CHARACTERIZATIONS OF PARALLEL IMBEDDINGS OF PROJECTIVE SPACES INTO SPACE FORMS BY CIRCLES

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Abstract. It is known that the parallel imbeddings of a complex or a quaternionic projective space into real space forms are the examples of planar geodesic submanifolds. Namely each geodesic on these projective spaces is mapped to a plane curve in the ambient real space form through the parallel imbeddings. Moreover, we know that some particular circles of positive curvature on these submanifolds are also mapped to plane curves. In this paper we consider the converse of this geometric property of such planar geodesic immersions.

1 Introduction.

Let M and $\tilde{M}$ be Riemannian manifolds and $f : M \to \tilde{M}$ be an isometric immersion. We recall the definition of planar geodesic immersions. If for each geodesic γ on the submanifold M the curve $f \circ \gamma$ is a plane curve in the ambient manifold $\tilde{M}$, that is $f \circ \gamma$ is locally contained in a 2-dimensional totally geodesic submanifold of $\tilde{M}$, the isometric immersion f is called a *planar geodesic* immersion.

By virtue of the result [6] planar geodesic submanifolds of a real space form $\tilde{M}^m(\tilde{c})(= E^m,\ S^m(\tilde{c})$ or $H^m(\tilde{c}))$ of curvature $\tilde{c}$ are completely classified. If M is a planar geodesic submanifold of $\tilde{M}^m(\tilde{c})$, then M is totally umbilic in $\tilde{M}^m(\tilde{c})$ or a compact symmetric space of rank one through parallel immersions (for details, see Theorem A). We here pay attention to these parallel imbeddings of three projective spaces, namely they are a real projective space RP^n, a complex projective space CP^n and a quaternionic projective space QP^n. We shall investigate the extrinsic shape $f \circ \gamma$ for a circle γ on $KP^n (K = R, C, Q)$ in the ambient real space form $\tilde{M}^m(\tilde{c})$ under the parallel imbedding f.

In this paper we first show that for each circle γ of positive curvature on RP^n the curve $f \circ \gamma$ is a helix of proper order 3 or 4 in $\tilde{M}^m(\tilde{c})$, so that it is never a plane curve in the ambient space (see Proposition 1). On the contrary, we know that some circles of CP^n or QP^n are mapped to plane curves in the ambient space $\tilde{M}^m(\tilde{c})$ under parallel imbeddings (see Propositions 2 and 3).

Our purpose of this paper is to give some characterizations of parallel imbeddings of a complex projective space and a quaternionic projective space by observing the extrinsic shape of particular circles (Theorems 1 and 2). Our main results are improvements of Theorems B and C.

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2 Preliminaries.

We first review the definition of circles. A curve $\gamma = \gamma(s)$, parametrized by its arclength s, in a Riemannian manifold M is called a *circle* if there exist a field

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$Y = Y(s)$ of unit vectors along γ and a nonnegative constant k which satisfy

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = kY \\ \nabla_{\dot\gamma}Y = -k\dot\gamma, \end{cases} \tag{2.1}$$

where $\dot\gamma$ denotes the unit tangent vector of γ and $\nabla_{\dot\gamma}$ the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. The constant k is called the *curvature* of the circle. A circle of curvature zero is nothing but a geodesic. For an arbitrary point x, an arbitrary orthonormal pair (u, v) of vectors at x and a positive k, there exists locally a unique circle $\gamma = \gamma(s)$ with initial condition that $\gamma(0) = x$, $\dot\gamma(0) = u$ and $Y(0) = v$. For detail, see [4] .

For later use, we prepare the following lemma ([4]).

Lemma 1. *A circle $\gamma = \gamma(s)$ satisfies the following differential equation*

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle\dot\gamma = 0, \tag{2.2}$$

where $\langle\ ,\ \rangle$ denotes the Riemannian metric of M. Conversely, if a curve $\gamma = \gamma(s)$ satisfies (2.2)*, then it is a circle.*

We next review the Frenet formula for a smooth Frenet curve in a Riemannian manifold M. A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s is called a *Frenet curve of proper order* d if there exist orthonormal frame fields $\{V_1 = \dot\gamma, \cdots, V_d\}$ along γ and positive functions $\kappa_1(s), \cdots, \kappa_{d-1}(s)$ satisfying the following system of ordinary equations

$$\nabla_{\dot\gamma}V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \cdots, d, \tag{2.3}$$

where $V_0 \equiv V_{d+1} \equiv 0$. We call Equation (2.3) the Frenet formula for the Frenet curve γ. The functions $\kappa_j(s)$ $(j = 1, \cdots, d-1)$ and the orthonormal frame $\{V_1, \cdots, V_d\}$ are called the curvatures and the Frenet frame of γ, respectively.

