ON n-INNER PRODUCTS, n-NORMS, AND THE CAUCHY-SCHWARZ INEQUALITY

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Abstract. Our observation on the Cauchy-Schwarz inequality in an inner product space and 2-inner product space suggests how the concepts of inner products and 2-inner products, as well as norms and 2-norms, can be generalized to those of n-inner products and n-norms for any $n \in \mathbf{N}$. In this paper, we offer a definition of n-inner products which is simpler than (but equivalent to) the one formulated by Misiak [9]. We also reprove the Cauchy-Schwarz inequality and give a necessary and sufficient condition for the equality.

1. Introduction

We are already familiar with inner products and norms. So, let us begin with the definition of 2-inner products and 2-norms.

Let X be a real vector space of dimension $d \geq 2$. A *2-inner product* on X is a function $\langle \cdot|\cdot,\cdot\rangle : X \times X \times X \to \mathbf{R}$ satisfying the following properties:

(I1) $\langle x|y,y\rangle \geq 0$ for all $x, y \in X$; $\langle x|y,y\rangle = 0$ if and only if x and y are linearly dependent;
(I2) $\langle x|y,y\rangle = \langle y|x,x\rangle$ for all $x, y \in X$;
(I3) $\langle x|y,z\rangle = \langle x|z,y\rangle$ for all $x, y, z \in X$;
(I4) $\langle x|y,\alpha z\rangle = \alpha\langle x|y,z\rangle$ for all $x, y, z \in X$ and $\alpha \in \mathbf{R}$;
(I5) $\langle x|y,z+z'\rangle = \langle x|y,z\rangle + \langle x|y,z'\rangle$ for all $x, y, z, z' \in X$.

The pair $(X, \langle \cdot|\cdot,\cdot\rangle)$ is called a *2-inner product space* (see [2] and [3]). Note that, for generalization purpose, we use a slightly different notation for 2-inner products.

Meanwhile, a *2-norm* on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbf{R}$ satisfying the following properties:

(N1) $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(N2) $\|x,y\| = \|y,x\|$ for all $x, y \in X$;
(N3) $\|x,\alpha y\| = |\alpha|\,\|x,y\|$ for all $x, y \in X$ and $\alpha \in \mathbf{R}$;
(N4) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$ for all $x, y, z \in X$.

The pair $(X, \|\cdot,\cdot\|)$ is called a *2-normed space* (see [4]).

If X is equipped with an inner product $\langle \cdot,\cdot\rangle$, then we can define a norm $\|\cdot\|$ on X by $\|x\| := \langle x,x\rangle^{\frac{1}{2}}$. One of the properties of the norm is that it satisfies the triangle inequality

$$\|x+y\| \leq \|x\| + \|y\|,$$

which is easy to prove by using the Cauchy-Schwarz inequality

$$\langle x,y\rangle^2 \leq \|x\|^2\|y\|^2.$$

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By rewriting it as a determinantal inequality involving a 2×2 Gram matrix

$$\begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{vmatrix} \geq 0,$$

we see that the Cauchy-Schwarz inequality holds since the matrix is positive semidefinite (see [6], pp. 407–408, for Gram matrices).

At the same time, we can also define a 2-inner product $\langle\cdot|\cdot,\cdot\rangle$ on X by

$$\langle x|y,z\rangle := \begin{vmatrix} \langle x,x\rangle & \langle x,z\rangle \\ \langle y,x\rangle & \langle y,z\rangle \end{vmatrix}$$

from which we obtain a 2-norm $\|\cdot,\cdot\|$ on X defined by $\|x,y\| := \langle x|y,y\rangle^{\frac{1}{2}}$, that is,

$$\|x,y\| = \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{vmatrix}^{\frac{1}{2}}.$$

Let us examine this 2-norm. As usual, the properties (N1), (N2) and (N3) are easy to check. To verify the property (N4) or the triangle inequality, it suffices to prove the Cauchy-Schwarz inequality

$$\langle x|y,z\rangle^2 \leq \|x,y\|^2\|x,z\|^2.$$

But, again, by rewriting it as

$$\begin{vmatrix} \langle x|y,y\rangle & \langle x|y,z\rangle \\ \langle x|z,y\rangle & \langle x|z,z\rangle \end{vmatrix} \geq 0,$$

and noting that the matrix is positive semidefinite, that is,

$$\begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} \langle x|y,y\rangle & \langle x|y,z\rangle \\ \langle x|z,y\rangle & \langle x|z,z\rangle \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \langle x|\alpha y+\beta z, \alpha y+\beta z\rangle \geq 0$$

for any $\alpha,\beta \in \mathbf{R}$, we see that the Cauchy-Schwarz inequality holds.

