GENERALIZED GOTTLIEB GROUPS AND GENERALIZED WANG HOMOMORPHISMS

YEON SOO YOON

Received February 26, 2001; revised May 25, 2001

ABSTRACT. We define and study a generalized Wang homomorphism closely related to the generalized Gottlieb group. We show that the existence of an element in the generalized Gottlieb group whose image under the Hurewicz map is nonzero is a sufficient condition for the vanishing of the mod p Wu numbers of X. Also, we can show that the image $\lambda^q_{\hat{\alpha}}(k)$ of k under the generalized Wang homomorphism is an obstruction to a map $S^n \times E_{kf} \to X$ lifting to a map $S^n \times E_{kf} \to E_k$, where E_k is the fibration induced by $k \in H^q(X;\pi)$

1. INTRODUCTION

Let $L(A,X)$ be the space of maps from A to X with the compact open topology. For a based map $f : A \to X$, $L(A,X;f)$ will denote the path component of $L(A,X)$ containing f. Let $\omega : L(A,X;f) \to X$ be the evaluation map given by $\omega(\alpha) = \alpha(*)$ for $\alpha \in L(A,X;f)$, where $*$ is a base point of A. Gottlieb [G1] studied the evaluation subgroup $\omega_\#(\pi_n(L(X,X;1))) = G_n(X)$ which is closely related to the Wang homomorphism $\lambda^*_{\hat{\alpha}} : H^*(X;\pi) \to H^{*-n}(X;\pi)$. Following [G2], any element $\hat{\alpha} \in \pi_n(L(X,X;1),1)$, or equivalently, $\bar{\alpha} : S^n \times X \to X$ may be viewed as a clutching map ([S], p. 455) along the equator of S^{n+1} which constructs a fibration $X \xrightarrow{i} E \xrightarrow{p} S^{n+1}$. Such a fibration has the Wang exact sequence corresponding to $\hat{\alpha}$;

$$\cdots \to H^q(E;\pi) \xrightarrow{i^*} H^q(X;\pi) \xrightarrow{\lambda^q_{\hat{\alpha}}} H^{q-n}(X;\pi) \to H^{q+1}(E;\pi) \to \cdots$$

and $\lambda^q_{\hat{\alpha}}$ is called *the Wang homomorphism for* $\hat{\alpha}$. Lupton and Oprea [LO] studied certain obstructions arising in symplectic geometry using some properties of the Wang homomorphism $\lambda^*_{\hat{\alpha}}$ for $\hat{\alpha} \in \pi_1(L(X,X;1),1)$. In [GNO], some properties of the Wang homomorphism $\lambda^*_{\hat{\alpha}}$ for $\hat{\alpha} \in \pi_n(L(X,X;f),f)$ were studied and it was shown that this homomorphism $\lambda^*_{\hat{\alpha}}$ is closely related to the spherical Lefschetz and Euler characteristics. In this paper, we generalize this homomorphism to $\lambda^*_{\hat{\alpha}} : H^*(X;\pi) \to H^{*-n}(A;\pi)$ for $\hat{\alpha} \in \pi_n(L(A,X;f),f)$ and study some properties of the generalized Wang homomorphism $\lambda^*_{\hat{\alpha}} : H^*(X;\pi) \to H^{*-n}(A;\pi)$ closely related to the generalized Gottlieb group $\omega_\#(\pi_n(L(A,X;f),f)) = G_n(A,f,X)$. In section 2, we introduce and study some properties of the generalized Wang homomorphism $\lambda^*_{\hat{\alpha}} : H^*(X;\pi) \to H^{*-n}(A;\pi)$ for $\hat{\alpha} \in \pi_n(L(A,X;f),f)$. We can obtain some connections between the generalized Gottlieb groups and the generalized Wang homomorphisms. We show that $\lambda^n_{\hat{\alpha}} = 0$ if and only if $h(\alpha) = 0$, where $\alpha = \omega_\#(\hat{\alpha}) \in G_n(A,f,X)$ and $h : \pi_n(Y) \to H_n(Y;\mathbb{Z}) \to H_n(Y;\pi)$ is the composition of the Hurewicz homomorphism tensored with π. Also we show that if $f : A \to X$ has a right homotopy inverse

2000 *Mathematics Subject Classification.* 55R05, 55N45, 57P10.

Key words and phrases. the generalized Gottlieb groups, the generalized Wang homomorphisms, the mod p Wu numbers.

This work was supported by Korea Research Foundation Grant(KRF-99-D00060).

$g : X \to A$, then the Wang homomorphism $\lambda^q_{(g^\natural)_\#(\hat\alpha)}$ for $(g^\natural)_\#(\hat\alpha) \in L(X,X;1)$ can be represented by the composition of g^* and the generalized Wang homomorphism $\lambda^q_{\hat\alpha}$, where $g^\natural : L(A,X;f) \to L(X,X;fg)$ is the map given by $g^\natural(f') = f'g$. On the other hand, Byun [B] defined the mod p Wu numbers and obtained a result in which the Wang homomorphism is closely related to the vanishing of the mod p Wu numbers. In section 3, we can replace Byun's condition with a weaker one for the mod p Wu numbers to vanish. A map $f : A \to X$ is called *cohomologically injective or c-injective over* π if $f^* : H^*(X;\pi) \to H^*(A;\pi)$ is an injective map. We show that for a c-injective map $f : A \to X$ over Z_p and the same dimensional path connected Z_p-Poincaré spaces X, A, if there is an element $\alpha \in G_1(A,f,X)$ such that the image $h(\alpha)$ of α under the Hurewicz map is nonzero, then all the mod p Wu numbers of X vanish. In section 4, we can know that $\lambda^q_{\hat\alpha}(k)$ can be considered as an obstruction to the map $\bar\alpha(1 \times p_{kf}) : S^n \times E_{kf} \to X$ lifting to a map $S^n \times E_{kf} \to E_k$, where E_k is the pullback of $k : X \to K(\pi,q)$ and $\epsilon : PK(\pi,q) \to K(\pi,q)$, and E_{kf} is the pullback of $kf : A \to K(\pi,q)$ and $\epsilon : PK(\pi,q) \to K(\pi,q)$. In fact, we show that there exists a map $\phi : S^n \times E_{kf} \to E_k$ such that $\phi_{|E_{kf}} \sim \tilde f$ and the diagram

$$\begin{array}{ccc} S^n \times E_{kf} & \xrightarrow{\ \phi\ } & E_k \\ {\scriptstyle 1\times p_{kf}}\downarrow & & \downarrow{\scriptstyle p_k} \\ S^n \times A & \xrightarrow{\ \bar\alpha\ } & X \end{array}$$

commutes if and only if $\lambda^q_{\hat\alpha}(k) = 0 \in H^{q-n}(A;\pi)$. The special case in which $f = 1_X$ and $A = X$, is a result of Gottlieb ([G1], Theorem 6.3). Throughout this paper, space means a space of the homotopy type of a locally finite connected CW complex. The base point as well as the constant map will be denoted by $*$. For simplicity, we use the same symbol for a map and its homotopy class.

Acknowledgment. The author would like to express his gratitude to John Oprea and Gregory Lupton for valuable conversations during this research.