A Frenet curve is called a *Frenet curve of order* d if it is a Frenet curve of proper order $r(\le d)$. For a Frenet curve of order d which is of proper order $r(\le d)$, we use the convention in (2.3) that $\kappa_j \equiv 0$ $(r \le j \le d-1)$, and $V_j \equiv 0$ $(r+1 \le j \le d)$. We call a smooth Frenet curve a *helix* when all its curvatures are constant. A helix of order 1 is nothing but a geodesic and a helix of order 2 is a circle. The following is an improvement of Lemma 1.

Lemma 2 ([3]). *A Frenet curve $\gamma = \gamma(s)$ of order* 2 *satisfies the following differential equation*

$$\kappa(s)(\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle\dot\gamma) = \dot\kappa(s)\nabla_{\dot\gamma}\dot\gamma, \tag{2.4}$$

where $\kappa(s) = \|\nabla_{\dot\gamma}\dot\gamma\|$. Conversely, if a Frenet curve $\gamma = \gamma(s)$ satisfies (2.4)*, then it is of order* 2.

In this paper a curve means a smooth Frenet curve. We next recall the notion of isotropic immersions ([5]). Let M and $\tilde M$ be Riemannian manifolds and $f : M \to \tilde M$ be an isometric immersion. We denote by σ the second fundamental form of f. Then the immersion f is said to be λ-*isotropic* at $x \in M$ if $\|\sigma(X, X)\|/\|X\|^2 (= \lambda)$ is constant for each nonzero $X \in T_x(M)$ of M at x. If the isometric immersion is isotropic at every point, then the immersion is isotropic. Note that a totally umbilic immersion is isotropic, but not *vice versa*.

Finally we review fundamental equations in submanifold theory. Let M be an n-dimensional Riemannian submanifold of $\tilde{M}^{n+p}$ with metric $\langle\ ,\ \rangle$. We denote by ∇ and $\tilde{\nabla}$ the covariant differentiations of M and $\tilde{M}$, respectively. Then the second fundamental form σ of the immersion is defined by

$$\sigma(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y, \tag{2.5}$$

where X and Y are vector fields tangent to M. For a vector field ξ normal to M, we write

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^{\perp}\xi, \tag{2.6}$$

where $-A_\xi X$ (resp. $\nabla_X^{\perp}\xi$) denotes the tangential (resp. the normal) component of $\tilde{\nabla}_X\xi$. We define the covariant differentiation $\bar{\nabla}$ of the second fundamental form σ with respect to the connection in (tangent bundle) + (normal bundle) as follows:

$$(\bar{\nabla}_X\sigma)(Y,Z) = \nabla_X^{\perp}(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.7}$$

The second fundamental form σ is said to be *parallel* if $(\bar{\nabla}_X\sigma)(Y,Z) = 0$ for all tangent vector fields X, Y and Z on M.

3 Planar geodesic immersions. Let $\tilde{M}^m(\tilde{c})$ be an m-dimensional complete simply connected real space form of curvature $\tilde{c}$. It is well-known that $\tilde{M}^m(\tilde{c})$ is isometric to E^m, $S^m(\tilde{c})$ or $H^m(\tilde{c})$, according as $\tilde{c}$ is zero, positive or negative.

The following theorem classifies all planar geodesic submanifolds in a real space form.

Theorem A ([6]). *Let M^n be a Riemannian submanifold of a real space form $\tilde{M}^{n+p}(\tilde{c})$ through an isometric immersion f. Suppose that f is a planar geodesic immersion. Then M^n is totally umbilic in $\tilde{M}^{n+p}(\tilde{c})$ or M^n is locally congruent to a compact symmetric space of rank one which is immersed into some totally umbilic submanifold of $\tilde{M}^{n+p}(\tilde{c})$ through the parallel minimal immersion. This parallel minimal immersion is locally equivalent either to the first standard minimal imbedding of one of the compact symmetric spaces of rank one or to the second standard minimal immersion of a sphere.*

4 Extrinsic shape of circles on RP^n. Let γ be a circle on a real projective space RP^n. We shall study the extrinsic shape $f\circ\gamma$ through the parallel imbedding f in the ambient real space form. Note that every circle on RP^n is locally contained in some totally geodesic RP^2 of RP^n. So it is enough to study the case of $n = 2$. It follows from Theorem A that for each geodesic γ on RP^n the curve $f\circ\gamma$ is a plane curve which is nothing but a circle of the same positive curvature in the ambient space (see [6]). The following clarifies the extrinsic shape of circles of positive curvature on RP^n in $\tilde{M}(\tilde{c})$.