Alternatively, one may observe that, under the assumption $x \neq 0$, the Cauchy-Schwarz inequality

$$\begin{vmatrix} \langle x,x\rangle & \langle x,z\rangle \\ \langle y,x\rangle & \langle y,z\rangle \end{vmatrix}^2 \leq \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{vmatrix} \begin{vmatrix} \langle x,x\rangle & \langle x,z\rangle \\ \langle z,x\rangle & \langle z,z\rangle \end{vmatrix}$$

is equivalent to

$$\begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle & \langle x,z\rangle \\ \langle y,x\rangle & \langle y,y\rangle & \langle y,z\rangle \\ \langle z,x\rangle & \langle z,y\rangle & \langle z,z\rangle \end{vmatrix} \geq 0$$

(see [5]). Since the matrix is positive semidefinite, the inequality follows and we also see that the equality holds if and only if x, y and z are linearly dependent.

The above observation on the Cauchy-Schwarz inequality in an inner product space and 2-inner product space suggests how the concepts of inner products and 2-inner products, as well as norms and 2-norms, can be generalized to those of n-inner products and n-norms for any $n \in \mathbf{N}$. In this paper, we shall offer a definition of n-inner products which is slightly simpler than (but equivalent to) the one offered by Misiak [9]. We shall also reprove the Cauchy-Schwarz inequality and give a necessary and sufficient condition for the equality. For related work, see another paper of Misiak [10].

2. An natural example of n-inner products and n-norms

We shall first show that we can actually define an n-inner product and accordingly an n-norm on any inner product space provided the dimension is sufficiently large.

Let $n \in \mathbf{N}$ and $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space of dimension $d \geq n$. Define the following function $\langle \cdot, \dots, \cdot | \cdot, \cdot \rangle$ on $X \times \cdots \times X$ ($n+1$ factors) by

$$\langle x_1, \dots, x_{n-1} | y, z \rangle := \begin{vmatrix} \langle x_1, x_1 \rangle & \dots & \langle x_1, x_{n-1} \rangle & \langle x_1, z \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle x_{n-1}, x_1 \rangle & \dots & \langle x_{n-1}, x_{n-1} \rangle & \langle x_{n-1}, z \rangle \\ \langle y, x_1 \rangle & \dots & \langle y, x_{n-1} \rangle & \langle y, z \rangle \end{vmatrix}.$$

Then one may check that this function satisfies the following five properties:

(I1) $\langle x_1, \dots, x_{n-1} | x_n, x_n \rangle \geq 0$; $\langle x_1, \dots, x_{n-1} | x_n, x_n \rangle = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(I2) $\langle x_1, \dots, x_{n-1} | x_n, x_n \rangle = \langle x_{i_1}, \dots, x_{i_{n-1}} | x_{i_n}, x_{i_n} \rangle$ for every permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$;

(I3) $\langle x_1, \dots, x_{n-1} | y, z \rangle = \langle x_1, \dots, x_{n-1} | z, y \rangle$;

(I4) $\langle x_1, \dots, x_{n-1} | y, \alpha z \rangle = \alpha \langle x_1, \dots, x_{n-1} | y, z \rangle$;

(I5) $\langle x_1, \dots, x_{n-1} | y, z + z' \rangle = \langle x_1, \dots, x_{n-1} | y, z \rangle + \langle x_1, \dots, x_{n-1} | y, z' \rangle$.