2. Generalized Gottlieb groups and Generalized Wang homomorphisms

In 1949, H. C. Wang [W] obtained the following result; For a fibration $X \xrightarrow{i} E \xrightarrow{p} S^{n+1}$ over a sphere, there is an exact sequence

$$\cdots \to H^q(E;\pi) \xrightarrow{i^*} H^q(X;\pi) \xrightarrow{\lambda^q_{\hat\alpha}} H^{q-n}(X;\pi) \to H^{q+1}(E;\pi) \to \cdots,$$

which is called *the Wang sequence*. On the other hand, D. H. Gottlieb [G2] considered that for any element $\hat\alpha \in \pi_n(L(X,X;1),1)$, the Wang homomorphism for $\hat\alpha$ as follows;

Represent S^{n+1} as the union of closed hemisphere E^{n+1}_- and E^{n+1}_+ with intersection S^n. For any element $\hat\alpha \in \pi_n(L(X,X;1),1)$, let $E_{\hat\alpha}$ be the space obtained from $(E^{n+1}_- \times X)\coprod(E^{n+1}_+ \times X)$ by identifying $(y,x) \in E^{n+1}_- \times X$ with $(y,\hat\alpha(y)(x)) \in E^{n+1}_+ \times X$ for $y \in S^n$ and $x \in X$. These identifications are compatible with the projections $E^{n+1}_- \times X \to E^{n+1}_-$ and $E^{n+1}_+ \times X \to E^{n+1}_+$. Thus we have a fibration $X \xrightarrow{i} E_{\hat\alpha} \xrightarrow{p_{\hat\alpha}} S^{n+1}$ with the clutching map $\bar\alpha : S^n \times X \to X$ for $p_{\hat\alpha}$. Such a fibration has the Wang exact sequence associated to it;

$$\cdots \to H^q(E;\pi) \xrightarrow{i^*} H^q(X;\pi) \xrightarrow{\lambda^q_{\hat\alpha}} H^{q-n}(X;\pi) \to H^{q+1}(E;\pi) \to \cdots$$

and $\lambda^q_{\hat\alpha}$ is called *the Wang homomorphism for* $\hat\alpha$ [G2].

There is a beautiful connection between the Wang homomorphism $\lambda^q_{\hat\alpha} : H^q(X;\pi) \to H^{q-n}(X;\pi)$ for $\hat\alpha$ and the clutching map $\bar\alpha : S^n \times X \to X$ for $p_{\hat\alpha}$ as follows;

For any $x \in H^q(X;\pi)$, $\bar{\alpha}^*(x) = 1 \times x + \bar{U} \times \lambda^q_{\hat{\alpha}}(x)$, where $\bar{U} \in H^n(S^n)$ is the generator (Confer [S] pp.456).

Now we would like to generalize this Wang homomorphism to $\lambda^*_{\hat{\alpha}} : H^*(X;\pi) \to H^{*-n}(A;\pi)$ for $\hat{\alpha} \in \pi_n(L(A,X;f),f)$ using a similar relation with the above fact and study some properties of the generalized Wang homomorphism $\lambda^*_{\hat{\alpha}} : H^*(X;\pi) \to H^{*-n}(A;\pi)$ closely related to the generalized Gottlieb group $G_n(A,f,X)$ defined by Woo and Kim [WK].

Definition 2.1. Let $f : A \to X$ be a based map. A based map $\alpha : S^n \to X$ is called *f-cyclic* if there is a map $\bar{\alpha} : S^n \times A \to X$ such that $\bar{\alpha}j \sim \nabla(\alpha \vee f) : S^n \vee A \to X$, where $j : S^n \vee A \to S^n \times A$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We say that $\bar{\alpha}$ is an associated map to α. The set of all homotopy class of f-cyclic maps from S^n to X is denoted by $G_n(A,f,X)$.

Also, a based map $\alpha : S^n \to X$ is called *cyclic* if α is 1_X-cyclic. The Gottlieb group denoted $G_n(X)$ is the set of all homotopy class of cyclic maps from S^n to X [G1].

Remark 2.2.

(1) $G_n(X) = \cap\{G_n(A,f,X) \mid f : A \to X$ is a map and A is a space$\}$. For $\alpha \in G_n(X)$, there is an associated map $\bar{\alpha} : S^n \times X \to X$. The composition

$$A \times S^n \overset{f\times 1}{\to} X \times S^n \overset{\bar{\alpha}}{\to} X$$

establishes that $\alpha \in G_n(A,f,X)$. Since f is arbitrary, $\alpha \in \cap_f G_n(A,f,X)$. On the other hand, if we take $A = X$ and $f = 1 : X \to X$, then the converse holds.

(2) $G_n(X,1,X) = G_n(X)$ and $G_n(A,*,X) = \pi_n(X)$.

(3) In general, $G_n(X) \subset G_n(A,f,X) \subset \pi_n(X)$ for any map $f : A \to X$. It is well known that $G_5(S^5) = 2\mathbb{Z}$ [G1] and $G_n(X,i_1,X \times Y) \cong G_n(X) \oplus \pi_n(Y)$ [LW]. Consider the inclusion $i_1 : S^5 \to S^5 \times S^5$ and the projection $p_1 : S^5 \times S^5 \to S^5$. Then we know, from the above fact, that $G_5(S^5 \times S^5) \cong 2\mathbb{Z} \oplus 2\mathbb{Z} \neq G_5(S^5,i_1,S^5 \times S^5) \cong 2\mathbb{Z} \oplus \mathbb{Z} \neq \pi_5(S^5 \times S^5) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Since A is a locally compact, any continuous map $\hat{\alpha} : (S^n,*) \to (L(A,X;f),f)$ corresponds to a continuous map $\bar{\alpha} : S^n \times A \to X$, where $\bar{\alpha}(s,a) = \hat{\alpha}(s)(a)$. Thus we have the following proposition.

Proposition 2.3. [WK] *Let $\omega : L(A,X;f) \to X$ be the evaluation map. Then $\omega_\#(\pi_n(L(A,X;f))) = G_n(A,f,X)$.*

For a based map $f : (A,*) \to (X,*)$, let $\alpha \in G_n(A,f,X)$. From Proposition 2.3, there is an element $\hat{\alpha} \in \pi_n(L(A,X;f),f)$ such that $\omega_\#(\hat{\alpha}) = \alpha$. By the exponential law, there is a map $\bar{\alpha} : S^n \times A \to X$ given by $\bar{\alpha}(s,a) = \hat{\alpha}(s)(a)$ or immediately by definition of $G_n(A,f,X)$, there is a map $\bar{\alpha} : S^n \times A \to X$ such that $\bar{\alpha}j \sim \nabla(\alpha \vee f)$, where $j : S^n \vee A \to S^n \times A$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. Now we are going to study some properties in cohomology of the associated map $\bar{\alpha} : S^n \times A \to X$. Let π be a commutative ring with a unit. By the Künneth formula and the fact $H^*(S^n;\mathbb{Z})$ has no torsion, we have

$$H^q(S^n \times A;\pi) \cong H^0(S^n;\mathbb{Z}) \otimes H^q(A;\pi) \oplus H^n(S^n;\mathbb{Z}) \otimes H^{q-n}(A;\pi).$$

Thus if $x \in H^q(S^n \times A;\pi)$, we may write $x = 1 \times z + \bar{U} \times y$, where $\bar{U} \in H^n(S^n)$ is the generator such that $\langle \bar{U}, U \rangle = 1$ and $U \in H_n(S^n)$ is the fundamental homology class. Let $i_1 : S^n \to S^n \times A, i_2 : A \to S^n \times A$ be inclusions and $p_1 : S^n \times A \to S^n, p_2 : S^n \times A \to A$ be projections. Then $p_1^*(\bar{U}) = \bar{U} \times 1, p_2^*(z) = 1 \times z$. Now since $p_1 i_1 = 1, i_1^*(z' \times 1) = z'$ and $i_1^*(z' \times z) = 0$ unless $z \in H^0(A;\pi)$. Since $p_2 i_2 = 1, i_2^*(1 \times z) = z$ and $i_2^*(z' \times z) = 0$ unless $z' \in H^0(S^n)$. Thus since $\bar{\alpha} i_2 = f$, $\bar{\alpha}^*(x) = 1 \times f^*(x) + \bar{U} \times y$. From now on, we shall denote y by $\lambda^q_{\hat{\alpha}}(x)$.