Proposition 1. *Let $f = f_2\circ f_1 : RP^2(\frac{c}{3}) \xrightarrow{f_1} S^4(c) \xrightarrow{f_2} \tilde{M}^{2+p}(\tilde{c})$ be an isometric parallel imbedding of $RP^2(\frac{c}{3})$ into a real space form $\tilde{M}^{2+p}(\tilde{c})$ $(c \geq \tilde{c})$. Here f_1 is the first standard minimal imbedding of $RP^2(\frac{c}{3})$ into $S^4(c)$ and f_2 is a totally umbilic imbedding of $S^4(c)$ into $\tilde{M}^{2+p}(\tilde{c})$. Then*

(I) *When $c = \tilde{c}$,*

- **(i)** *f maps each circle of curvature $\frac{\sqrt{c}}{\sqrt{6}}$ to a helix of proper order 3 of curvatures $\frac{\sqrt{c}}{\sqrt{2}}, \sqrt{c}$.*

(ii) *f maps each circle of positive curvature $k \neq \frac{\sqrt{c}}{\sqrt{6}}$ to a helix of proper order 4 of curvatures* $\frac{\sqrt{3k^2+c}}{\sqrt{3}}, \frac{3k\sqrt{c}}{\sqrt{3k^2+c}}, \frac{|6k^2-c|}{\sqrt{3(3k^2+c)}}$.

(II) *When $c > \tilde{c}$,*
f maps each circle of positive curvature k to a helix of proper order 4 of curvatures $\frac{\sqrt{3k^2+4c-3\tilde{c}}}{\sqrt{3}}, \frac{3k\sqrt{c}}{\sqrt{3k^2+4c-3\tilde{c}}}, \frac{\sqrt{4(3k^2+c)^2-3\tilde{c}(12k^2+c)}}{\sqrt{3(3k^2+4c-3\tilde{c})}}$.

Proof. Let $f : M^n(c_1) \to \tilde{M}^{n+p}(c_2)$ be a λ-isotropic immersion. Then the equation of Gauss for the second fundamental form σ of f is given by

$$\begin{aligned}\langle \sigma(X,Y), \sigma(Z,W)\rangle - \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ = (c_1 - c_2)(\langle X,Y\rangle\langle Z,W\rangle - \langle X,W\rangle\langle Y,Z\rangle)\end{aligned} \tag{4.1}$$

for any vector fields X, Y, Z, W on the submanifold $M^n(c_1)$. On the other hand, exchanging X and Y in (4.1), we get

$$\begin{aligned}\langle \sigma(X,Y), \sigma(Z,W)\rangle - \langle \sigma(X,Z), \sigma(Y,W)\rangle \\ = (c_1 - c_2)(\langle X,Y\rangle\langle Z,W\rangle - \langle X,Z\rangle\langle Y,W\rangle).\end{aligned} \tag{4.2}$$

Since f is λ-isotropic, for all vector fields X on M, we have $\langle \sigma(X,X), \sigma(X,X)\rangle = \lambda^2\langle X,X\rangle\langle X,X\rangle$, which is equivalent to

$$\begin{aligned}&\langle \sigma(X,Y), \sigma(Z,W)\rangle + \langle \sigma(X,Z), \sigma(Y,W)\rangle + \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ &= \lambda^2(\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle Y,W\rangle + \langle X,W\rangle\langle Y,Z\rangle).\end{aligned} \tag{4.3}$$

Summing up (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned}&\langle \sigma(X,Y), \sigma(Z,W)\rangle \\ &\quad = \frac{c_1 - c_2}{3}\,(\,2\langle X,Y\rangle\langle Z,W\rangle - \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle) \\ &\quad\quad + \frac{\lambda^2}{3}(\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle Y,W\rangle + \langle X,W\rangle\langle Y,Z\rangle).\end{aligned}$$

