b \leq a \in P$, P an ideal of I, we have $b \in P$. □

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