Definition 2.4. For any $\hat{\alpha} \in \pi_n(L(A,X;f),f)$, define a map $\lambda^q_{\hat{\alpha}} : H^q(X;\pi) \to H^{q-n}(A;\pi)$ by $\bar{\alpha}^*(x) = 1 \times f^*(x) + \bar{U} \times \lambda^q_{\hat{\alpha}}(x)$. Since $\bar{\alpha}^*(x+y) = \bar{\alpha}^*(x) + \bar{\alpha}^*(y)$, we can easily show that $\lambda^q_{\hat{\alpha}} : H^q(X;\pi) \to H^{q-n}(A;\pi)$ is a group homomorphism. We call this homomorphism a *generalized Wang homomorphism*.

Note that in case $A = X$ and $f = 1$, the generalized Wang homomorphism $\lambda^*_{\hat{\alpha}}$ for $\hat{\alpha}$ is exactly the Wang homomorphism.

Here we use Spanier's conventions for the cup product, cap product and slant product [S]. Recall that the slant product $\backslash : H_n(X;G) \otimes H^q(X \times Y; G') \to H^{q-n}(Y; G' \otimes G)$ is given by $\langle z \backslash u, c\rangle = \sum \langle u, z_i \otimes c\rangle \otimes g_i$, where $z = \sum z_i \otimes g_i \in H_n(X;G)$ ([S], p. 351). We can easily know that for the fundamental class $U \in H_n(S^n)$, the map $U\backslash\ : H^q(S^n \times A;\pi) \to H^{q-n}(A;\pi)$ given by $(U\backslash\ )(\gamma) = U\backslash\gamma$ for $\gamma \in H^q(S^n \times A;\pi)$, is a homomorphism.

Lemma 2.5. *For each q, $\lambda^q_{\hat{\alpha}} = (U\backslash\ )(\bar{\alpha}^*) : H^q(X;\pi) \xrightarrow{\bar{\alpha}^*} H^q(S^n \times A;\pi) \xrightarrow{U\backslash} H^{q-n}(A;\pi)$.*

Proof. Let $x \in H^q(X;\pi)$. Since $\deg U = n > 0$, $U\backslash(1 \times f^*(x)) = 0$. Thus $(U\backslash\ )(\bar{\alpha}^*)(x) = U\backslash\bar{\alpha}^*(x) = U\backslash(1 \times f^*(x)) + U\backslash(\bar{U} \times \lambda^q_{\hat{\alpha}}(x)) = 0 + \langle \bar{U}, U\rangle \lambda^q_{\hat{\alpha}}(x) = \lambda^q_{\hat{\alpha}}(x)$. □

Let $E : L(A,X) \times A \to X$ be the evaluation map. Then define a homomorphism $\bar{\lambda}^q : H_n(L(A,X);\pi) \to \mathrm{Hom}(H^q(X;\pi), H^{q-n}(A;\pi))$ by $\bar{\lambda}^q(\beta)(z) = \beta\backslash E^*(z)$, where $\backslash : H_n(L(A,X);\pi) \otimes H^q(L(A,X) \times A;\pi) \to H^{q-n}(A;\pi)$ is the slant product. Let $h : \pi_n(Y) \to H_n(Y;\mathbb{Z}) \to H_n(Y;\pi)$ be the composition of the Hurewicz homomorphism tensored with π.

Proposition 2.6. *For each $\hat{\alpha} \in \pi_n(L(A,X;f),f)$, $\lambda^q_{\hat{\alpha}} = \bar{\lambda}^q(h(\hat{\alpha}))$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} S^n \times A & \xrightarrow{\hat{\alpha}\times 1} & L(A,X) \times A \\ \bar{\alpha}\big\downarrow & & E\big\downarrow \\ X & \xrightarrow{=} & X. \end{array}$$

Thus $\lambda^q_{\hat{\alpha}}(x) = U\backslash\bar{\alpha}^*(x) = U\backslash(\hat{\alpha}\times 1_X)^*E^*(x) = \hat{\alpha}_*(U)\backslash E^*(x) = h(\hat{\alpha})\backslash E^*(x) = \bar{\lambda}^q(h(\hat{\alpha}))(x)$. □

Note that $\lambda^q_{\hat{\alpha}} = 0 : H^q(X;\pi) \to H^{q-n}(A;\pi)$ for all $q < n$ since $H^{q-n}(A;\pi) = 0$.

Corollary 2.7. *If $\hat{\alpha} \in Ker(h : \pi_n(L(A,X;f),f) \to H_n(L(A,X);\pi))$, then $\lambda^q_{\hat{\alpha}} = 0 : H^q(X;\pi) \to H^{q-n}(A;\pi)$ for all $q \geq n$.*

For any $r > 0$ and any $\hat{\beta} \in \pi_r(L(A,X;f),f)$, let $\omega_\#(\hat{\beta}) = \beta : S^r \to X$. Then $\omega_*\hat{\beta}_* = \beta_* : H_r(S^r) \xrightarrow{\hat{\beta}_*} H_r(L(A,X);\pi) \xrightarrow{\omega_*} H_r(X;\pi)$. Therefore, we have the following commutative diagram;

$$\begin{array}{ccc} \pi_r(L(A,X;f),f) & \xrightarrow{h} & H_r(L(A,X);\pi) \\ \omega_\#\big\downarrow & & \omega_*\big\downarrow \\ \pi_r(X,x_0) & \xrightarrow{h} & H_r(X;\pi). \end{array}$$

Thus we can obtain a necessary and sufficient condition for $\lambda^n_{\hat{\alpha}} = 0$ as follows.

Theorem 2.8. *$h\omega_\#(\hat{\alpha}) = \omega_* h(\hat{\alpha}) = 0 \in H_n(X;\pi)$ if and only if $\lambda^n_{\hat{\alpha}} = 0 : H^n(X;\pi) \to H^0(A;\pi)$ for any field π.*

Proof. Let $x \in H^n(X;\pi)$. Since $\bar{\alpha}^*(x) = 1 \times f^*(x) + \bar{U} \times \lambda^n_{\hat{\alpha}}(x) = 1 \times f^*(x) + \lambda^n_{\hat{\alpha}}(x) \cdot \bar{U} \times 1$ and $\alpha = \bar{\alpha} i_1$, $(\omega_\#(\hat{\alpha}))^*(x) = \alpha^*(x) = i_1^* \bar{\alpha}^*(x) = i_1^*(1 \times f^*(x)) + i_1^*(\lambda^n_{\hat{\alpha}}(x) \cdot \bar{U} \times 1) = \lambda^n_{\hat{\alpha}}(x) \cdot \bar{U}$. Thus $\lambda^n_{\hat{\alpha}}(x) = \langle \lambda^n_{\hat{\alpha}}(x) \cdot \bar{U}, U \rangle = \langle (\omega_\#(\hat{\alpha}))^*(x), U \rangle = \langle x, (\omega_\#(\hat{\alpha}))_*(U) \rangle = \langle x, h(\omega_\#(\hat{\alpha})) \rangle$. Thus $\lambda^n_{\hat{\alpha}} = 0$ if and only if $h(\omega_\#(\hat{\alpha})) = 0$. Since $\omega_* h = h \omega_\#$, we have proved the theorem. □

