L^2-BEHAVIOUR OF SOLUTIONS TO THE LINEAR HEAT AND WAVE EQUATIONS IN EXTERIOR DOMAINS

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ABSTRACT. Uniform L^2-decay of solutions for the linear heat equations will be given. In order to derive the L^2-decay of solutions, the modified method of Morawetz [6] will be used and we shall show that the L^2 norm of solutions decays like $O(t^{-1})$ as $t \to +\infty$ for some kinds of weighted initial data. Furthermore, by the same argument, one can also derive the L^2-bound and L^2-decay for weak solutions of the linear free and dissipative wave equations, respectively.

1. Introduction.

Let Ω be an exterior domain in $\mathbf{R}^N$ ($N \geq 2$) with a compact C^2-boundary $\partial\Omega$. Without loss of generality, we may assume $0 \notin \overline{\Omega}$. In this paper, first we are concerned with the initial-boundary value problem

$$u_t(t,x) - \Delta u(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{1.3}$$

Throughout this paper, $\|\cdot\|$ means the usual $L^2(\Omega)$-norm. Furthermore, we set

$$(f,g) = \int_\Omega f(x)g(x)\,dx.$$

First of all, we shall state the well-posedness to the problem (1.1)-(1.3) (c.f. Cazenave and Haraux [2, Proposition 3.5.3]).

Proposition 1.1. *For each $u_0 \in H_0^1(\Omega)$, there exists a unique solution $u(t,x)$ in the class*

$$C([0,\infty); H_0^1(\Omega)) \bigcap C^1((0,\infty); L^2(\Omega)) \bigcap C((0,\infty); H^2(\Omega))$$

to the problem (1.1)-(1.3) *satisfying*

$$\frac{1}{2}\|\nabla u(t,\cdot)\|^2 + \int_0^t \|u_t(s,\cdot)\|^2\,ds = \frac{1}{2}\|\nabla u_0\|^2 \quad \text{on } [0,\infty), \tag{1.4}$$

$$(u_t(t,\cdot), u(t,\cdot)) + \|\nabla u(t,\cdot)\|^2 = 0 \quad \text{on } [0,\infty). \tag{1.5}$$

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For the equation (1.1), as will be seen in the proof of Corollary 1.3 below, we can easily derive the decay of energy $\|\nabla u(t,\cdot)\|^2$. But then, it seems unknown at least for the exterior problem whether L^2-norm of the solution $u(t,x)$ to the problem (1.1)-(1.3) decays or not. Since we treat the continuous orbit $\{u(t,\cdot)\}_{t\geq 0}$ in the phase space $H_0^1(\Omega)$, it is quite natural to investigate L^2-decay of solutions. On the other hand, in the case when $\Omega = \mathbf{R}^N$, although we can use the explicit formula through the fundamental solution for the heat operator $\frac{\partial}{\partial t} - \Delta$ (see Racke [10, Lemma 11.6]), our result will make sense (at least) of an alternative proof.

The first purpose of this paper is to derive a certain decay rate of L^2-norm of the solution to the problem (1.1)-(1.3) with the weighted initial data in an “exterior domain”. Our argument is based on the (modified) method of Morawetz [6] (for another use of Morawetz' method, see also Ikehata and Matsuyama [4] and Nakao [8]), and the so called Hardy inequality (see Dan and Shibata [3]). Before introducing our main theorem, we must define a function $d(x)$ as follows:

$$d(x) = \begin{cases} |x|, & N \geq 3, \\ |x|\log(B|x|), & N = 2, \end{cases} \tag{1.6}$$

where $B > 0$ is a constant such that $\inf_{x\in\Omega}|x| \geq \dfrac{2}{B} > 0$. Then, based on Proposition 1.1 our main result reads as follows.

Theorem 1.2. *Let $N \geq 2$ and assume that the initial data u_0 belongs to $H_0^1(\Omega)$ and further satisfies $\|d(\cdot)u_0\| < +\infty$. Then, the solution $u(t,x)$ to the problem* (1.1)-(1.3) *satisfies*

$$(1+t)\|u(t,\cdot)\|^2 \leq C^*\|d(\cdot)u_0\|^2$$

for all $t \geq 0$, where $C^* > 0$ is a certain constant.

