*Scientiae Mathematicae Japonicae Online*, **Vol. 4**, (2001), 55–66 

# ON SEMISIMPLE $U(\sigma)$-ALGEBRAS*

Yoshiaki TANIGUCHI and Kenji ATSUYAMA

Received April 18, 2000; revised April 24, 2001

Abstract. The $U(\sigma)$-algebras are new triple systems, which are obtained by extending the concept of Freudenthal-Kantor triple systems introduced by I.L. Kantor [9] and K. Yamaguti [12]. In this paper, for $U(\sigma)$-algebras we define the semisimplicity and radicals and show that any semisimple $U(\sigma)$-algebra is decomposed into the direct sum of $\sigma$-simple ideals. We also give a formula which describes a relationship between the trace form of a semisimple $U(\sigma)$-algebra $U$ and the Killing form of the Lie algebra associated with $U$.

## Introduction

A triple system $U$ with trilinear product $(xyz)$ is called generalized Jordan triple system if the identity $(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz))$ is valid for all $u, v, x, y, z \in U$. This definition was given by I.L. Kantor [9]. Starting from a given generalized Jordan triple system $U$, he constructed a certain graded Lie algebra $\mathfrak{L} = \sum_{i=-\infty}^{\infty} U_i$ which is now called the Kantor's algebra of $U$. On the other hand, B.N. Allison [1] and W. Hein [6], [7] gave the concept of $\mathfrak{J}$-ternary algebra. It was based on results of H. Freudenthal [5] about the geometry of exceptional Lie groups. Reforming the axioms of $\mathfrak{J}$-ternary algebra, K. Yamaguti [12] defined $U(\varepsilon)$-algebras for $\varepsilon = \pm 1$, and later, he called them Freudenthal-Kantor triple systems. In our paper [4], we extended the concept of Freudenthal-Kantor triple systems by replacing $\varepsilon = \pm 1$ with automorphisms $\sigma$'s of triple systems, and constructed a graded Lie algebra of the 2nd order from a $U(\sigma)$-algebra via an Lie triple system. A $U(\mathrm{Id})$-algebra is nothing but generalized Jordan triple system of the 2nd order and a $U(-\mathrm{Id})$-algebra particularly is called a Freudenthal triple system. Our concern is the semisimplicity of $U(\sigma)$-algebra. N. Kamiya [8] defined the radical of a Freudenthal-Kantor triple system and studied about the semisimplicity of Freudenthal-Kantor triple systems. In this paper, we will define the semisimplicity and the radical of any $U(\sigma)$-algebra and show that any semisimple $U(\sigma)$-algebra is decomposed into the direct sum of $\sigma$-simple ideals (Theorem 2.8). Our next concern is to generalize some results of H. Asano and S. Kaneyuki [3] on generalized Jordan triple systems to the case of $U(\sigma)$-algebras. We introduced the tace form $\gamma$ of a $U(\sigma)$-algebra in [4]. We give a formula which describes a relationship between the trace form $\gamma$ of the $U(\sigma)$-algebra $U$ and the Killing form of the graded Lie algebra $\mathfrak{L}$ associated with $U$ (Theorem 3.2).

Throughout this paper, it is assumed that any vector space is finite dimensional vector space over a field of characteristic different from two.

---

2000 *Mathematics Subject Classification.* 17A40, 17B70, 17C10.

*Key words and phrases.* generalized Jordan triple system, Lie triple system, graded Lie algebra.

*Research supported in part by the Grand in Aid for Fundamental Scientific Research of Ministry of Education, Science and Culture (C) 09640078.