On the other hand, we have a bilinear map from $\pi_n(L(A,X;f),f) \times H^q(X;\pi) \to H^{q-n}(A;\pi)$ given by $(\hat{\alpha}, k) \to \lambda^q_{\hat{\alpha}}(k)$. If we fix $\hat{\alpha} \in \pi_n(L(A,X;f),f)$, we get a generalized Wang homomorphism $\lambda^q_{\hat{\alpha}} : H^q(X;\pi) \to H^{q-n}(A;\pi)$. If we fix $k \in H^q(X;\pi)$, we get a homomorphism $\Lambda_k : \pi_n(L(A,X;f),f) \to H^{q-n}(A;\pi)$ given by $\Lambda_k(\hat{\alpha}) = \lambda^q_{\hat{\alpha}}(k)$. We call Λ_k *a generalized Gnaw homomorphism.*

Proposition 2.9. *Let $k : X \to Y$ and $l : Y \to K(\pi,q)$ be maps. Then the diagram*

$$\begin{array}{ccc} \pi_n(L(A,X;f),f) & \xrightarrow{(k_\natural)_\#} & \pi_n(L(A,Y;kf),kf) \\ \Lambda_{lk}\downarrow & & \Lambda_l \downarrow \\ H^{q-n}(A;\pi) & = & H^{q-n}(A;\pi) \end{array}$$

commutes, where $k_\natural : L(A,X;f) \to L(A,Y;kf)$ is the map given by $k_\natural(f') = kf'$.

Proof. Let $\hat{\alpha} \in \pi_n(L(A,X;f),f)$. Then $\Lambda_l(k_\natural)_\#(\hat{\alpha}) = \Lambda_l((k_\natural)_\#(\hat{\alpha})) = \lambda^q_{(k_\natural)_\#(\hat{\alpha})}(l) = U \backslash (k\bar{\alpha})^*(l) = U \backslash \bar{\alpha}^* k^*(l) = U \backslash \bar{\alpha}^*(lk) = \lambda^q_{\hat{\alpha}}(lk) = \Lambda_{lk}(\hat{\alpha})$. □

Let ι_q be the fundamental class of $H^q(K(\pi,q);\pi)$. Then it is well known by Thom [T] that $\Lambda_{\iota_q} : \pi_n(L(X,K(\pi,q);k),k) \cong H^{q-n}(X;\pi)$ is an isomorphism. Taking $Y = K(\pi,q)$, $l = 1_{K(\pi,q)}$, $A = X$ and $f = 1_X$, we have the following corollary.

Corollary 2.10. [G2] *For $k : X \to K(\pi,q)$, $\Lambda_k = \Lambda_{\iota_q}(k_\natural)_\# : \pi_n(L(X,X;1),1) \overset{(k_\natural)_\#}{\to} \pi_n(L(X,K(\pi,q);k),k) \overset{\Lambda_{\iota_q}}{\to} H^{q-n}(X;\pi)$.*

The first commutative diagram of the following proposition comes from the proof of Proposition 2.9.

Proposition 2.11. *Let $j : B \to A, k : X \to Y$ be maps and $\hat{\alpha} \in \pi_n(L(A,X;f),f)$. Then the diagram*

$$\begin{array}{ccccc} H^q(Y;\pi) & \xrightarrow{k^*} & H^q(X;\pi) & = & H^q(X;\pi) \\ \lambda^q_{(k_\natural)_\#(\hat{\alpha})}\downarrow & & \lambda^q_{\hat{\alpha}}\downarrow & & \lambda^q_{(j^\natural)_\#(\hat{\alpha})}\downarrow \\ H^{q-n}(A;\pi) & = & H^{q-n}(A;\pi) & \xrightarrow{j^*} & H^{q-n}(B;\pi) \end{array}$$

commutes, where $j^\natural : L(A,X;f) \to L(B,X;fj)$ is the map given by $j^\natural(f') = f'j$.

Proof. We only show that the second diagram is commutative. Let $x \in H^q(X;\pi)$. Then $\lambda^q_{(j^\natural)_\#(\hat{\alpha})}(x) = U \backslash (\bar{\alpha}(1 \times j))^*(x) = U \backslash (1 \times j)^* \bar{\alpha}^*(x) = j^* U \backslash \bar{\alpha}^*(x) = j^* \lambda^q_{\hat{\alpha}}(x)$. □

Corollary 2.12. *Let $g : X \to A$ be a map and $\hat{\alpha} \in \pi_n(L(A,X;f),f)$. Then the following diagram is commutative;*

$$\begin{array}{ccccc} H^q(A;\pi) & \xrightarrow{g^*} & H^q(X;\pi) & = & H^q(X;\pi) \\ \lambda^q_{(g_\natural)_\#(\hat{\alpha})}\downarrow & & \lambda^q_{\hat{\alpha}}\downarrow & & \lambda^q_{(g^\natural)_\#(\hat{\alpha})}\downarrow \\ H^{q-n}(A;\pi) & = & H^{q-n}(A;\pi) & \xrightarrow{g^*} & H^{q-n}(X;\pi). \end{array}$$

The above corollary says that for any element $\hat{\alpha} \in \pi_n(L(A, X; f), f)$ and a map $g : X \to A$, the homomorphism $\lambda^q_{(g^\natural)_\#(\hat{\alpha})}$ for $(g^\natural)_\#(\hat{\alpha}) \in L(X, X; fg)$ is the composition $H^q(X;\pi) \xrightarrow{\lambda^q_{\hat{\alpha}}} H^{q-n}(A;\pi) \xrightarrow{g^*} H^{q-n}(X;\pi)$ and the homomorphism $\lambda^q_{(g_\natural)_\#(\hat{\alpha})}$ for $(g_\natural)_\#(\hat{\alpha}) \in L(A, A; gf)$ is the composition $H^q(A;\pi) \xrightarrow{g^*} H^q(X;\pi) \xrightarrow{\lambda^q_{\hat{\alpha}}} H^{q-n}(A;\pi)$. Thus if $f : A \to X$ has a right homotopy inverse $g : X \to A$, then the Wang homomorphism $\lambda^q_{(g^\natural)_\#(\hat{\alpha})}$ for $(g^\natural)_\#(\hat{\alpha}) \in L(X, X; 1)$ can be represented by the composition of g^* and the generalized Wang homomorphism $\lambda^q_{\hat{\alpha}}$.