Corollary 1.3. *Under the same assumptions as in Theorem* 1.2*, one has*

$$(1+t)\|u(t,\cdot)\|_{H^1}^2 \leq C(\|u_0\|_{H^1}^2 + \|d(\cdot)u_0\|^2)$$

for all $t \geq 0$ with some constant $C > 0$, where $\|\cdot\|_{H^1}$ denotes the usual $H_0^1(\Omega)$-norm.

The second purpose of this paper is to derive L^2-bound for the free wave equation:

$$u_{tt}(t,x) - \Delta u(t,x) = 0, \quad (t,x) \in (0,\infty)\times\Omega, \tag{1.7}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{1.8}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{1.9}$$

Generally speaking, in studying the local energy decay of solutions to the problem (1.7)-(1.9) it seems essential to derive the L^2-bound for solutions (see [6] and [8]). Further, since $\|\nabla u(t,\cdot)\|$ is bounded for all $t \geq 0$, from the point of view of the dynamical system, it is important to know whether the L^2-norm of the solution to the problem (1.7)-(1.9) is bounded or not for the initial data $\{u_0, u_1\} \in H_0^1(\Omega)\times L^2(\Omega)$. In this occasion, we shall proceed our argument based on the energy identity.

Proposition 1.4. *For each* $\{u_0, u_1\} \in H_0^1(\Omega) \times L^2(\Omega)$*, there exists a unique solution* $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ *to the problem* (1.7)-(1.9) *such that*

$$\frac{1}{2}\|\nabla u(t,\cdot)\|^2 + \frac{1}{2}\|u_t(t,\cdot)\|^2 = \frac{1}{2}\|\nabla u_0\|^2 + \frac{1}{2}\|u_1\|^2.$$

As in the results of Morawetz [6] and Nakao [8], we can see that the boundedness of $\|u(t,\cdot)\|$ plays an essential role in deriving the local energy decay of solutions to the wave equations with the compactly supported initial data $\{u_0, u_1\}$. However, the following result implies that the L^2-boundedness holds true without any compactness of the support of the initial data. If, in particular, we impose the initial data as $u_1 = 0$, then we can obtain the L^2-bound for the solution without any condition of compact support for the initial data u_0. Generalizing these observations, we have the following result.

Theorem 1.5. *Let* $N \geq 2$ *and assume the initial data* $\{u_0, u_1\}$ *belongs to* $H_0^1(\Omega) \times L^2(\Omega)$ *and further satisfies* $\|d(\cdot)u_1\| < +\infty$*. Then, the solution* $u(t,x)$ *to the problem* (1.7)-(1.9) *satisfies*

$$\|u(t,\cdot)\|^2 \leq \|u_0\|^2 + C\|d(\cdot)u_1\|^2$$

for all $t \geq 0$ *with a certain constant* $C > 0$.

In [3] Dan and Shibata have investigated the asymptotic behaviour of the dissipative wave equations in an exterior domain and proved that if the initial data $\{u_0, u_1\}$ has a compact support, then the local energy $E_{loc}(t)$ and L^2-norm of solutions decay like $C(\|u_0\|_{H^1}^2 + \|u_1\|^2)(1+t)^{-N}$ as $t \to +\infty$. The proof in [3] is based on a spectral analysis and the Poincaré type inequality (see [3, Lemma 2.3]), and the dissipative term u_t plays an essential role.

Our final result is concerned with the L^2-decay of solutions for the following dissipative wave equation:

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.10}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{1.11}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{1.12}$$

In [4], we have derived the uniform L^2-decay of a solution to the problem (1.10)-(1.12) for an appropriately selected initial data. Compared with (especially) the Cauchy problem in $\mathbf{R}^N$ for the dissipative wave equation (1.10) treated by Kawashima *et al.* [5], our result is much stronger than [5] in the sense that if we apply the result of [5] to our problem, we can only derive the L^2-bound for the solution with the same initial data as in [4]. On the contrary, if the initial data $\{u_0, u_1\}$ has the weight $d(x)$, we can derive the L^2-decay of the solution to the problem (1.10)-(1.12). To state the result, we need the well-posedness of the problem (1.10)-(1.12).