Accordingly, we can define $\|\cdot, \dots, \cdot\|$ on $X \times \cdots \times X$ (n factors) by

$$\|x_1, \dots, x_n\| := \langle x_1, \dots, x_{n-1} | x_n, x_n \rangle^{1/2},$$

that is,

$$\|x_1, \dots, x_n\| = \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{\frac{1}{2}}.$$

For $n = 1$, we know that $\| \cdot \|$ is a norm, while for $n = 2$, $\|\cdot, \cdot\|$ defines a 2-norm. Note further that for $n = 1$, $\|x_1\|$ gives the length of x_1, while for $n = 2$, $\|x_1, x_2\|$ represents the area of the parallelogram spanned by x_1 and x_2. One may also observe that, for $n = 3$ and $X = \mathbf{R}^3$, $\|x_1, x_2, x_3\|$ is nothing but the volume of the parallelepiped spanned by x_1 x_2 and x_3, that is,

$$\|x_1, x_2, x_3\| = |x_1 \cdot (x_2 \times x_3)|.$$

Thus, in general, $\|x_1, \dots, x_n\|$ can be interpreted as the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in X. Further, it satisfies the following four properties:

(N1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(N2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(N3) $\|x_1, \dots, x_{n-1}, \alpha x_n\| = |\alpha| \, \|x_1, \dots, x_n\|$;

(N4) $\|x_1, \dots, x_{n-1}, y + z\| \leq \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, z\|$.

Again, the first three properties are easy to see. To prove the last property or the triangle inequality, we need to establish the Cauchy-Schwarz inequality. Indeed, we have the following:

Fact 2.1 (The Cauchy-Schwarz Inequality). *For all $x_1, \dots, x_{n-1}, y, z \in X$, we have*

$$\langle x_1, \dots, x_{n-1} | y, z \rangle^2 \leq \|x_1, \dots, x_{n-1}, y\|^2 \|x_1, \dots, x_{n-1}, z\|^2, \tag{1}$$

and the equality holds if and only if $x_1, \dots, x_{n-1}, y, z$ are linearly dependent.

Proof. First observe that the inequality may be rewritten as

$$\begin{vmatrix} \langle x_1,\dots,x_{n-1}|y,y\rangle & \langle x_1,\dots,x_{n-1}|y,z\rangle \\ \langle x_1,\dots,x_{n-1}|z,y\rangle & \langle x_1,\dots,x_{n-1}|z,z\rangle \end{vmatrix} \ge 0,$$

which obviously holds since the matrix is positive semidefinite.

Next, suppose that we have the equality

$$\begin{vmatrix} \langle x_1,\dots,x_{n-1}|y,y\rangle & \langle x_1,\dots,x_{n-1}|y,z\rangle \\ \langle x_1,\dots,x_{n-1}|z,y\rangle & \langle x_1,\dots,x_{n-1}|z,z\rangle \end{vmatrix} = 0.$$

If $\langle x_1,\dots,x_{n-1}|y,y\rangle = 0$ or $\langle x_1,\dots,x_{n-1}|z,z\rangle = 0$, then $x_1,\dots,x_{n-1},y,z$ are linearly dependent. Otherwise, there exists a $\beta \ne 0$ such that

$$\langle x_1,\dots,x_{n-1}|y,z\rangle = \beta\langle x_1,\dots,x_{n-1}|y,y\rangle$$

and

$$\langle x_1,\dots,x_{n-1}|z,z\rangle = \beta\langle x_1,\dots,x_{n-1}|z,y\rangle.$$

Hence

$$\langle x_1,\dots,x_{n-1}|y,\beta y - z\rangle = 0 \ \text{ and } \ \langle x_1,\dots,x_{n-1}|z,\beta y - z\rangle = 0,$$

and so

$$\langle x_1,\dots,x_{n-1}|\beta y - z,\beta y - z\rangle = 0.$$

But this implies that $x_1,\dots,x_{n-1},\beta y - z$ are linearly dependent, and so are $x_1,\dots,x_{n-1}$, y, z.

Conversely, suppose that $x_1,\dots,x_{n-1},y,z$ are linearly dependent. If $x_1,\dots,x_{n-1}$ are linearly dependent, then the right-hand side of (1) equals zero and so does the left-hand side. So suppose that $x_1,\dots,x_{n-1}$ are linearly independent. Since the equation

$$\alpha_1 x_1 + \cdots + \alpha_{n-1}x_{n-1} + \beta y + \gamma z = 0$$

has a non-trivial solution, we must have β or $\gamma \ne 0$. Without loss of generality, assume that $\gamma \ne 0$ so that