3. Generalized Gottlieb groups and the mod p Wu numbers

Byun [B] defined the mod p Wu numbers and obtained a result in which the Gottlieb group is closely related to a condition for the vanishing of the mod p Wu numbers. In this section, we show that the generalized Gottlieb group is closely related to a condition for the vanishing of the mod p Wu numbers. Thus we can replace Byun's condition with a weaker one for the mod p Wu numbers to vanish. By *an m-dimensional Z_p-Poincaré complex*, we will mean a CW complex X for which there is a homology class $U_X \in H_m(X;\pi)$ such that the map $\cap U_X : H^*(X; Z_p) \to H_{m-*}(X; Z_p)$ is an isomorphism. Consider an m-dimensional Z_p-Poincaré complex X with a prime integer p. Let $\{p\} = 1$ if $p = 2$ and $\{p\} = 2(p-1)$ if $p > 2$. Let $P^k : H^q(X; Z_p) \to H^{q+\{p\}k}(X; Z_p)$ be the k-th Steenrod square. It is customary to denote it by Sq^k when $p = 2$. Using the universal coefficient theorem and Poincaré isomorphism, there is an isomorphism ϕ: $H^q(X; Z_p) \to \mathrm{Hom}(H^{m-q}(X; Z_p), Z_p)$ given by $\phi(x)(y) = \langle x \cup y, U_X\rangle$, where $x \in H^q(X; Z_p)$, $y \in H^{m-q}(X; Z_p)$. Consider an m-dimensional Z_p-Poincaré complex X with a prime integer p and the homomorphism $H^{m-k\{p\}}(X; Z_p) \xrightarrow{P^k} H^m(X; Z_p) \xrightarrow{\langle\ ,U_X\rangle} Z_p$. Then, by the above fact, there exists an unique cohomology class $v_k \in H^{\{p\}k}(X; Z_p)$ such that $x \cup v_k = P^k_X(x)$ for any $x \in H^{m-\{p\}k}(X; Z_p)$. Thus we have the following lemma.

Lemma 3.1. *Let X be an m-dimensional Z_p-Poincaré complex with a prime integer p. Then there exists an unique cohomology class $v_k \in H^{\{p\}k}(X; Z_p)$ such that $x \cup v_k = P^k_X(x)$ for any $x \in H^{m-\{p\}k}(X; Z_p)$, where P^k_X is the k-th mod p Steenrod power.*

Set $v = 1 + v_1 + v_2 + \cdots$. Let $P = 1 + P^1 + P^2 + \cdots$ be the mod p total Steenrod power. The *mod p total Wu class q* and *the k-th Wu class q_k* are defined by the equation $q = Pv = 1 + q_1 + q_2 + \cdots$, that is, $q_k = \sum_{i+j=k} P^i(v_j) \in H^{\{p\}k}(X; Z_p)$. Also, *the mod p Wu numbers* are defined [B] as $q_I = \langle q_{\iota_1} q_{\iota_2} \cdots q_{\iota_l}, U_X\rangle$ for any sequence of natural numbers $I = (\iota_1, \iota_2, \cdots, \iota_l)$ such that $\{p\}(\iota_1 + \iota_2 + \cdots + \iota_l) = m$. In particular, when $p = 2$ and X is a manifold, these characteristic numbers are none other than the usual Stiefel-Whitney numbers of X. When $p = 2$, the classical terminology calls q_k the Stiefel-Whitney class of X and $v_k(X)$ the Wu class of X. Let π be a commutative ring with a unit. A map $f : A \to X$ is called *cohomologically injective or c-injective over* π if $f^* : H^*(X;\pi) \to H^*(A;\pi)$ is an injective map. Given a c-injective map $f : A \to X$ over π, *a f-derivation of degree $-n$ from $H^*(X;\pi)$ to $H^*(A;\pi)$* is a group homomorphism $D_f : H^*(X;\pi) \to H^{*-n}(A;\pi)$ such that $D_f(u \cup v) = D_f(u) \cup f^*(v) + (-1)^{n|u|} f^*(u) \cup D_f(v)$ for any $u, v \in H^*(X;\pi)$. Here $|u|$ means dim u. We can obtain the following lemma by modification of Byun's proof.

Lemma 3.2. *Let X and A be m-dimensional Z_p-Poincaré spaces for a prime p. Assume $f : A \to X$ is a c-injective map over Z_p and there is a f-derivation D_f of deg $-n$ from $H^*(X; Z_p)$ to $H^*(A; Z_p)$ which commutes with the mod p Steenrod power. Also assume there is a class $\phi \in H^n(X; Z_P)$ such that $D_f(\phi) = 1, P^i(f^*(\phi)) = 0$ for any i such that $0 < i < n$ if $p = 2$ and $0 < i < n/2$ if $p > 2$, where P^i's are the i-th mod p Steenrod powers. Then all the mod p Wu numbers of X vanish.*

Proof. First of all, we will show that $f^*(\phi)\cup D_f(q)=0$. Let $m=dim\ X$. From Lemma 3.1, there exists a unique cohomology class $v_k\in H^{\{p\}k}(X;Z_p)$ such that $x\cup v_k=P_X^k(x)$ for any $x\in H^{m-\{p\}k}(X;Z_p)$. Then $f^*(\phi)\cup D_fP_X^k(x)=f^*(\phi)\cup D_f(x\cup v_k)=(f^*(\phi)\cup D_f(x))\cup f^*(v_k)+(-1)^{2n|x|}f^*(x)\cup(f^*(\phi)\cup D_f(v_k))$. Thus we have

$$f^*(\phi)\cup D_fP_X^k(x)=f^*(x)\cup(f^*(\phi)\cup D_f(v_k))+(f^*(\phi)\cup D_f(x))\cup f^*(v_k).\ (*)$$

Case 1. Assume $p=2$ or n is even. Then, noting that $Pf^*(\phi)=f^*(\phi)+f^*(\phi)^p$ from the hypothesis, we obtain, by Cartan's formula, $P^k(f^*(\phi)\cup D_f(x))=\sum_i P^{k-i}f^*(\phi)\cup P_A^iD_f(x)=f^*(\phi)\cup P_A^kD_f(x)+f^*(\phi)^p\cup P_A^{k-\delta}D_f(x)$, where $\delta=n$ if $p=2$ and $\delta=n/2$ if $p>2$. From the equation (*), we have $P^k(f^*(\phi)\cup D_f(x))-f^*(\phi)^p\cup P^{k-\delta}D_f(x)=f^*(\phi)\cup P^kD_f(x)=f^*(x)\cup(f^*(\phi)\cup D_f(v_k))+(f^*(\phi)\cup D_f(x))\cup f^*(v_k)$. Since A is also an m-dimensional Z_p-Poincaré complex, there exists a unique cohomology class $\bar v_k\in H^{\{p\}k}(A;Z_p)$ such that $a\cup\bar v_k=P_A^k(a)$ for any $a\in H^{m-\{p\}k}(A;Z_p)$. Since $P_A^kf^*(x)=f^*P_X^k(x)=f^*(x\cup v_k)=f^*(x)\cup f^*(v_k)$ and $f^*(x)\in H^{\{p\}k}(A;Z_p)$, we know that $\bar v_k=f^*(v_k)$ and $P^k(f^*(\phi)\cup D_f(x))=(f^*(\phi)\cup D_f(x))\cup f^*(v_k)$. Thus we have $f^*(\phi)^p\cup P_A^{k-\delta}D_f(x)=-f^*(x)\cup(f^*(\phi)\cup D_f(v_k))$. Note that $\phi^p\cup P^{k-\delta}(x)=0$ for dimensional reasons and that $D_f(\phi^p)=0$. Thus we have $0=D_f(\phi^p\cup P_X^{k-\delta}(x))=D_f(\phi^p)\cup f^*P_X^{k-\delta}(x)+(-1)^{n|\phi^p|}f^*(\phi)^p\cup D_fP_X^{k-\delta}(x)=f^*(\phi)^p\cup P_A^{k-\delta}D_f(x)$. Since $f^*(\phi)^p\cup P_A^{k-\delta}D_f(x)=-f^*(x)\cup(f^*(\phi)\cup D_f(v_k))$, we know $(f^*(\phi)\cup D_f(v_k))\cup f^*(x)=0$ for any x ($|x|=m-\{p\}k$). By Lemma 3.1, we have that $f^*(\phi)\cup D_f(v_k)=0$ and, therefore, that $f^*(\phi)\cup D_f(v)=0$. Thus $0=P(0)=P(f^*(\phi)\cup D_f(v))=Pf^*(\phi)\cup PD_f(v)=(f^*(\phi)+f^*(\phi)^p)\cup PD_f(v)=(f^*(\phi)\cup PD_f(v))+(f^*(\phi)^p\cup PD_f(v))$. It follows that $f^*(\phi)\cup PD_f(v)=-f^*(\phi)^p\cup PD_f(v)=(-1)^2f^*(\phi)^{2(p-1)+1}\cup PD_f(v)=\cdots=(-1)^Nf^*(\phi)^{N(p-1)+1}\cup PD_f(v)=0$, where N is any sufficiently large integer. Thus we have $f^*(\phi)\cup D_f(q)=f^*(\phi)\cup D_fP(v)=f^*(\phi)\cup PD_f(v)=0$.