Proposition 1.6. *For each* $\{u_0, u_1\} \in H_0^1(\Omega) \times L^2(\Omega)$*, there exists a unique solution* $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ *to the problem* (1.10)-(1.12) *such that*

$$\frac{1}{2}\|\nabla u(t,\cdot)\|^2 + \frac{1}{2}\|u_t(t,\cdot)\|^2 + \int_0^t \|u_t(s,\cdot)\|^2\,ds = \frac{1}{2}\|\nabla u_0\|^2 + \frac{1}{2}\|u_1\|^2, \tag{1.13}$$

$$\frac{d}{dt}(u_t(t,\cdot), u(t,\cdot)) + \|\nabla u(t,\cdot)\|^2 + (u_t(t,\cdot), u(t,\cdot)) = \|u_t(t,\cdot)\|^2. \tag{1.14}$$

Our final result reads as follows.

Theorem 1.7. *Let $N \geq 2$ and assume the initial data $\{u_0, u_1\} \in H_0^1(\Omega) \times L^2(\Omega)$ further satisfies $\|d(\cdot)(u_0 + u_1)\| < +\infty$. Then, the solution $u(t,x)$ to the problem* (1.10)-(1.12) *satisfies*

$$(1.15) \qquad (1+t)\|u(t,\cdot)\|^2 \leq C\{\|u_0\|_{H^1}^2 + \|u_1\|^2 + \|d(\cdot)(u_0+u_1)\|^2\}$$

for all $t \geq 0$ with a constant $C > 0$ independent of $t \in [0,\infty)$.

Remark 1.8. *If $u_0 + u_1 = 0$ in $L^2(\Omega)$, then this result coincides with that of* [4]. *On the other hand, if $\operatorname{supp} u_0 \cup \operatorname{supp} u_1$ is compact, then $\|d(\cdot)(u_0 + u_1)\| < +\infty$ obviously follows. Thus, in the framework of compactly supported initial data, the decay estimate* (1.15) *is also valid.*

2. Proof of Theorem 1.2. In this section, we shall prove our main Theorem 1.2. First, we shall prepare the Hardy inequality (see [3]).

Lemma 2.1. *For each $u \in H_0^1(\Omega)$, it holds that*

$$\int_\Omega \frac{|u(x)|^2}{d(x)^2}\,dx \leq C\|\nabla u\|^2,$$

where $d(x)$ is the function defined in (1.6).

Lemma 2.2. *For each $u_0 \in H_0^1(\Omega)$, the solution $u(t,x)$ to the problem* (1.1)-(1.3) *satisfies*

$$(1+t)\|u(t,\cdot)\|^2 \leq \|u_0\|^2 + \int_0^t \|u(s,\cdot)\|^2\,ds$$

for all $t > 0$.

Proof. Multiplying the both sides of the identity (1.5) by $1+t$ and integrating it over $[0,t]$, one has

$$\int_0^t (1+s)\frac{1}{2}\frac{d}{ds}\|u(s,\cdot)\|^2\,ds + \int_0^t (1+s)\|\nabla u(s,\cdot)\|^2\,ds = 0.$$

Then integrating by parts, we see

$$\int_0^t (1+s)\frac{1}{2}\frac{d}{ds}\|u(s,\cdot)\|^2\,ds = \frac{1+t}{2}\|u(t,\cdot)\|^2 - \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\int_0^t \|u(s,\cdot)\|^2\,ds,$$

which completes the proof of Lemma 2.2. ∎

The following lemma plays an essential role in our argument and Theorem 1.2 is an immediate consequence of Lemmas 2.2 and 2.3 below.

Lemma 2.3. *Under the assumptions as in Theorem* 1.2, *one has*

$$\int_0^t \|u(s,\cdot)\|^2\,ds \leq C^*\|d(\cdot)u_0\|^2$$

for all $t > 0$ with some constant $C^* > 0$.