Since our isometric imbedding f given by the assumption of Proposition 1 is a $\frac{\sqrt{4c-3\tilde{c}}}{\sqrt{3}}$-isotropic (parallel) imbedding, it satisfies that

$$\begin{aligned}\langle \sigma(X,Y), \sigma(Z,W)\rangle \quad = \quad &\frac{2c-3\tilde{c}}{3}\langle X,Y\rangle\langle Z,W\rangle \\ &+ \frac{c}{3}(\langle X,W\rangle\langle Y,Z\rangle + \langle X,Z\rangle\langle Y,W\rangle).\end{aligned} \tag{4.4}$$

Let $\gamma = \gamma(s)$ be a circle of curvature k (> 0) in $RP^2(\frac{c}{3})$. We denote by $\tilde{\nabla}$ the covariant differentiation of $\tilde{M}^{2+p}(\tilde{c})$. Then it follows from (2.1), (2.5), (2.6) and (2.7) that

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \kappa_1 V_2, \tag{4.5}$$

where

$$\kappa_1 = \frac{\sqrt{3k^2 + 4c - 3\tilde{c}}}{\sqrt{3}} \tag{4.6}$$

and

$$V_2 = \frac{\sqrt{3}}{\sqrt{3k^2+4c-3\tilde{c}}}(kY + \sigma(\dot{\gamma},\dot{\gamma})). \tag{4.7}$$

Differentiating (4.7), we obtain

$$\tilde{\nabla}_{\dot{\gamma}} V_2 = -\kappa_1 \dot{\gamma} + \frac{3\sqrt{3}k}{\sqrt{3k^2+4c-3\tilde{c}}}\sigma(\dot{\gamma}, Y).$$

Therefore, from (4.4) if we put

$$\kappa_2 = \frac{3k\sqrt{c}}{\sqrt{3k^2+4c-3\tilde{c}}} \tag{4.8}$$

and

$$V_3 = \frac{\sqrt{3}}{\sqrt{c}}\sigma(\dot{\gamma}, Y), \tag{4.9}$$

then we have

$$\tilde{\nabla}_{\dot{\gamma}} V_2 = -\kappa_1 \dot{\gamma} + \kappa_2 V_3. \tag{4.10}$$

Similarly, differentiating (4.9), we obtain

$$\tilde{\nabla}_{\dot{\gamma}} V_3 = -\kappa_2 V_2 + \kappa_3 V_4, \tag{4.11}$$

where

$$\kappa_3 = \frac{\sqrt{4(3k^2+c)^2 - 3\tilde{c}(12k^2+c)}}{\sqrt{3(3k^2+4c-3\tilde{c})}} \tag{4.12}$$

and

$$\begin{aligned} V_4 \quad = \quad & \frac{(6k^2-4c+3\tilde{c})cY - 3k(3k^2+c-3\tilde{c})\sigma(\dot{\gamma},\dot{\gamma})}{\sqrt{c(3k^2+4c-3\tilde{c})\{4(3k^2+c)^2-3\tilde{c}(12k^2+c)\}}} \\ & + \frac{3k(3k^2+4c-3\tilde{c})\sigma(Y,Y)}{\sqrt{c(3k^2+4c-3\tilde{c})\{4(3k^2+c)^2-3\tilde{c}(12k^2+c)\}}}. \end{aligned} \tag{4.13}$$

Finally, differentiating (4.13), we find

$$\tilde{\nabla}_{\dot{\gamma}} V_4 = -\kappa_3 V_3. \tag{4.14}$$

From $(4.5), (4.6), (4.8), (4.10), (4.11), (4.12)$ and (4.14) we get (i),(ii) of (I) and (II).

$\square$

5 Extrinsic shape of Kähler circles and quaternionic circles. We here review the definition of some particular circles in a Kähler manifold and a quaternionic Kähler manifold.