## §1. $U(\sigma)$-algebras and graded Lie algebras

Let $U$ be a vector space over a field $F$ and let $B : U \times U \times U \longrightarrow U$ be a trilinear mapping. Then the pair $(U, B)$ (or $U$) is called a *triple system* over $F$. We shall often write $(xyz)$ (or $[xyz]$) in stead of $B(x, y, z)$. For any subspaces $V_i$ $(i = 1, 2, 3)$ of $U$, we denote by $(V_1V_2V_3)$ the subspace spanned by all elements of the form $(x_1x_2x_3)$ for $x_i \in V_i$. A subspace $I$ of $U$ is called an *ideal* if $(UUI) + (UIU) + (IUU) \subset I$ is valid. The whole space $U$ and $\{0\}$ are called the trivial ideals. A triple system $U$ is said to be *simple* if $(UUU) \neq \{0\}$ and $U$ has no non-trivial ideal. An endomorphism $D$ of $U$ is called a *derivation* if $D(xyz) = (Dx\,y\,z) + (x\,Dy\,z) + (x\,y\,Dz),\ x, y, z \in U$. We denote by $\mathfrak{D}(U)$ the set of all derivations of $U$. $\mathfrak{D}(U)$ is a Lie algebra under the usual Lie product: $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$. For $x, y \in U$, let us define the endomorphisms $L(x, y),\ R(x, y),\ K(x, y)$ on $U$ by

$$L(x, y)z := (xyz), \quad R(x, y)z := (zxy), \quad K(x, y)z := (xzy) - (yzx).$$

A *Lie triple system* (or *LTS* simply) is a triple system $T$ with trilinear product $[xyz]$ satisfying the following conditions for $u, v, x, y, z \in T$:

(L1) $[xxy] = 0$,
(L2) $[xyz] + [yzx] + [zxy] = 0$,
(L3) $[uv[xyz]] = [[uvx]yz] + [x[uvy]z] + [xy[uvz]]$.

The condition (L3) shows that $L(x, y)$ is a derivation of $T$, which is called an *inner derivation*. We denote by $L(T, T)$ the space spanned by all inner derivations of $T$. A vector space direct sum

$$\mathfrak{L} = T \oplus L(T, T)$$

becomes a Lie algebra with respect to the product

$$[x_1 + D_1, x_2 + D_2] = D_1x_2 - D_2x_1 + [D_1, D_2] + L(x_1, x_2),$$

where $x_i \in T,\ D_i \in L(T, T)$ $(i = 1, 2)$. The Lie algebra $\mathfrak{L}$ is called the *standard enveloping Lie algebra* of $T$.

**Definition.** A triple system $(U, B)$ is called a *$U(\sigma)$-algebra* if there exists an automorphism $\sigma$ of $(U, B)$ satisfying the following identities:

(U1) $[L(u, v), L(x, y)] = L(L(u, v)x, y) - L(x, L(v, \sigma u)y)$,
(U2) $K(K(u, v)x, y) = L(y, x)K(u, v) + K(u, v)L(x, \sigma y)$,

where $u, v, x, y \in U$.

The $U(\pm\mathrm{Id})$-algebras are nothing but the *Freudenthal-Kantor triple systems* $U(\varepsilon),\ \varepsilon = \pm 1$ (cf. [12]), particularly, the $U(\mathrm{Id})$-algebras are the *generalized Jordan triple systems* (or *GJTS* simply) *of the 2nd order* (cf. [9]) and the $U(-\mathrm{Id})$-algebras are the *Freudenthal triple systems* (cf. [5]).

Let $(U, B)$ be a GJTS of the 2nd order. A non-singular linear transformation $\varphi$ is called a *weak automorphism* of $(U, B)$ if there exists a linear transformation $\overline{\varphi}$ of $U$ such that

$$\varphi B(x, y, z) = B(\varphi x, \overline{\varphi} y, \varphi z), \quad \overline{\varphi} B(x, y, z) = B(\overline{\varphi} x, \varphi y, \overline{\varphi} z).$$

For a weak automorphism $\varphi$ of $(U, B)$, we define a new triple product in $U$ by

$$B_\varphi(x, y, z) := B(x, \varphi y, z).$$

Then $(U, B_\varphi)$ becomes a $U(\sigma)$-algebra for $\sigma = (\overline{\varphi}\varphi)^{-1}$ and is called the *$\varphi$-modification* of $(U, B)$ (cf. [4]). The notion of $\varphi$-modification was defined by H. Asano [2] for an involutive automorphism $\varphi$ of a GJTS $(U, B)$. In this case, the $\varphi$-modification is also a GJTS of the 2nd order.