Proof. As in [4], set

$$w(t,x) = \int_0^t u(s,x)\,ds.$$

Then, since $\|\Delta u(t,\cdot)\| = \|u_t(t,\cdot)\| \in L^1_{loc}(0,\infty)$ holds (c.f. [2, Proposition 3.5.3]), it follows from the identity (1.4) that $w \in C^1([0,\infty); H_0^1(\Omega)) \cap C((0,\infty); H^2(\Omega))$ satisfies

$$w_t(t,\cdot) - \Delta w(t,\cdot) = u_0 \quad \text{on} \ \ (0,\infty),$$

$$w(0,x) = 0, \quad x \in \Omega,$$

$$w|_{\partial\Omega} = 0, \quad t \in (0,\infty),$$

so that

$$\|w_t(t,\cdot)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla w(t,\cdot)\|^2 = (u_0, w_t(t,\cdot)). \tag{2.1}$$

Hence, noting $w_t = u$ and integrating the both sides of the identity (2.1), one has

$$\begin{aligned}\int_0^t \|u(s,\cdot)\|^2\,ds + \frac{1}{2}\|\nabla w(t,\cdot)\|^2 &= \int_0^t (u_0, w_t(s,\cdot))\,ds \\ &= (u_0, w(t,\cdot)) - (u_0, w(0,\cdot)) \\ &= (u_0, w(t,\cdot)). \end{aligned} \tag{2.2}$$

Here we note that

$$|(u_0, w(t,\cdot))| \le \left(\int_\Omega |u_0(x)|^2 d(x)^2\,dx\right)^{1/2} \left(\int_\Omega \frac{|w(t,x)|^2}{d(x)^2}\,dx\right)^{1/2}. \tag{2.3}$$

Then, combining Lemma 2.1 with the inequality (2.3), we have

$$\begin{aligned}|(u_0, w(t,\cdot))| &\le C\|d(\cdot)u_0\|\,\|\nabla w(t,\cdot)\| \\ &\le \frac{C}{2\varepsilon}\|d(\cdot)u_0\|^2 + \frac{C\varepsilon}{2}\|\nabla w(t,\cdot)\|^2, \end{aligned} \tag{2.4}$$

where $C > 0$ is a constant appearing in Lemma 2.1 and $\varepsilon > 0$ is an arbitrarily fixed real number. Thus, it follows from (2.2) and (2.4) that

$$\int_0^t \|u(s,\cdot)\|^2\,ds + \frac{1}{2}(1 - C\varepsilon)\|\nabla w(t,\cdot)\|^2 \le \frac{C}{2\varepsilon}\|d(\cdot)u_0\|^2.$$

Taking $\varepsilon > 0$ so small that $0 < \varepsilon < \frac{1}{C}$, one obtains the desired inequality. ∎

Secondly, let us prove Corollary 1.3. The essential part of the proof of Corollary 1.3 lies in deriving the decay of $\|\nabla u(t,\cdot)\|$.

Proof of Corollary 1.3. Since the function $t \mapsto \|\nabla u(t,\cdot)\|^2$ is monotone decreasing because of the energy identity (1.4), we have

$$\begin{aligned}\frac{d}{dt}\{(1+t)\|\nabla u(t,\cdot)\|^2\} &= \|\nabla u(t,\cdot)\|^2 + (1+t)\frac{d}{dt}\|\nabla u(t,\cdot)\|^2 \\ &\le \|\nabla u(t,\cdot)\|^2. \end{aligned} \tag{2.5}$$

Integrating the inequality (2.5) over $[0,t]$, we see

$$(1+t)\|\nabla u(t,\cdot)\|^2 \le \|\nabla u_0\|^2 + \int_0^t \|\nabla u(s,\cdot)\|^2\,ds. \tag{2.6}$$

On the other hand, it follows from the identity (1.5) that

$$\int_0^t \|\nabla u(s,\cdot)\|^2\,ds + \frac{1}{2}\|u(t,\cdot)\|^2 = \frac{1}{2}\|u_0\|^2. \tag{2.7}$$