$$z = a_1 x_1 + \cdots + a_{n-1}x_{n-1} + by$$

for some scalars $a_1,\dots,a_{n-1},b \in \mathbf{R}$. From its definition, we have $\langle x_1,\dots,x_{n-1}|y,x_k\rangle = \langle x_1,\dots,x_{n-1}|z,x_k\rangle = 0$ for each $k = 1,\dots,n-1$. Hence

$$\langle x_1,\dots,x_{n-1}|y,z\rangle = \langle x_1,\dots,x_{n-1}|y,by\rangle = b\langle x_1,\dots,x_{n-1}|y,y\rangle$$

and

$$\langle x_1,\dots,x_{n-1}|z,z\rangle = \langle x_1,\dots,x_{n-1}|by,by\rangle = b^2\langle x_1,\dots,x_{n-1}|y,y\rangle,$$

and therefore the equality follows. □

Moreover, as it can be predicted from our introductory observation, we have the following:

Fact 2.2. *The Cauchy-Schwarz inequality* (1) *is equivalent to*

$$\begin{vmatrix} \langle x_1,x_1\rangle & \dots & \langle x_1,y\rangle & \langle x_1,z\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle y,x_1\rangle & \dots & \langle y,y\rangle & \langle y,z\rangle \\ \langle z,x_1\rangle & \dots & \langle z,y\rangle & \langle z,z\rangle \end{vmatrix} \ge 0. \tag{2}$$

To prove Fact 2.2, we shall use some facts about symmetric matrices. For 2×2 matrices $A_2 = [a_{ij}]$, we have $|A_2| = a_{11}a_{22} - a_{12}a_{21}$. Particularly, when $a_{12} = a_{21}$, we have $|A_2| = a_{11}a_{22} - a_{12}^2$, and so, for instance, $|A_2| \ge 0$ is equivalent to $a_{12}^2 \le a_{11}a_{22}$. For larger matrices, we have the following:

Fact 2.3. *Suppose that $A_N = [a_{ij}]$ is an $N \times N$ matrix ($N \geq 3$) such that the determinants of the sub-matrices $A_k = [a_{ij}]_{i,j=1,\dots,k}$ ($k = 1, \dots, N-2$) are all non-zero. Then we have*

$$|A_{N-2}|\,|A_N| = |M_{N-1,N-1}|\,|M_{NN}| - |M_{N-1,N}|\,|M_{N,N-1}|, \tag{3}$$

where M_{ij} denotes the $(N-1)\times(N-1)$ matrix obtained from A_N by deleting the i-th row and j-th column. In particular, if A_N is symmetric, then

$$|A_{N-2}|\,|A_N| = |M_{N-1,N-1}|\,|M_{NN}| - |M_{N-1,N}|^2.$$

Proof. The proof is elementary. One can just use Gaussian elimination to reduce A_N into the following form

$$\begin{bmatrix} * & * & \dots & * & * & * \\ 0 & * & \dots & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & * & * & * \\ 0 & 0 & \dots & 0 & * & * \\ 0 & 0 & \dots & 0 & * & * \end{bmatrix},$$

and then compare both sides of (3). □

We are now ready to prove Fact 2.2.

Proof of Fact 2.2. First note that the Cauchy-Schwarz inequality says that

$$\begin{vmatrix} \langle x_1, x_1\rangle & \dots & \langle x_1, x_{n-1}\rangle & \langle x_1, z\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle x_{n-1}, x_1\rangle & \dots & \langle x_{n-1}, x_{n-1}\rangle & \langle x_{n-1}, z\rangle \\ \langle y, x_1\rangle & \dots & \langle y, x_{n-1}\rangle & \langle y, z\rangle \end{vmatrix}^2 \leq$$

$$\begin{vmatrix} \langle x_1, x_1\rangle & \dots & \langle x_1, x_{n-1}\rangle & \langle x_1, y\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle x_{n-1}, x_1\rangle & \dots & \langle x_{n-1}, x_{n-1}\rangle & \langle x_{n-1}, y\rangle \\ \langle y, x_1\rangle & \dots & \langle y, x_{n-1}\rangle & \langle y, y\rangle \end{vmatrix} \begin{vmatrix} \langle x_1, x_1\rangle & \dots & \langle x_1, x_{n-1}\rangle & \langle x_1, z\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle x_{n-1}, x_1\rangle & \dots & \langle x_{n-1}, x_{n-1}\rangle & \langle x_{n-1}, z\rangle \\ \langle z, x_1\rangle & \dots & \langle z, x_{n-1}\rangle & \langle z, z\rangle \end{vmatrix}.$$