Let $\mathbb{H}$ be the set of all quaternion numbers and define a triple product in $\mathbb{H}$ by

$$B(x, y, z) := x\overline{y}z + z\overline{y}x - y\overline{x}z,$$

where $\overline{x}$ denotes the conjugate quaternion of $x$. Then it is easy to verify that the triple system $(\mathbb{H}, B)$ is a GJTS of the 2nd order. Moreover, it is easily seen that the mapping $\varphi : x \mapsto ax$ is an automorphism of $(\mathbb{H}, B)$ for a fixed quaternion number $a$ such that $|a| = 1$. Therefore $(\mathbb{H}, B_\varphi)$ becomes a $U(\sigma)$-algebra for $\sigma = \varphi^{-2}$. If $a = \pm 1$, $(\mathbb{H}, B_\varphi)$ is a GJTS of the 2nd order and if $a$ is a pure quaternion number, $(\mathbb{H}, B_\varphi)$ is an FTS.

Let $U$ be a $U(\sigma)$-algebra, and let us consider the vector space direct sum

$$T = T(U) = U \oplus U.$$

An element $a \oplus x$ of $T(U)$ is also denoted as $\begin{pmatrix} a \\ x \end{pmatrix}$ in column vector form. Define a trilinear product in $T(U)$ by

$$\left[\begin{pmatrix} a \\ x \end{pmatrix}\begin{pmatrix} b \\ y \end{pmatrix}\begin{pmatrix} c \\ z \end{pmatrix}\right] := \begin{pmatrix} L(a,y)c - L(b,x)c + K(a,b)z \\ K(x,y)\sigma c + L(x,\sigma b)z - L(y,\sigma a)z \end{pmatrix} \tag{1.1}$$

$$= \begin{pmatrix} L(a,y) - L(b,x) & K(a,b) \\ K(x,y)\sigma & L(x,\sigma b) - L(y,\sigma a) \end{pmatrix}\begin{pmatrix} c \\ z \end{pmatrix},$$

where $a, b, c, x, y, z \in U$. Then $T(U)$ becomes an LTS with respect to this product ([4] Proposition 2.2). The Lie triple system $T(U)$ is called the *LTS associated with* $U$. By $\mathfrak{L}(U)$ we denote the standard enveloping Lie algebra of the LTS $T(U)$. Let $\mathfrak{L}_i$ $(i = 0, \pm 1, \pm 2)$ be subspaces of $\mathfrak{L}(U)$ as follows:

$\mathfrak{L}_{-2}$ is the subspace spanned by all operators $L(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix})$,

$\mathfrak{L}_{-1}$ is the subspace spanned by all elements $a \oplus 0 \in T(U)$,

$\mathfrak{L}_0$ is the subspace spanned by all operators $L(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix})$,

$\mathfrak{L}_1$ is the subspace spanned by all elements $0 \oplus x \in T(U)$,

$\mathfrak{L}_2$ is the subspace spanned by all operators $L(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix})$.

Then it follows that

$$\mathfrak{L}(U) = \sum_{i=-2}^{2} \mathfrak{L}_i, \quad [\mathfrak{L}_i, \mathfrak{L}_j] \subset \mathfrak{L}_{i+j},$$

that is, $\mathfrak{L}(U)$ is a graded Lie algebra (or GLA simply) of the 2nd order, which is called the *GLA associated with* $U$. We have obviously

$$L(T, T) = \mathfrak{L}_{-2} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_2, \quad T(U) = \mathfrak{L}_{-1} \oplus \mathfrak{L}_1. \tag{1.2}$$

Define a linear transformation $\theta$ of $T(U)$ by

$$\theta\begin{pmatrix} a \\ x \end{pmatrix} = \begin{pmatrix} \sigma^{-1}x \\ a \end{pmatrix}.$$

It is easy to check that $\theta$ is an automorphism of the LTS $T(U)$. By putting

$$\theta(L(X,Y)) = L(\theta(X),\theta(Y)),\ X,Y \in T(U),$$

this automorphism can be extended to an automorphism $\theta$ (we use the same symbol) of $\mathfrak{L}(U)$. Then the automorphism $\theta$ is grade-reversing, that is, $\theta(\mathfrak{L}_i) = \mathfrak{L}_{-i}$ $(i = 0, \pm 1, \pm 2)$.