Case 2. Assume both p and n are odd. Then $Pf^*(\phi)=f^*(\phi)$ from the hypothesis and, therefore, $P_A^k(f^*(\phi)\cup D_f(x))=\sum_i P_A^{k-i}f^*(\phi)\cup P_A^iD_f(x)=f^*(\phi)\cup P_A^kD_f(x)$. Together with equation (*), we obtain $P_A^k(f^*(\phi)\cup D_f(x))=f^*(x)\cup(f^*(\phi)\cup D_f(v_k))+(f^*(\phi)\cup D_f(x))\cup f^*(v_k)$. From the property of $f^*(v_k)$, we know that $P_A^k(f^*(\phi)\cup D_f(x))=(f^*(\phi)\cup D_f(x))\cup f^*(v_k)$. Therefore we have that $f^*(x)\cup(f^*(\phi)\cup D_f(v_k))=0$ for any x ($|x|=m-2(p-1)k$). By Lemma 3.1, we know that $f^*(\phi)\cup D_f(v_k)=0$ and, therefore, that $f^*(\phi)\cup D_f(v)=0$. Thus $0=P_A(f^*(\phi)\cup D_f(v_k))=P_Af^*(\phi)\cup P_AD_f(v)=f^*(\phi)\cup D_fP_X(v)=f^*(\phi)\cup D_f(q)$. This proves the assertion $f^*(\phi)\cup D_f(q)=0$.

Secondly, if u is a cohomology class with $D_f(\phi\cup u)=0$, then $0=D_f(\phi\cup u)=f^*(u)+(-1)^{n^2}f^*(\phi)\cup D_f(u)$. Thus we know that $f^*(u)=(-1)^{n^2+1}f^*(\phi)\cup D_f(u)=(-1)^{(n+1)^2}f^*(\phi)\cup D_f(u)=(-1)^{n+1}f^*(\phi)\cup D_f(u)$. In particular, if $\phi\cup u=0$, then $f^*(u)=(-1)^{n+1}f^*(\phi)\cup D_f(u)$.

Now we will prove this lemma. Let $I=(\iota_1,\iota_2,\cdots,\iota_l)$ be a sequence of natural numbers such that $\{p\}(\iota_1+\iota_2+\cdots+\iota_l)=m$. Set $q_I=q_{\iota_1}q_{\iota_2}\cdots q_{\iota_l}$. From the fact $\phi\cup q_I\in H^{m+n}(X:Z_p)=0$, we have that $f^*(q_I)=(-1)^{n+1}f^*(\phi)\cup D_f(q_I)=(-1)^{n+1}f^*(\phi)\cup\sum_j(-1)^{(|q_{\iota_1}|+\cdots+|q_{\iota_{j-1}}|)|q_{\iota_j}|}D_f(q_{\iota_j})f^*(q_{\iota_1})\cdots\widehat{f^*(q_{\iota_j})}\cdots f^*(q_{\iota_l})$. Since $|q_{\iota_j}|$ is even ($p>2$) and $(-1)^{(|q_{\iota_1}|+\cdots+|q_{\iota_{j-1}}|)|q_{\iota_j}|}\equiv 1\ (mod\,p=2)$, we can get $\sum_j(-1)^{(|q_{\iota_1}|+\cdots+|q_{\iota_{j-1}}|)|q_{\iota_j}|}D_f(q_{\iota_j})f^*(q_{\iota_1})\cdots\widehat{f^*(q_{\iota_j})}\cdots f^*(q_{\iota_l})=\sum_j D_f(q_{\iota_j})f^*(q_{\iota_1})\cdots\widehat{f^*(q_{\iota_j})}\cdots f^*(q_{\iota_l})$. Thus we know, from the fact $f^*(\phi)\cup D_f(q)=0$, that $f^*(q_I)=(-1)^{n+1}\sum_j(f^*(\phi)\cup D_f(q_{\iota_j}))f^*(q_{\iota_1})\cdots\widehat{f^*(q_{\iota_j})}\cdots f^*(q_{\iota_l})=0$. Since f is c-injective over π, $q_I=0$ and $q_I(X)=\langle q_{\iota_1}q_{\iota_2}\cdots q_{\iota_l},U_X\rangle=0$. This proves the lemma. □

For any $\hat{\alpha} \in \pi_n(L(A,X;f),f)$, there is a map $\lambda^*_{\hat{\alpha}} : H^*(X;\pi) \to H^{*-n}(A;\pi)$. If $f : A \to X$ is a c-injective map over π, then the following lemma shows that $\lambda^*_{\hat{\alpha}}$ is an f-derivation of degree $-n$.

Lemma 3.3. (1) *If $f : A \to X$ is a c-injective map over π, $\lambda^*_{\hat{\alpha}}(u \cup v) = \lambda^*_{\hat{\alpha}}(u) \cup f^*(v) + (-1)^{n|u|} f^*(u) \cup \lambda^*_{\hat{\alpha}}(v)$ for any $u, v \in H^*(X;\pi)$.*
(2) *If $\pi = Z_p$, we have $\lambda^*_{\hat{\alpha}} P^k = P^k \lambda^*_{\hat{\alpha}}$ for any integer k.*

Proof. (1) We have $\bar{\alpha}^*(u \cup v) = 1 \times f^*(u \cup v) + \bar{U} \times \lambda^*_{\hat{\alpha}}(u \cup v)$. On the other hand, $\bar{\alpha}^*(u \cup v) = \bar{\alpha}^*(u) \cup \bar{\alpha}^*(v) = (1 \times f^*(u) + \bar{U} \times \lambda^*_{\hat{\alpha}}(u)) \cup (1 \times f^*(v) + \bar{U} \times \lambda^*_{\hat{\alpha}}(v)) = 1 \times f^*(u \cup v) + \bar{U} \times ((\lambda^*_{\hat{\alpha}}(u) \cup f^*(v)) + (-1)^{n|u|}(f^*(u) \cup \lambda^*_{\hat{\alpha}}(v)))$. Thus $\lambda^*_{\hat{\alpha}}(u \cup v) = (\lambda^*_{\hat{\alpha}}(u) \cup f^*(v)) + (-1)^{n|u|}(f^*(u) \cup \lambda^*_{\hat{\alpha}}(v))$.
(2) Since $P^k \bar{\alpha}^*(u) = \bar{\alpha}^* P^k(u)$ and $P^k f^*(u) = f^* P^k(u)$, we have that $1 \times P^k f^*(u) + \bar{U} \times P^k \lambda^*_{\hat{\alpha}}(u) = P^k(1 \times f^*(u) + \bar{U} \times \lambda^*_{\hat{\alpha}}(u)) = P^k \bar{\alpha}^*(u) = \bar{\alpha}^* P^k(u) = 1 \times f^* P^k(u) + \bar{U} \times \lambda^*_{\hat{\alpha}} P^k(u) = 1 \times P^k f^*(u) + \bar{U} \times \lambda^*_{\hat{\alpha}} P^k(u)$. Hence $P^k \lambda^*_{\hat{\alpha}}(u) = \lambda^*_{\hat{\alpha}} P^k(u)$. □

We know, from Corollary 2.12 and the above lemma (1), that if $f : A \to X$ has a right homotopy inverse $g : X \to A$, then $g^* \lambda^q_{\hat{\alpha}}$ is the Wang derivation of degree $-n$ of $H^*(X;\pi)$. Now, Lemma 3.2 and Lemma 3.3 together prove the following theorem.