Hence, we see from (2.6) and (2.7) that

$$(1+t)\|\nabla u(t,\cdot)\|^2 \le \|\nabla u_0\|^2 + \frac{1}{2}\|u_0\|^2 \le \|u_0\|_{H^1}^2. \tag{2.8}$$

Combining (2.8) with the estimate in Theorem 1.2, we have the desired result. ∎

Finally, let us remark the Cauchy problem in $\mathbf{R}^N$ $(N \ge 3)$ for the heat equation (1.1):

$$u_t(t,x) - \Delta u(t,x) = 0, \quad (t,x) \in (0,\infty)\times\mathbf{R}^N, \tag{2.9}$$

$$u(0,x) = u_0(x), \quad x \in \mathbf{R}^N. \tag{2.10}$$

Then we can obtain the similar result to the problem (1.1)-(1.3) by using the Hardy inequality replaced by the following one.

Lemma 2.4. (Benci and Cerami [1]) *Let $N \ge 3$. For each $u \in H^1(\mathbf{R}^N)$, it holds that*

$$\int_{\mathbf{R}^N} \frac{|u(x)|^2}{(1+|x|)^2}\,dx \le C\|\nabla u\|^2$$

with a certain constant $C > 0$.

Our result concerning the Cauchy problem reads as follows.

Theorem 2.5. *Let $N \ge 3$ and assume that the initial data $u_0 \in H^1(\mathbf{R}^N)$ further satisfies $\|(1+|x|)u_0\| < +\infty$. Then, the solution $u(t,x)$ to the problem (2.9)-(2.10) in the sense of Proposition 1.1 replaced by $\Omega = \mathbf{R}^N$ satisfies*

$$(1+t)\|u(t,\cdot)\|^2 \le C^*\|(1+|x|)u_0\|^2$$

for all $t > 0$, where $C^* > 0$ is a certain constant.

3. Proof of Theorems 1.5 and 1.7.

In this section, we shall prove Theorems 1.5 and 1.7. This is done by modifying the arguments in the previous section. To begin with, let us prove Theorem 1.5.

Proof of Theorem 1.5. Set

$$w(t,x) = \int_0^t u(s,x)\,ds.$$

Then, $w \in C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))$ satisfies

$$w_{tt}(t,x) - \Delta w(t,x) = u_1, \quad (t,x) \in (0,\infty)\times\Omega, \tag{3.1}$$

$$w(0,x)=0, \quad w_t(0,x)=u_0(x), \quad x\in\Omega, \tag{3.2}$$

$$w|_{\partial\Omega}=0, \quad t\in(0,\infty) \tag{3.3}$$

and

$$\frac{1}{2}\|w_t(t,\cdot)\|^2+\frac{1}{2}\|\nabla w(t,\cdot)\|^2=\frac{1}{2}\|u_0\|^2+\int_0^t(u_1,w_t(s,\cdot))\,ds. \tag{3.4}$$

Since

$$\int_0^t(u_1,w_t(s,\cdot))\,ds=\int_0^t\frac{d}{dt}(u_1,w(s,\cdot))\,ds=(w(t,\cdot),u_1),$$

it follows from the identity (3.4) that

$$\frac{1}{2}\|w_t(t,\cdot)\|^2+\frac{1}{2}\|\nabla w(t,\cdot)\|^2=\frac{1}{2}\|u_0\|^2+(u_1,w(t,\cdot)). \tag{3.5}$$

On the other hand, by the similar argument to the proof of the inequality (2.4) we see that

$$|(u_1,w(t,\cdot))|\le\frac{C_1}{\varepsilon}\|d(\cdot)u_1\|^2+\frac{C_2\varepsilon}{2}\|\nabla w(t,\cdot)\|^2 \tag{3.6}$$

with some constants $C_i>0$ $(i=1,2)$, where we have just used the Hardy inequality in Lemma 2.1. Thus, (3.4)-(3.6) imply

$$\frac{1}{2}\|w_t(t,\cdot)\|^2+\frac{1}{2}(1-C_2\varepsilon)\|\nabla w(t,\cdot)\|^2\le\frac{1}{2}\|u_0\|^2+\frac{C_1}{\varepsilon}\|d(\cdot)u_1\|^2.$$