Let γ be a circle in a Kähler manifold M. Then we see from (2.1) that $\langle J\dot{\gamma}, Y\rangle$ is constant along γ, where J is the complex structure of M. Therefore it makes sense to define a *Kähler circle* as a circle γ satisfying the condition that $\dot{\gamma}$ and Y span a holomorphic plane, that is, $Y = J\dot{\gamma}$ or $Y = -J\dot{\gamma}$. Note that if γ is a Kähler circle, then (2.1) reduces to

$$\nabla_{\dot{\gamma}}\dot{\gamma} = kJ\dot{\gamma} \quad \text{or} \quad \nabla_{\dot{\gamma}}\dot{\gamma} = -kJ\dot{\gamma}.$$

The extrinsic shape of Kähler circles through the parallel imbedding of a complex projective space is known as follows ([2]):

Proposition 2. *Let $f = f_2 \circ f_1 : CP^n(\frac{2n}{n+1}c) \xrightarrow{f_1} S^{n(n+2)-1}(c) \xrightarrow{f_2} \tilde{M}^{2n+p}(\tilde{c})$ be an isometric parallel imbedding of $CP^n(\frac{2n}{n+1}c)$ into a real space form $\tilde{M}^{2n+p}(\tilde{c})$ $(c \geq \tilde{c})$. Here f_1 is the first standard minimal imbedding of $CP^n(\frac{2n}{n+1}c)$ into $S^{n(n+2)-1}(c)$* and *f_2 is a totally umbilic imbedding of $S^{n(n+2)-1}(c)$ into $\tilde{M}^{2n+p}(\tilde{c})$. Then f maps every Kähler circle γ of $CP^n(\frac{2n}{n+1}c)$ to a circle in $\tilde{M}^{2n+p}(\tilde{c})$, so that in particular the curve $f \circ \gamma$ is a plane curve in the ambient space $\tilde{M}^{2n+p}(\tilde{c})$.*

The following is a characterization of the parallel imbedding f in Proposition 2.

Theorem B([2]). *Let M be a non-flat Kähler manifold of real dimension $2n$ (≥ 4) which is immersed into a real space form $\tilde{M}^{2n+p}(\tilde{c})$. If there exists $k > 0$ and all Kähler circles of curvature k on M are mapped to circles in $\tilde{M}^{2n+p}(\tilde{c})$, then M is locally congruent to a complex projective space imbedded into some sphere in $\tilde{M}^{2n+p}(\tilde{c})$ through the first standard minimal imbedding.*

Let M be a quaternionic Kähler manifold with local basis $\{I, J, K\}$ of quaternionic structure and let γ be a circle in M. Then I, J and K satisfy

$$\begin{cases} \nabla_{\dot{\gamma}} I = qJ - rK \\ \nabla_{\dot{\gamma}} J = -qI + pK \\ \nabla_{\dot{\gamma}} K = rI - pJ, \end{cases} \tag{5.1}$$

for some functions p, q, r along γ. We see from (2.1) and (5.1) that $\langle Y, I\dot{\gamma}\rangle^2 + \langle Y, J\dot{\gamma}\rangle^2 + \langle Y, K\dot{\gamma}\rangle^2$ is constant along γ ([1]). Therefore it makes sense to consider a circle γ satisfying the condition that Y is a linear combination of $I\dot{\gamma}, J\dot{\gamma}$ and $K\dot{\gamma}$ at each point of γ. In fact, Y is a linear combination of $I\dot{\gamma}, J\dot{\gamma}$ and $K\dot{\gamma}$ if and only if $\langle Y, I\dot{\gamma}\rangle^2 + \langle Y, J\dot{\gamma}\rangle^2 + \langle Y, K\dot{\gamma}\rangle^2 = 1$. A *quaternionic circle* is, by definition, a circle with such a property.

The extrinsic shape of quaternionic circles through the parallel imbedding of a quaternionic projective space is known as follows ([2]):

Proposition 3. *Let $g = g_2 \circ g_1 : QP^n(\frac{2n}{n+1}c) \xrightarrow{g_1} S^{n(2n+3)-1}(c) \xrightarrow{g_2} \tilde{M}^{4n+p}(\tilde{c})$ be an isometric parallel imbedding of $QP^n(\frac{2n}{n+1}c)$ into a real space form $\tilde{M}^{4n+p}(\tilde{c})$ $(c \geq \tilde{c})$. Here g_1 is the first standard minimal imbedding of $QP^n(\frac{2n}{n+1}c)$ into $S^{n(2n+3)-1}(c)$ and g_2 is a totally umbilic imbedding of $S^{n(2n+3)-1}(c)$ into $\tilde{M}^{4n+p}(\tilde{c})$. Then g maps every quaternionic circle γ of $QP^n(\frac{2n}{n+1}c)$ to a circle in $\tilde{M}^{4n+p}(\tilde{c})$, so that in particular the curve $g \circ \gamma$ is a plane curve in the ambient space $\tilde{M}^{4n+p}(\tilde{c})$.*

The following is a characterization of the parallel imbedding g in Proposition 3.