**Proposition 1.1.** If a $U(\sigma)$-algebra $U$ satisfies $(UUU) = U$, then $[T(U)T(U)T(U)] = T(U)$ is valid.

**Proof.** Since $(UUU) = U$, we have

$$[\mathfrak{L}_0, \mathfrak{L}_{-1}] = \{[L(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}), \begin{pmatrix} c \\ 0 \end{pmatrix}]\}_{span} = \{\begin{pmatrix} L(a,y)c \\ 0 \end{pmatrix}\}_{span} = \begin{pmatrix} (UUU) \\ 0 \end{pmatrix} = \begin{pmatrix} U \\ 0 \end{pmatrix} = \mathfrak{L}_{-1}.$$

Using this equality, also we have

$$[\mathfrak{L}_0, \mathfrak{L}_1] = [\theta(\mathfrak{L}_0), \theta(\mathfrak{L}_{-1})] = \theta([\mathfrak{L}_0, \mathfrak{L}_{-1}]) = \theta(\mathfrak{L}_{-1}) = \mathfrak{L}_1.$$

Hence we get

$$[L(T(U),T(U)),T(U)] \supset [\mathfrak{L}_0, \mathfrak{L}_{-1} \oplus \mathfrak{L}_1] = \mathfrak{L}_{-1} \oplus \mathfrak{L}_1 = T(U).$$

Since the converse inclusion is clear, we obtain $[L(T(U),T(U)),T(U)] = T(U)$. □

Let $U$ be a $U(\sigma)$-algebra. The bilinear form $\gamma$ on $U$ defined by

$$\gamma(x,y) := \frac{1}{2}\mathrm{Tr}\{2R(y,x) + 2R(\sigma x,y) - L(x,y) - L(y,\sigma x)\} \tag{1.3}$$

is called the *trace form* of $U$ [4]. Let $\alpha$ and $\beta$ be the Killing forms of $\mathfrak{L}(U)$ and $T(U)$ respectively. It is well known (see [11]) that $\alpha(X,Y) = 2\beta(X,Y)$ for $X,Y \in T(U)$. From [4] Lemma 2.3, we have

$$\alpha(\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix}) = 2\beta(\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix}) = 2\{\gamma(a,y) + \gamma(b,x)\}, \tag{1.4}$$

where $a,b,x,y \in U$. We note that the trace form $\gamma$ is neither symmetric nor anti-symmetric except in case $\sigma = \pm\mathrm{Id}$. But the right non-degeneration of $\gamma$ is equivalent to the left one. Therefore we say that $\gamma$ is *non-degenerate* if $\gamma$ is right non-degenerate or left non-degenerate. From (1.4) and Theorem 2.1 in [11], the non-generations of $\alpha$, $\beta$ and $\gamma$ are equivalent each other. The following lemma will be needed later on.

**Lemma 1.2** ([4] Lemma 2.4)**.** For any $u,v,x,y \in U$, the following identities hold:

(1.5) $\gamma(L(u,v)x,y) = \gamma(x,L(v,\sigma u)y)$,
(1.6) $\gamma(R(u,v)x,y) = \gamma(x,R(\sigma v,u)y)$,

(1.7) $\gamma(y,x)=\gamma(\sigma^{-1}x,y)=\gamma(x,\sigma y)$.

**Proposition 1.3.** If the trace form $\gamma$ of a $U(\sigma)$-algebra $U$ is identically zero, then the GLA $\mathfrak{L}(U)$ associated with $U$ is solvable.