Taking $\varepsilon>0$ so small above, one has

$$\|u(t,\cdot)\|^2=\|w_t(t,\cdot)\|^2\le\|u_0\|^2+2C_1\varepsilon^{-1}\|d(\cdot)u_1\|^2,$$

which completes the proof of Theorem 1.5. ∎

Finally, we shall prove Theorem 1.7. Indeed, setting

$$w(t,x)=\int_0^t u(s,x)\,ds,$$

we see that $w\in C^1([0,\infty);H_0^1(\Omega))\cap C^2([0,\infty);L^2(\Omega))$ satisfies

$$w_{tt}(t,x)-\Delta w(t,x)+w_t(t,x)=u_1+u_0, \quad (t,x)\in(0,\infty)\times\Omega,$$

$$w(0,x)=0, \quad w_t(0,x)=u_0(x), \quad x\in\Omega,$$

$$w|_{\partial\Omega}=0, \quad t\in(0,\infty)$$

and

$$\begin{aligned}&\frac{1}{2}\|w_t(t,\cdot)\|^2+\frac{1}{2}\|\nabla w(t,\cdot)\|^2+\int_0^t\|w_t(s,\cdot)\|^2\,ds\\ =\;&\frac{1}{2}\|u_0\|^2+\int_0^t(u_1+u_0,w_t(s,\cdot))\,ds.\end{aligned} \tag{3.7}$$

Since

$$\int_0^t (u_1+u_0, w_t(s,\cdot))\,ds = \int_0^t \frac{d}{ds}(u_1+u_0, w(s,\cdot))\,ds = (w(t,\cdot), u_1+u_0),$$

in a completely analogous way to the proof of Theorem 1.5 we see from the identity (3.7) that

$$\begin{aligned} &\frac{1}{2}\|w_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla w(t,\cdot)\|^2 + \int_0^t \|w_t(s,\cdot)\|^2\,ds \\ \le\ &\frac{1}{2}\|u_0\|^2 + C_3\|d(\cdot)(u_0+u_1)\|^2 + \frac{C_4\varepsilon}{2}\|\nabla w(t,\cdot)\|^2, \end{aligned}$$

which implies

$$\frac{1}{2}\|u(t,\cdot)\|^2 + \frac{1}{2}(1-C_4\varepsilon)\|\nabla w(t,\cdot)\|^2 + \int_0^t \|u(s,\cdot)\|^2\,ds \le \frac{1}{2}\|u_0\|^2 + C_3\|d(\cdot)(u_0+u_1)\|^2$$

for $\varepsilon > 0$ with some constants $C_i > 0$ $(i = 3, 4)$. Taking $\varepsilon > 0$ so small, we arrive at the following estimate.

Lemma 3.1. *Under the same assumptions as in Theorem* 1.7*, it holds that*

$$\frac{1}{2}\|u(t,\cdot)\|^2 + \int_0^t \|u(s,\cdot)\|^2\,ds \le \frac{1}{2}\|u_0\|^2 + C_3\|d(\cdot)(u_0+u_1)\|^2$$

for all $t > 0$.

On the other hand, we have the following estimate which has been proved in [4, Lemma 3.1]. For the convenience of the readers we shall prove it briefly.

Lemma 3.2. *Under the assumptions as in Theorem* 1.7*, one has*

$$(1+t)\|u(t,\cdot)\|^2 \le C_5\{\|u_0\|_{H^1}^2 + \|u_1\|^2\} + \int_0^t \|u(s,\cdot)\|^2\,ds \tag{3.8}$$

for all $t > 0$*, where* $C_5 > 0$ *is a constant.*

Proof. Multiplying the both sides of (1.14) by t and integrating it over $[0, t]$, one has

$$\begin{aligned} &\int_0^t s\frac{d}{ds}(u_s(s,\cdot), u(s,\cdot))\,ds - \int_0^t s\|u_s(s,\cdot)\|^2\,ds \\ &+ \int_0^t s\|\nabla u(s,\cdot)\|^2\,ds + \frac{1}{2}\int_0^t s\frac{d}{ds}\|u(s,\cdot)\|^2\,ds = 0. \end{aligned}$$