Theorem C([2]). *Let M be a non-flat quaternionic Kähler manifold of real dimension $4n$ (≥ 8) which is immersed into a real space form $\tilde{M}^{4n+p}(\tilde{c})$. If there exists $k > 0$ and all quaternionic circles of curvature k on M are mapped to circles in $\tilde{M}^{4n+p}(\tilde{c})$, then M is locally congruent to a quaternionic projective space imbedded into some sphere in $\tilde{M}^{4n+p}(\tilde{c})$ through the first standard minimal imbedding.*

6 Characterization of parallel imbeddings of CP^n. We consider the converse of Proposition 2 to obtain a characterization of the parallel imbedding of a complex projective space. First we prove the following.

Theorem 1. *Let M be a non-flat Kähler manifold of real dimension $2n$ (≥ 4) which is immersed into a real space form $\tilde{M}^{2n+p}(\tilde{c})$. If there exists $k > 0$ and all Kähler circles of curvature k on M are mapped to plane curves in $\tilde{M}^{2n+p}(\tilde{c})$, then M is locally congruent to a complex projective space imbedded into some sphere in $\tilde{M}^{2n+p}(\tilde{c})$ through the first standard minimal imbedding.*

Proof. We denote by $f : M \to \tilde{M}^{2n+p}(\tilde{c})$ the isometric immersion which satisfies our assumption. Let x be any point of M. In the following, take and fix a unit vector $X \in T_xM$.

Let $\gamma = \gamma(s)$ be a Kähler circle of curvature k on M satisfying the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = kJ\dot{\gamma},\ |s| < \epsilon$ for some $\epsilon > 0$ with initial condition that $\gamma(0) = x,\ \dot{\gamma}(0) = X$ and $(\nabla_{\dot{\gamma}}\dot{\gamma})(0) = kJX$. Needless to say, the curve γ satisfies (2.2). By assumption the curve $f \circ \gamma$ is a plane curve in $\tilde{M}^{2n+p}(\tilde{c})$, hence from (2.4) it satisfies the differential equation

$$\kappa(s)(\tilde{\nabla}_{\dot{\gamma}}(\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}) + \langle\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}, \tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\rangle\dot{\gamma}) = \dot{\kappa}(s)\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}, \tag{6.1}$$

where $\kappa(s) = \|\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\|$ and $\tilde{\nabla}$ is the covariant differentiation of $\tilde{M}^{2n+p}(\tilde{c})$. We here note that $\kappa(s) > 0$ for any s, that is, the curve $f \circ \gamma$ is of proper order 2. Indeed, suppose that the Frenet curve $f \circ \gamma$ satisfies $\kappa \equiv 0$. This implies that the curve $f \circ \gamma$ is a geodesic in the ambient space $\tilde{M}^{2n+p}(\tilde{c})$, so that the curve $\gamma = \gamma(s)$ is a geodesic in M, which is a contradiction. It follows from (2.5), (2.6) and (2.7) that

$$\tilde{\nabla}_{\dot{\gamma}}(\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}) = \nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}\dot{\gamma}) + 3k\sigma(J\dot{\gamma}, \dot{\gamma}) - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma}). \tag{6.2}$$

We find from (2.2), (2.5), (6.1) and (6.2) that

$$\begin{aligned} &\kappa(s)(3k\sigma(J\dot{\gamma}, \dot{\gamma}) - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma}) + \|\sigma(\dot{\gamma}, \dot{\gamma})\|^2\dot{\gamma}) \\ &= \dot{\kappa}(s)(kJ\dot{\gamma} + \sigma(\dot{\gamma}, \dot{\gamma})). \end{aligned} \tag{6.3}$$

Considering the tangential component and the normal component for the submanifold M in Equation (6.3), we obtain the following:

$$\kappa(s)(-A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + \|\sigma(\dot{\gamma}, \dot{\gamma})\|^2\dot{\gamma}) = \dot{\kappa}(s)kJ\dot{\gamma}. \tag{6.4}$$

$$\kappa(s)(3k\sigma(J\dot{\gamma}, \dot{\gamma}) + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma})) = \dot{\kappa}(s)\sigma(\dot{\gamma}, \dot{\gamma}). \tag{6.5}$$