Theorem 3.4. *Let p be a prime, X and A m-dimensional path connected Z_p-Poincaré spaces. Assume $f : A \to X$ is a c-injective map over Z_p and there is a class $\phi \in H^n(X;Z_p)$ such that $\langle \phi, h(\alpha) \rangle = 1$ for some element $\alpha \in G_n(A,f,X)$ and $P^i(f^*(\phi)) = 0$ for any i such that $0 < i < n$ if $p = 2$ and $0 < i < n/2$ if $p > 2$, where P^i's are the i-th mod p Steenrod powers. Then all the mod p Wu numbers of X vanish.*

The following corollary says that, from Proposition 2.3 and Theorem 2.8, the nontriviality of the generalized Wang homomorphism is a sufficient condition for the vanishing of the mod p Wu numbers.

Corollary 3.5. *Let $f : A \to X$ be a c-injective map over Z_p, where X and A have the same dimensional path connected Z_p-Poincaré spaces. If there is an element $\alpha \in G_1(A,f,X)$ such that $h(\alpha) \neq 0$, then all the mod p Wu numbers of X vanish.*

Proof. Since the map $H^1(X;Z_p) \to \mathrm{Hom}(H_1(X;Z_p),Z_p)$ defined by the Kronecker product is an isomorphism, there is a $\phi \in H^1(X;Z_p)$ with $\langle \phi, h(\alpha) \rangle = 1$. This proves the corollary. □

Taking $f = 1$ and $A = X$, we get a result of Byun [B].

Corollary 3.6. [B] *Let X be a path connected Z_p-Poincare space. If there is an element $\alpha \in G_1(X)$ whose image under the Hurewicz map $h : \pi_1(X) \to H_1(X;Z_p)$ is not zero, then all the mod p Wu numbers vanish.*

4. Generalized Wang homomorphisms and Lifting Gottlieb groups

Let $k : X \to Y$ be a map and PY the space of paths in Y which begin at $*$. Let $\epsilon : PY \to Y$ be the fibration given by evaluating a path at its end point. Let $p_k : E_k \to X$ be the fibration induced by $k : X \to Y$ from ϵ. Let $f : A \to X$ be a map. Then we can also consider the fibration $p_{kf} : E_{kf} \to A$ induced by $kf : A \to Y$ from ϵ. The following lemma is standard.

Lemma 4.1. *A map $g : B \to X$ can be lifted to a map $B \to E_k$ if and only if $kg \sim *$.*

In [Y], we showed that the following theorem is true for the case of $f = 1_X$ and $A = X$. It is can be easily extended as follows.

Theorem 4.2. *Let $\bar{\alpha} : B \times A \to X$ be an associated map to $\alpha \in G(B; A, f, X)$. Then there exists a map $\phi : B \times E_{kf} \to E_k$ such that $\phi_{|E_{kf}} \sim \tilde{f}$ and the diagram*

$$\begin{array}{ccc} B \times E_{kf} & \xrightarrow{\phi} & E_k \\ {\scriptstyle (1\times p_{kf})}\downarrow & & {\scriptstyle p_k}\downarrow \\ B \times A & \xrightarrow{\bar{\alpha}} & X \end{array}$$

commutes if and only if $k\bar{\alpha}(1 \times p_{kf}) \sim *$.

Proof. If such a ϕ exists, we know, from Lemma 4.1, that $k\bar{\alpha}(1 \times p_{kf}) \sim *$. Conversely, suppose $k\bar{\alpha}(1 \times p_{kf}) \sim *$. By Lemma 4.1, there is a map $\phi' : B \times E_{kf} \to E_k$ such that $p_k\phi' = \bar{\alpha}(1 \times p_{kf})$. Then $p_k\phi'_{|E_{kf}} = \bar{\alpha}(1 \times p_{kf})_{|E_{kf}} \sim p_k\tilde{f}$. It is known ([MT], Proposition 2, p. 109) that for maps $g_1, g_2 : C \to E_k$, $p_kg_1 \sim p_kg_2$ if and only if there is a map $\gamma : C \to \Omega Y$ such that $g_1 \sim \mu(g_2 \times \gamma)\Delta$, where $\mu : E_k \times \Omega Y \to E_k$ is given by $\mu((a, \eta), \omega) = (a, \omega + \eta)$ and $\Delta : C \to C \times C$ is the diagonal map. Thus for maps $\tilde{f}, \phi'_{|E_{kf}} : E_{kf} \to E_k$, there is a map $\gamma : E_{kf} \to \Omega Y$ such that $\tilde{f} \sim \mu(\phi'_{|E_{kf}} \times \gamma)\Delta$. Let $\gamma' = \gamma p_2 : B \times E_{kf} \to \Omega Y$, where $p_2 : B \times E_{kf} \to E_{kf}$ is the projection. Consider the map $\phi = \mu(\phi' \times \gamma')\Delta_{B\times E_{kf}} : B \times E_{kf} \to E_k$. Then $p_k\phi = p_k\phi' = \bar{\alpha}(1 \times p_{kf}), \phi_{|E_{kf}} \sim \mu(\phi'_{|E_{kf}} \times \gamma)\Delta_{E_{kf}} \sim \tilde{f}$. This proves the theorem. □

The following corollary says that $\lambda^q_{\hat{\alpha}}(k)$ can be considered as an obstruction to the map $\bar{\alpha}(1 \times p_{kf}) : S^n \times E_{kf} \to X$ lifting to a map $S^n \times E_{kf} \to E_k$.

Corollary 4.3. *Let $\hat{\alpha} \in \pi_n(L(A, X; f), f)$ and $\bar{\alpha} : S^n \times A \to X$ be the map given by $\bar{\alpha}(s, a) = \hat{\alpha}(s)(a)$. Let $k \in H^q(X; \pi)$, where $q \geq 2$. Then there exists a map $\phi : S^n \times E_{kf} \to E_k$ such that $\phi_{|E_{kf}} \sim \tilde{f}$ and the diagram*

$$\begin{array}{ccc} S^n \times E_{kf} & \xrightarrow{\phi} & E_k \\ {\scriptstyle 1\times p_{kf}}\downarrow & & {\scriptstyle p_k}\downarrow \\ S^n \times A & \xrightarrow{\bar{\alpha}} & X \end{array}$$

commutes if and only if $\lambda^q_{\hat{\alpha}}(k) = 0 \in H^{q-n}(A; \pi)$.