Integrating by parts, we see that

$$\int_0^t s\frac{d}{ds}(u_s(s,\cdot), u(s,\cdot))\,ds = t(u_t(t,\cdot), u(t,\cdot)) - \frac{1}{2}\|u(t,\cdot)\|^2 + \frac{1}{2}\|u_0\|^2$$

and

$$\int_0^t s\frac{d}{ds}\|u(s,\cdot)\|^2\,ds = t\|u(t,\cdot)\|^2 - \int_0^t \|u(s,\cdot)\|^2\,ds.$$

Thus, we get

$$
\begin{aligned}
& t(u_t(t,\cdot),u(t,\cdot)) + \frac{1}{2}\|u_0\|^2 + \int_0^t s\|\nabla u(s,\cdot)\|^2\,ds + \frac{t}{2}\|u(t,\cdot)\|^2 \\
= \; & \frac{1}{2}\|u(t,\cdot)\|^2 + \int_0^t s\|u_s(s,\cdot)\|^2\,ds + \frac{1}{2}\int_0^t \|u(s,\cdot)\|^2\,ds. \qquad (3.9)
\end{aligned}
$$

On the other hand, it is easy to prove (cf. [4] or [5])

$$
\|u(t,\cdot)\|^2 \le I_0, \quad (1+t)E(t) \le E(0) + \beta, \quad \int_0^t E(s)\,ds \le \beta, \tag{3.10}
$$

where we set $\beta = \frac{1}{2}E(0) + \frac{1}{8}I_0$ with $I_0 = 2\|u_0\|^2 + (u_0,u_1) + 8E(0)$ and

$$
E(t) = \frac{1}{2}\{\|\nabla u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2\}.
$$

Therefore, integrating by parts with respect to t, one gets

$$
\beta \ge \int_0^t E(s)\,ds = tE(t) - \int_0^t sE'(s)\,ds = tE(t) + \int_0^t s\|u_s(s,\cdot)\|^2\,ds,
$$

from which it follows that

$$
\int_0^t s\|u_s(s,\cdot)\|^2\,ds \le \beta, \tag{3.11}
$$

where we have just used the relation $E'(t) = -\|u_t(t,\cdot)\|^2$ (see (1.13)). Note that

$$
-(u_t(t,\cdot),u(t,\cdot)) \le \|u_t(t,\cdot)\|\|u(t,\cdot)\| \le \frac{1}{4}\|u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2. \tag{3.12}
$$

From the estimates (3.9), (3.11) and (3.12) it follows that

$$
\begin{aligned}
\frac{1}{2}\|u(t,\cdot)\|^2 + \int_0^t s\|u_s(s,\cdot)\|^2\,ds + \frac{1}{2}\int_0^t \|u(s,\cdot)\|^2\,ds \quad & \ge \quad t(u_t(t,\cdot),u(t,\cdot)) + \frac{t}{2}\|u(t,\cdot)\|^2 \\
& \ge \quad \frac{t}{4}\|u(t,\cdot)\|^2 - t\|u_t(t,\cdot)\|^2,
\end{aligned}
$$

which implies that

$$
t\|u_t(t,\cdot)\|^2 + \frac{1}{2}\int_0^t \|u(s,\cdot)\|^2\,ds + \int_0^t s\|u_s(s,\cdot)\|^2\,ds + \frac{1}{2}\|u(t,\cdot)\|^2 \ge \frac{t}{4}\|u(t,\cdot)\|^2.
$$

Noting that $(1+t)E(t) \le \beta + E(0)$ implies $t\|u_t(t,\cdot)\|^2 \le 2(\beta + E(0))$ and combining it with the estimates (3.10), we arrive at the desired inequality (3.8). ∎

Therefore, Theorem 1.7 is an immediate consequence of Lemmas 3.1 and 3.2.

Remark 3.3. *Using Lemma* 2.4 *instead of Lemma* 2.1*, we can obtain the similar result associated with Theorem* 2.5 *for the Cauchy problem in* $\mathbf{R}^N$ *to the equation* (1.7) *or* (1.10)*. So we shall omit the details.*

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