Note that

$$\kappa(s) = \|\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\| = \sqrt{k^2 + \|\sigma(\dot{\gamma}, \dot{\gamma})\|^2} > 0.$$

Hence

$$\begin{aligned}\kappa(s)\dot\kappa(s)=\frac{1}{2}\frac{d}{ds}\kappa(s)^2 &= \frac{1}{2}\frac{d}{ds}\langle\sigma(\dot\gamma,\dot\gamma),\sigma(\dot\gamma,\dot\gamma)\rangle\\ &= \langle\nabla^{\perp}_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma)),\sigma(\dot\gamma,\dot\gamma)\rangle\\ &= \langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)+2k\sigma(J\dot\gamma,\dot\gamma),\sigma(\dot\gamma,\dot\gamma)\rangle.\end{aligned}$$

Thus, at $s=0$ we get the following:

$$\kappa(0)=\sqrt{k^2+\|\sigma(X,X)\|^2}. \tag{6.6}$$

$$\kappa(0)\dot\kappa(0)=\langle(\bar\nabla_X\sigma)(X,X)+2k\sigma(JX,X),\sigma(X,X)\rangle. \tag{6.7}$$

Evaluating (6.5) at $s=0$, we find

$$\kappa(0)(3k\sigma(JX,X)+(\bar\nabla_X\sigma)(X,X))=\dot\kappa(0)\sigma(X,X). \tag{6.8}$$

It follows from (6.7) and (6.8) that

$$\begin{aligned}&3k\kappa(0)^2\sigma(JX,X)-2k\langle\sigma(JX,X),\sigma(X,X)\rangle\sigma(X,X)\\ &\quad=\langle(\bar\nabla_X\sigma)(X,X),\sigma(X,X)\rangle\sigma(X,X)-\kappa(0)^2(\bar\nabla_X\sigma)(X,X).\end{aligned} \tag{6.9}$$

We here apply the above discussion to another Kähler circle $\nabla_{\dot\gamma}\dot\gamma=-kJ\dot\gamma$ with initial condition that $\gamma(0)=x$, $\dot\gamma(0)=X$ and $(\nabla_{\dot\gamma}\dot\gamma)(0)=-kJX$, we find the following corresponding to (6.9):

$$\begin{aligned}&-3k\kappa(0)^2\sigma(JX,X)+2k\langle\sigma(JX,X),\sigma(X,X)\rangle\sigma(X,X)\\ &\quad=\langle(\bar\nabla_X\sigma)(X,X),\sigma(X,X)\rangle\sigma(X,X)-\kappa(0)^2(\bar\nabla_X\sigma)(X,X).\end{aligned} \tag{6.9$'$}$$

Hence, from (6.9) and (6.9)$'$ we can see that

$$3\kappa(0)^2\sigma(JX,X)-2\langle\sigma(JX,X),\sigma(X,X)\rangle\sigma(X,X)=0.$$

This, together with (6.6), yields

$$(3k^2+\|\sigma(X,X)\|^2)\langle\sigma(JX,X),\sigma(X,X)\rangle=0.$$

As $k>0$, we get

$$\langle\sigma(JX,X),\sigma(X,X)\rangle=0. \tag{6.10}$$

We here evaluate (6.4) at $s=0$. Then we find

$$\kappa(0)(-A_{\sigma(X,X)}X+\|\sigma(X,X)\|^2X)=\dot\kappa(0)kJX. \tag{6.11}$$

Taking the inner product of the both sides of (6.11) and JX, we have

$$-\kappa(0)\langle\sigma(JX,X),\sigma(X,X)\rangle=\dot\kappa(0)k. \tag{6.12}$$

We get $\dot\kappa(0)=0$ from (6.10) and (6.12). Since x is arbitrary, this guarantees that $\dot\kappa(s)=0$ for any s, so that $\kappa(s)$ is constant along the curve $f\circ\gamma$. Namely the curve $f\circ\gamma$ is a circle in $\tilde M^{2n+p}(\tilde c)$. Therefore from Theorem B we get the conclusion. □

7 Characterization of parallel imbeddings of QP^n. For a quaternionic Kähler manifold we obtain the following result similar to Theorem 1.