If $x_1, \dots, x_{n-1}$ are linearly dependent, then both (1) and (2) become the equality $0 = 0$. So suppose that $x_1, \dots, x_{n-1}$ are linearly independent. Then $|[\langle x_i, x_j\rangle]_{i,j=1,\dots,k}| > 0$ for each $k = 1, \dots, n-1$, and so, by Fact 2.3, the inequality is equivalent to

$$\begin{vmatrix} \langle x_1, x_1\rangle & \dots & \langle x_1, y\rangle & \langle x_1, z\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle y, x_1\rangle & \dots & \langle y, y\rangle & \langle y, z\rangle \\ \langle z, x_1\rangle & \dots & \langle z, y\rangle & \langle z, z\rangle \end{vmatrix} \geq 0,$$

since the $(n+1)\times(n+1)$ matrix is symmetric. □

3. A definition of n-inner products and n-norms

Inspired by our observations in the previous sections, we shall now generalize the concepts of inner products and 2-inner products as well as norms and 2-norms to those of n-inner products and n-norms for any $n \in \mathbf{N}$.

Let $n \in \mathbf{N}$ and X be a real vector space of dimension $d \geq n$. A function $\langle \cdot, \dots, \cdot | \cdot, \cdot \rangle$ on $X \times \cdots \times X$ ($n+1$ factors) satisfying the five properties (I1) – (I5) listed in §2 is called an *n-inner product* on X, and the pair $(X, \langle \cdot, \dots, \cdot | \cdot, \cdot \rangle)$ is called an *n-inner product space.*

Meanwhile, a function $\|\cdot,\ldots,\cdot\|$ on $X\times\cdots\times X$ (n factors) satisfying the four properties (N1) – (N4) listed in §2 is called an *n-norm* on X, and the pair $(X,\|\cdot,\ldots,\cdot\|)$ is called an *n-normed space*.

Note that our definition of n-inner products is slightly simpler than Misiak's [9]. To see that it is equivalent to Misiak's, one only needs to verify that

$$\langle x_1,\ldots,x_{n-1}|y,z\rangle = \langle x_{i_1},\ldots,x_{i_{n-1}}|y,z\rangle$$

for every permutation $(i_1,\ldots,i_{n-1})$ of $(1,\ldots,n-1)$. But this will follow easily from the property (I2) and the polarization identity

$$\langle x_1,\ldots,x_{n-1}|y,z\rangle = \frac{1}{4}\big[\langle x_1,\ldots,x_{n-1}|y+z,y+z\rangle - \langle x_1,\ldots,x_{n-1}|y-z,y-z\rangle\big].$$

The following theorem confirms that Fact 2.1 is true in any n-inner product space.

Theorem 3.1 (The Cauchy-Schwarz Inequality). *Let $(X,\langle\cdot,\ldots,\cdot|\cdot,\cdot\rangle)$ be an n-inner product space. Then we have*

$$\langle x_1,\ldots,x_{n-1}|y,z\rangle^2 \le \langle x_1,\ldots,x_{n-1}|y,y\rangle\langle x_1,\ldots,x_{n-1}|z,z\rangle,$$

and the equality holds if and only if $x_1,\ldots,x_{n-1},y,z$ are linearly dependent.

Proof. The proof goes like that of Fact 2.1. The only difference is when we have to prove that, if $x_1,\ldots,x_{n-1},y,z$ are linearly dependent, then the equality holds. We note here that, for each $k=1,\ldots,n-1$, we have $\langle x_1,\ldots,x_{n-1}|x_k,x_k\rangle = 0$ and consequently

$$\langle x_1,\ldots,x_{n-1}|y,x_k\rangle^2 \le \langle x_1,\ldots,x_{n-1}|y,y\rangle\langle x_1,\ldots,x_{n-1}|x_k,x_k\rangle = 0,$$