**Proof.** Put $T=T(U)$ and $\mathfrak{L}=\mathfrak{L}(U)$. From (1.4), the Killing form $\alpha$ of $\mathfrak{L}$ is identically zero on $T$. Since $L(T,T)=[T,T]$, every element $D\in L(T,T)$ can be written as $D=\sum_i [X_i,Y_i]$ $(X_i,Y_i\in T)$. Then, for an arbitrary element $D'\in L(T,T)$, we have

$$\alpha(D,D')=\sum_i \alpha([X_i,Y_i],D')=\sum_i \alpha(X_i,[Y_i,D'])=0.$$

This means that $\alpha$ is identically zero on $L(T,T)$. Since $\mathfrak{L}=\sum \mathfrak{L}_i$ is a GLA, we have $\alpha(\mathfrak{L}_i,\mathfrak{L}_j)=0$ if $i+j\neq 0$. Hence we have $\alpha(L(T,T),T)=0$. Consequently, $\alpha$ is identically zero on $\mathfrak{L}$. Therefore $\mathfrak{L}$ is solvable. □

## §2. The semisimplicity of a $U(\sigma)$-algebras

In this section, we consider about the semisiplicity of a $U(\sigma)$-algebras. For this purpose, we define the radical of a $U(\sigma)$-algebra (cf. [10],[8]). Throughout this section, we assume that the base field is of characteristic zero.

For two ideals $I,J$ of $U$, we put

$$I*J=(IJU)+(JIU)+(IUJ)+(JUI)+(UIJ)+(UJI). \tag{2.1}$$

It is clear that $I*J=J*I$.

**Lemma 2.1.** If $I,J$ are $\sigma$-invariant ideals of a $U(\sigma)$-algebra $U$, then so is $I*J$.

**Proof.** By applying (U1) to an element $z$, we have

$$(uv(xyz))=(xy(uvz))+((uvx)yz)-(x(v\,\sigma u\,y)z). \tag{2.2}$$

In (2.2), let $u,v,z\in U,\ x\in I$ and $y\in J$. Since $I,J$ are ideals of $U$, we get

$$(uv(xyz))\in(IJ(UUU))+(((UUI)JU)+(I(UUJ)U)\subset(IJU)\subset I*J.$$

This means that $(UU(IJU))\subset I*J$. Similarly we can obtain that

$$(UU(IUJ))\subset I*J,\ (UU(UIJ))\subset I*J.$$

Permuting $I$ and $J$, we have

$$(UU(JIU))\subset I*J,\ (UU(JUI))\subset I*J,\ (UU(UJI))\subset I*J.$$

Consequently we get

$$(U\,U\,I*J)\subset I*J.$$

Equation (2.2) is rewritten as

$$((uvx)yz)=(uv(xyz))-(xy(uvz))+(x(v\,\sigma u\,y)z).$$

Since $I, J$ are $\sigma$-invariant, using this identity, we have

$$((IUJ)UU) \subset I*J,\ ((JUI)UU) \subset I*J,\ ((UIJ)UU) \subset I*J,\ ((UJI)UU) \subset I*J.$$

From (U2), we get

$$\begin{aligned}(2.3)\quad ((uxv)zy) = &(yz(uxv)) + (yx(uzv)) + (u(x\,\sigma y\,z)v) + ((vxu)zy)\\ &-(yz(vxu)) - (yx(vzu)) - (v(x\,\sigma y\,z)u).\end{aligned}$$

In (2.3), let $v, y, z \in U,\ u \in I$ and $x \in J$, then we have

$$\begin{aligned}((uxv)zy) \in &(UU(IJU)) + (UJ(IUU)) + (I(JUU)U) + ((UJI)UU) + (UU(UJI))\\ &+(UJ(UUI)) + (U(JUU)I)\\ &\subset (UU(I*J)) + (UJI) + (IJU) + ((I*J)UU) + (UU(I*J)) \subset I*J.\end{aligned}$$