Proof. From Theorem 4.2, it is sufficient to show that $k\bar{\alpha}(1 \times p_{kf}) \sim *$ if and only if $\lambda^q_{\hat{\alpha}}(k) = 0 \in H^{q-n}(A; \pi)$. It is known [HV] that if $p : E \to B$ is a fibration with $q - 2$ connected fibre F ($q \geq 2$), then for any coefficient group π, $p^* : H^i(B; \pi) \to H^i(E; \pi)$ is an isomorphism for $i \leq q - 2$ and a monomorphism for $i = q - 1$. Since $p_{kf} : E_{kf} \to A$ is a fibration with fibre $K(\pi, q - 1)$, $p^* : H^i(A; \pi) \to H^i(E_{kf}; \pi)$ is a monomorphism for all $i \leq q - 1$. Thus $\lambda^q_{\hat{\alpha}}(k) = 0 \in H^{q-n}(A; \pi)$ if and only if $p^*_{kf}\lambda^q_{\hat{\alpha}}(k) = 0 \in H^{q-n}(E_{kf}; \pi)$. We have that $\bar{\alpha}^*(k) = 1 \times f^*(k) + \bar{U} \times \lambda^q_{\hat{\alpha}}(k)$. Hence we have, from the property of cohomology cross product with respect to coboundary operator of a pair, that $\delta^*\bar{\alpha}^*(k) = \delta^*1 \times f^*(k) + \delta^*\bar{U} \times \lambda^q_{\hat{\alpha}}(k) \in H^{q+1}(D^{n+1} \times A, S^n \times A; \pi)$. Since $H^1(D^{n+1}, S^n; \mathbb{Z}) = H^1(S^{n+1}; \mathbb{Z}) = 0$, $\delta^*1 = 0$. Therefore $\delta^*\bar{\alpha}^*(k) = \delta^*\bar{U} \times \lambda^q_{\hat{\alpha}}(k)$. Since $(1_{D^{n+1}} \times p_{kf})^*\delta^*\bar{\alpha}^*(k) = \delta^*\bar{U} \times p^*_{kf}\lambda^q_{\hat{\alpha}}(k)$ and $\delta^*\bar{U}$ is a generator of $H^{n+1}(D^{n+1}, S^n; \mathbb{Z}) = \mathbb{Z}$, it follows that $p^*_{kf}\lambda^q_{\hat{\alpha}}(k) = 0 \in H^{q-n}(E_{kf}; \pi)$ if and only if $(1_{D^{n+1}} \times p_{kf})^*\delta^*\bar{\alpha}^*(k) = 0 \in H^{q+1}(D^{n+1} \times E_{kf}, S^n \times E_{kf}; \pi)$. Since $(1_{D^{n+1}} \times p_{kf})^*\delta^* = \delta^*(1_{S^n} \times p_{kf})^* : H^q(S^n \times A; \pi) \to H^{q+1}(D^{n+1} \times E_{kf}, S^n \times E_{kf}; \pi)$, $p^*_{kf}\lambda^q_{\hat{\alpha}}(k) = 0 \in H^{q-n}(E_{kf}; \pi)$ if and only if $\delta^*(k\bar{\alpha}(1_{S^n} \times p_{kf})) = \delta^*(1_{S^n} \times p_{kf})^*\bar{\alpha}^*(k) = 0 \in H^{q+1}(D^{n+1} \times E_{kf}, S^n \times E_{kf}; \pi)$. Thus we only show that $\delta^*(k\bar{\alpha}(1_{S^n} \times p_{kf})) = 0 \in H^{q+1}(D^{n+1} \times E_{kf}, S^n \times E_{kf}; \pi)$ if and only if $k\bar{\alpha}(1_{S^n} \times p_{kf}) \sim *$. If $k\bar{\alpha}(1_{S^n} \times p_{kf}) \sim *$, then clearly $\delta^*(k\bar{\alpha}(1_{S^n} \times p_{kf})) = 0$. Conversely, suppose $\delta^*(k\bar{\alpha}(1_{S^n} \times p_{kf})) = 0 \in$

$H^{q+1}(D^{n+1} \times E_{kf}, S^n \times E_{kf}; \pi)$. Since the sequence $H^q(D^{n+1} \times E_{kf}; \pi) \stackrel{(\iota \times 1)^*}{\to} H^q(S^n \times E_{kf}; \pi) \stackrel{\delta^*}{\to} H^{q+1}(D^{n+1} \times E_{kf}, S^n \times E_{kf}\pi)$ is exact, there is a map $F : cS^n \times E_{kf} \to K(\pi, q)$ such that $F_{|S^n \times E_{kf}} = k\bar{\alpha}(1_{S^n} \times p_{kf})$, where cS^n is the reduced cone of S^n. Thus we have a map $H : S^n \times E_{kf} \times I \to K(\pi, q)$ given by $H(s, e, t) = F([s, t], e)$. Then $H(\ , \ , 0) = k\bar{\alpha}(1_{S^n} \times p_{kf}), H(\ , \ , 1) = H(*, \ , 0) = kfp_{kf}p_2$. Since there is a map $\tilde{f}p_2 : S^n \times E_{kf} \to E_k$ satisfying $p_k(\tilde{f}p_2) \sim fp_{kf}p_2$, by Lemma 4.1, $H(\ , \ , 1) = kfp_{kf}p_2 \sim *$. Thus we have $k\bar{\alpha}(1_{S^n} \times p_{kf}) \sim *$. □

The special case in which $f = 1_X$ and $A = X$, is a result of Gottlieb ([G1], Theorem 6.3).

References

[B] Y. Byun, *On vanishing of characteristic numbers in Poincare complexes*, Trans. Amer. Math. Soc. 348(8)(1996), 3085–3095.

[GNO] R. Geoghegan, A. Nicas and J. Oprea, *Higher Lefschetz traces and spherical Euler characteristics*, Trans. Amer. Math. Soc. 348(5)(1996), 2039–2062.

[G1] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. 91(1969), 729–756.

[G2] D. H. Gottlieb, *Applications of bundle map theory(1)*, Trans. Amer. Math. Soc. 171(1972), 23–50.

[HV] I. G. Halbhavi and K. Varadarajan, *Gottlieb sets and duality in homotopy theory*, Canad. J. Math.,27(5)(1975), 1042–1055.

[H] P. Hilton, *Homotopy Theory and Duality*, Gordon and Breach Science Publishers, Inc., 1965.

[LW] K. Y. Lee and M. H. Woo, *Generalized evaluation subgroups of product spaces relative to a factor*, Proc. Amer. Math. Soc.,124(7)(1996), 2255–2260.

[LO] G. Lupton and J. Oprea, *Cohomologically symplectic spaces: Toral actions and the Gottlieb group*, Trans. Amer. Math. Soc. 347(1)(1995), 261–288.

[MT] R. E. Mosher and M. C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, New York, 1968.

[S] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[T] R. Thom, *L'Homologie des Espaces Fonctionels*, Colloque de Topologie algebrique Louvain, 1957.

[W] H. C. Wang, *The homotopy groups of the fibre bundles over a sphere*, Duke Math. J.,16(1949), 33–38.

[WK] M. H. Woo and J. R. Kim, *Certain subgroups of homotopy groups*, J. Korean Math. Soc.,21(2)(1984), 109–120.

[Y] Y. S. Yoon, *Lifting Gottlieb sets and duality* , Proc. Amer. Math. Soc. 119(4)(1993), 1315–1321.

Department of Mathematics, Hannam University, Taejon 306-791, Korea.
E-mail address: yoon@math.hannam.ac.kr