which implies that $\langle x_1,\ldots,x_{n-1}|y,x_k\rangle = 0$. The same is true when y is replaced by z. Thus, if $z = a_1x_1+\cdots+a_{n-1}x_{n-1}+by$ for some $a_1,\ldots,a_{n-1},\ b\in\mathbf{R}$, then

$$\langle x_1,\ldots,x_{n-1}|y,z\rangle = \langle x_1,\ldots,x_{n-1}|y,by\rangle = b\langle x_1,\ldots,x_{n-1}|y,y\rangle$$

and

$$\langle x_1,\ldots,x_{n-1}|z,z\rangle = \langle x_1,\ldots,x_{n-1}|by,by\rangle = b^2\langle x_1,\ldots,x_{n-1}|y,y\rangle,$$

and hence the equality follows. □

Corollary 3.2. *On an n-inner product space $(X,\langle\cdot,\ldots,\cdot|\cdot,\cdot\rangle)$, the following function*

$$\|x_1,\ldots,x_n\| := \langle x_1,\ldots,x_{n-1}|x_n,x_n\rangle^{\frac{1}{2}}$$

defines an n-norm. In particular, the triangle inequality

$$\|x_1,\ldots,x_{n-1},y+z\| \le \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|$$

holds for all $x_1,\ldots,x_{n-1},y,z\in X$.

Corollary 3.3. *Let $(X,\langle\cdot,\ldots,\cdot|\cdot,\cdot\rangle)$ be an n-inner product space. If $x_1,\ldots,x_{n-1},y,z$ are linearly dependent in X, then*

$$\|x_1,\ldots,x_{n-1},y+z\| = \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|$$

or

$$\|x_1,\ldots,x_{n-1},y-z\| = \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|.$$

Conversely, if one of the above two equalities holds, then $x_1,\ldots,x_{n-1},y,z$ are linearly dependent in X.

Proof. Suppose that $x_1, \dots, x_{n-1}, y, z$ are linearly dependent in X. As before, we may assume that $z = a_1 x_1 + \dots + a_{n-1} x_{n-1} + by$ for some $a_1, \dots, a_{n-1}, b \in \mathbf{R}$. If $b \geq 0$, then we have

$$\begin{aligned} \|x_1, \dots, x_{n-1}, y + z\| &= \|x_1, \dots, x_{n-1}, (1 + b)y\| \\ &= (1 + b)\,\|x_1, \dots, x_{n-1}, y\| \\ &= \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, by\| \\ &= \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, z\|. \end{aligned}$$

If $b < 0$, then we have

$$\begin{aligned} \|x_1, \dots, x_{n-1}, y - z\| &= \|x_1, \dots, x_{n-1}, (1 - b)y\| \\ &= (1 - b)\,\|x_1, \dots, x_{n-1}, y\| \\ &= \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, by\| \\ &= \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, z\|. \end{aligned}$$

Therefore one of the two equalities must hold.

Conversely, without loss of generality, suppose that the equality

$$\|x_1, \dots, x_{n-1}, y + z\| = \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, z\|$$

holds. Squaring both sides, we get

$$\langle x_1, \dots, x_{n-1} | y, z\rangle = \|x_1, \dots, x_{n-1}, y\|\,\|x_1, \dots, x_{n-1}, z\|.$$

By Theorem 3.1, $x_1, \dots, x_{n-1}, y, z$ must be linearly dependent. □

The notion of n-normed spaces may be of independent interest. In an n-normed space $(X, \|\cdot, \dots, \cdot\|)$, we have, for instance, $\|x_1, \dots, x_n\| \geq 0$ and $\|x_1, \dots, x_{n-1}, x_n\| = \|x_1, \dots, x_{n-1}, x_n + \alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}\|$ for all $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_{n-1} \in \mathbf{R}$.

As in a 2-normed space, a sequence $x(k)$ in an n-normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be *convergent* to some $x \in X$ if $\lim_{k\to\infty} \|x_1, \dots, x_{n-1}, x(k) - x\| = 0$ for all $x_1, \dots, x_{n-1} \in X$. In such a case, we write $\lim_{k\to\infty} x(k) = x$ and call x the *limit* of $x(k)$. One may then show that, when $\lim_{k\to\infty} x(k)$ exists, it must be unique.

Many results in 2-normed spaces, such as fixed point theorems (see [1], [7] and [8]), may have analogues in n-normed spaces.

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