This means that $((IJU)UU) \subset I*J$. Permuting $I$ and $J$, we get $((JIU)UU) \subset I*J$. Therefore we have

$$((I*J)UU) \subset I*J.$$

Again rewriting (2.2), we have

$$(x(v\,\sigma u\,y)z) = (xy(uvz)) + ((uvx)yz) - (uv(xyz)).$$

In this identity, let $x, y, z \in U$ $v \in I$ and $u \in J$, then we have

$$\begin{aligned}(x(v\,\sigma u\,y)z) \in &(UU(JIU)) + ((JIU)UU) + (JI(UUU))\\ &\subset (UU(I*J)) + ((I*J)UU) + (JIU) \subset I*J.\end{aligned}$$

Since $\sigma(J) = J$, this means that $(U(IJU)U) \subset I*J$. Permuting $I$ and $J$, we have $(U(JIU)U) \subset I*J$. Similarly we can show that

$$(U(IUJ)U) \subset I*J,\ (U(JUI)U) \subset I*J,\ (U(UIJ)U) \subset I*J,\ (U(UJI)U) \subset I*J.$$

The above means that $(U(I*J)U) \subset I*J$ is valid. Therefore $I*J$ is an ideal of $U$. It is clear that $I*J$ is $\sigma$-invariant. □

For an $\sigma$-invariant ideal $I$ of $U$, we define a sequence of ideals of $U$ by

$$(2.4)\qquad I^{(0)} = I,\quad I^{(n)} = I^{(n-1)} * I^{(n-1)}\ (n \geq 1).$$

The ideal $I$ is called *solvable in* $U$ if there exists an integer $n$ such that $I^{(n)} = \{0\}$.

Let $U$ be a $U(\sigma)$-algebra and $I$ an ideal of $U$. Then the quotient space $U/I$ becomes a triple system with respect to the trilinear product $(\overline{x}\,\overline{y}\,\overline{z}) := \overline{(xyz)}$, where $\overline{x} = x+I,\ (x \in U)$. If $I$ is $\sigma$-invariant, then the mapping $\overline{\sigma} : \overline{x} \longmapsto \overline{\sigma(x)}$ is also an automorphism of $U/I$ satisfying the conditions (U1) and (U2). Hence the quotient space $U/I$ is a $U(\overline{\sigma})$-algebra.

**Proposition 2.2.** Let $U$ be a $U(\sigma)$-algebra and $I, J$ $\sigma$-invariant ideals of $U$.
(1) If $I$ and $U/I$ are solvable, then $U$ is solvable.
(2) If $I, J$ are solvable, then so is $I + J$.

**Proof.** (1) There exists an integer $n$ such that $(U/I)^{(n)} = \{0\}$. By $\pi$, we denote the canonical homomorphism of $U$ onto $U/I$. Then

$$\pi(U^{(n)}) = (U/I)^{(n)} = \{0\},$$

and therefore

$$U^{(n)} \subset I = \operatorname{Ker} \pi.$$

Since $I^{(m)} = \{0\}$ for some $m$, we have

$$U^{(n+m)} = (U^{(n)})^{(m)} \subset I^{(m)} = \{0\},$$

therefore $U$ is solvable.

(2) Obviously $I + J$ is a $\sigma$-invariant ideal. By the mathematical induction, it is easily seen that

$$(I + J)^{(n)} \subset I^{(n)} + J^{(n)} + I \cap J. \tag{2.5}$$

Since $I, J$ are solvable, we have $I^{(n)} = J^{(n)} = \{0\}$ for large enough $n$. Hence

$$(I + J)^{(n)} \subset I \cap J \subset I,$$

and

$$(I + J)^{(2n)} \subset I^{(n)} = \{0\}.$$

Therefore $I + J$ is solvable. □

From this proposition, we see that for any finite dimensional $U(\sigma)$-algebra $U$ there exists the unique maximal solvable $\sigma$-invariant ideal, which is called the *radical* of $U$. We denote it by $Rad(U)$. A $U(\sigma)$-algebra $U$ is said to be *semisimple* if $Rad(U) = \{0\}$.