Theorem 2. *Let M be a non-flat quaternionic Kähler manifold of real dimension $4n\ (\geq 8)$ which is immersed into a real space form $\tilde{M}^{4n+p}(\tilde{c})$. If there exists $k>0$ and all quaternionic circles of curvature k on M are mapped to plane curves in $\tilde{M}^{4n+p}(\tilde{c})$, then M is locally congruent to a quaternionic projective space imbedded into some sphere in $\tilde{M}^{4n+p}(\tilde{c})$ through the first standard minimal imbedding.*

Proof. We denote by $f: M \to \tilde{M}^{4n+p}(\tilde{c})$ the isometric immersion which satisfies our assumption. Let x be any point of M. In the following, take and fix a unit vector $X \in T_xM$.

Let $\gamma = \gamma(s)$ be a quaternionic circle of curvature k with initial condition that $\gamma(0) = x$, $\dot\gamma(0) = X$ and $Y(0) = Y = \lambda IX + \mu JX + \nu KX$ on M. Then the circle γ satisfies the following equations:

$$\nabla_{\dot\gamma}\dot\gamma = kY(s),\ \nabla_{\dot\gamma}Y(s) = -k\dot\gamma,\ Y(s) = \lambda I\dot\gamma + \mu J\dot\gamma + \nu K\dot\gamma,$$

where λ, μ and ν are functions along γ satisfying $\lambda^2+\mu^2+\nu^2 = 1$. Then by an argument similar to the proof of Theorem 1, at the point x we obtain

$$\begin{aligned}(7.1)\qquad & 3k\kappa(0)^2\sigma(\lambda IX + \mu JX + \nu KX, X)\\ &\quad -2k\langle\sigma(\lambda IX + \mu JX + \nu KX, X), \sigma(X,X)\rangle\sigma(X,X)\\ &= \langle(\bar\nabla_X\sigma)(X,X), \sigma(X,X)\rangle\sigma(X,X) - \kappa(0)^2(\bar\nabla_X\sigma)(X,X)\end{aligned}$$

which corresponds to Equation (6.9).

Next, we shall study another quaternionic circle γ of the same curvature k with initial condition that $\gamma(0) = x$, $\dot\gamma(0) = X$ and $Y(0) = -Y$. Then we get the following identity corresponding to (7.1):

$$\begin{aligned}(7.1)'\qquad & -3k\kappa(0)^2\sigma(\lambda IX + \mu JX + \nu KX, X)\\ &\quad +2k\langle\sigma(\lambda IX + \mu JX + \nu KX, X), \sigma(X,X)\rangle\sigma(X,X)\\ &= \langle(\bar\nabla_X\sigma)(X,X), \sigma(X,X)\rangle\sigma(X,X) - \kappa(0)^2(\bar\nabla_X\sigma)(X,X).\end{aligned}$$

It follows from (7.1) and $(7.1)'$ that

$$\begin{aligned}& 3\kappa(0)^2\sigma(\lambda IX + \mu JX + \nu KX, X)\\ &\quad -2\langle\sigma(\lambda IX + \mu JX + \nu KX, X), \sigma(X,X)\rangle\sigma(X,X) = 0.\end{aligned}$$

This, together with (6.6), implies that

$$(3k^2 + \|\sigma(X,X)\|^2)\langle\sigma(\lambda IX + \mu JX + \nu KX, X), \sigma(X,X)\rangle = 0.$$

As $k>0$, we get

$$(7.2)\qquad \langle\sigma(\lambda IX + \mu JX + \nu KX, X), \sigma(X,X)\rangle = 0.$$

On the other hand, in (6.4) setting $\lambda I\dot\gamma + \mu J\dot\gamma + \nu K\dot\gamma$ in place of $J\dot\gamma$ and $s=0$, we obtain

$$\kappa(0)(-A_{\sigma(X,X)}X + \|\sigma(X,X)\|^2X) = \dot\kappa(0)k(\lambda IX + \mu JX + \nu KX).$$

Taking the inner product of this identity and the vector $\lambda IX + \mu JX + \nu KX$, we see that

$$-\kappa(0)\langle\sigma(\lambda IX + \mu JX + \nu KX, X), \sigma(X,X)\rangle = \dot\kappa(0)k.$$

This, combined with (7.2), shows $\dot\kappa(0) = 0$. Hence, from Theorem C we get the conclusion. □

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