**Proposition 2.3.** For any $U(\sigma)$-algebra $U$, the $U(\overline{\sigma})$-algebra $U/Rad(U)$ is semisimple.

**Proof.** Let $\pi$ be the canonical homomorphism of $U$ onto $U/Rad(U)$, and put $R = \pi^{-1}(Rad(U/Rad(U)))$. Obviously $R$ is an ideal of $U$ containing $Rad(U)$. Moreover $R$ is $\sigma$-invariant since $\pi \circ \sigma = \overline{\sigma} \circ \pi$. Since $R/Rad(U) = \pi(R) = Rad(U/Rad(U))$, $R/Rad(U)$ is solvable in $R/Rad(U)$. Since $Rad(U)$ is also solvable, from Proposition 2.2 (1), $R$ is solvable. Therefore we get $R \subset Rad(U)$. Hence $Rad(U/Rad(U)) = R/Rad(U) = \{0\}$. Thus $U/Rad(U)$ is semisimple. □

In an LTS $T$, by conditions (L1) and (L2), a subspace $A$ is an ideal of $T$ if and only if $[ATT] \subset A$. The derived series of an ideal $A$ of $T$ is defined by

$$A^{(0)} = A,\ A^{(n)} = [A^{(n-1)} T A^{(n-1)}]\ (n = 1, 2, 3, \cdots).$$

**Lemma 2.4.** Let $U$ be a $U(\sigma)$-algebra, and let $T(U)$ be the LTS associated with $U$. If $I$ is an $\sigma$-invariant ideal of $U$, then $I \oplus I$ is an ideal of $T(U)$. Furthermore the following relation is valid for any positive integer $n$:

$$(I \oplus I)^{(n)} = I^{(n)} \oplus I^{(n)}. \tag{2.6}$$

**Proof.** Since $I$ is $\sigma$-invariant, by (1.1) we have

$$\left[ \begin{pmatrix} I \\ I \end{pmatrix} \begin{pmatrix} U \\ U \end{pmatrix} \begin{pmatrix} U \\ U \end{pmatrix} \right] \subset \begin{pmatrix} (IUU) + (UIU) + (UUI) \\ (IUU) + (UUI) + (UIU) \end{pmatrix} \subset \begin{pmatrix} I \\ I \end{pmatrix}.$$

Therefore $I \oplus I$ is an ideal of $T$. We will prove (2.6) by induction on $n$. From (1.1), we have

$$\begin{pmatrix} (xyz) \\ 0 \end{pmatrix} = \left[ \begin{pmatrix} x \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} \right].$$

Hence, using (L1) and (L2), we have

$$\begin{pmatrix} (IIU) \\ 0 \end{pmatrix} \subset \left[\begin{pmatrix} I \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ I \end{pmatrix}\begin{pmatrix} U \\ 0 \end{pmatrix}\right] \subset \left[\begin{pmatrix} I \\ I \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}\begin{pmatrix} U \\ U \end{pmatrix}\right] \subset \left[\begin{pmatrix} I \\ I \end{pmatrix}\begin{pmatrix} U \\ U \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}\right],$$

that is, $(IIU) \oplus \{0\} \subset (I \oplus I)^{(1)}$. Similarly we obtain $(IUI) \oplus \{0\} \subset (I \oplus I)^{(1)}$ and $(UII) \oplus \{0\} \subset (I \oplus I)^{(1)}$. Consequently we have $I^{(1)} \oplus \{0\} \subset (I \oplus I)^{(1)}$. Since $\sigma(I) = I$ and

$$\begin{pmatrix} 0 \\ (x\,\sigma y\,z) \end{pmatrix} = \left[\begin{pmatrix} 0 \\ x \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ z \end{pmatrix}\right],$$

similarly we get $\{0\} \oplus I^{(1)} \subset (I \oplus I)^{(1)}$. Thus we have $I^{(1)} \oplus I^{(1)} \subset (I \oplus I)^{(1)}$. On the other hand, we obtain

$$(I \oplus I)^{(1)} = \left[\begin{pmatrix} I \\ I \end{pmatrix}\begin{pmatrix} U \\ U \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}\right] \subset \begin{pmatrix} I^{(1)} \\ I^{(1)} \end{pmatrix} = I^{(1)} \oplus I^{(1)}.$$

Thus we get $(I \oplus I)^{(1)} = I^{(1)} \oplus I^{(1)}$. In this, exchanging $I$ for $I^{(n-1)}$, we have

$$(I^{(n-1)} \oplus I^{(n-1)})^{(1)} = (I^{(n-1)})^{(1)} \oplus (I^{(n-1)})^{(1)}.$$

By the definition, $(I^{(n-1)})^{(1)} = I^{(n)}$. Therefore by the assumption of induction, we obtain

$$(I^{(n-1)} \oplus I^{(n-1)})^{(1)} = ((I \oplus I)^{(n-1)})^{(1)} = (I \oplus I)^{(n)}.$$

Thus we have $(I \oplus I)^{(n)} = I^{(n)} \oplus I^{(n)}$. □

**Lemma 2.5.** Let $U$ be a $U(\sigma)$-algebra, and let $T = T(U)$ be the LTS associated with $U$. Let $Rad(U)$ and $Rad(T)$ be the radicals of $U$ and $T$, respectively. Then we have

$$Rad(T) = Rad(U) \oplus Rad(U). \tag{2.7}$$

**Proof.** For large enough $n$, we have $Rad(U)^{(n)} = \{0\}$ and therefore, by Lemma 2.4, $(Rad(U) \oplus Rad(U))^{(n)} = Rad(U)^{(n)} \oplus Rad(U)^{(n)} = \{0\}$. Hence $Rad(U) \oplus Rad(U)$ is solvable and therefore $Rad(U) \oplus Rad(U) \subset Rad(T)$. We will prove the converse inclusion. We define an endomorphism $\tau$ of $T$ by

$$\tau\begin{pmatrix} a \\ x \end{pmatrix} = \begin{pmatrix} -a \\ x \end{pmatrix}.$$

Then we have

$$-\tau\left[\begin{pmatrix} a \\ x \end{pmatrix}\begin{pmatrix} b \\ y \end{pmatrix}\begin{pmatrix} c \\ z \end{pmatrix}\right] = \left[\tau\begin{pmatrix} a \\ x \end{pmatrix}\tau\begin{pmatrix} b \\ y \end{pmatrix}\tau\begin{pmatrix} c \\ z \end{pmatrix}\right].$$

Hence if $A$ is an ideal of $T$, then so is $\tau(A)$. Moreover we obtain $\tau(A)^{(n)} = \tau(A^{(n)})$ by induction on $n$. Therefore we have $\tau(Rad(T)) = Rad(T)$ since $\tau$ is non-singular. Hence if $\begin{pmatrix} a \\ x \end{pmatrix} \in Rad(T)$, then $\begin{pmatrix} -a \\ x \end{pmatrix} \in Rad(T)$. This implies $\begin{pmatrix} a \\ 0 \end{pmatrix} \in Rad(T)$ and $\begin{pmatrix} 0 \\ x \end{pmatrix} \in Rad(T)$. We denote by $R_1$ and $R_2$ the images of $Rad(T)$ by the projection of $T$ to $U \oplus \{0\}$ and $\{0\} \oplus U$ respectively, then $Rad(T) = R_1 \oplus R_2$. Put $\theta = \begin{pmatrix} 0 & \sigma^{-1} \\ 1 & 0 \end{pmatrix}$. Then $\theta$ is an automorphism of $T$. Therefore we have

$$Rad(T) = \theta^{-1}(Rad(T)) = \theta^{-1}(R_1 \oplus R_2) = R_2 \oplus \sigma(R_1).$$

This implies $R_1 = R_2$ and $\sigma(R_1) = R_2$. Therefore $Rad(T) = R_1 \oplus R_1$ and $\sigma(R_1) = R_1$, that is, $R_1$ is $\sigma$-invariant. By the definition of the triple product of $T$, we have

$$\begin{pmatrix} (R_1UU) \\ 0 \end{pmatrix} = \left[ \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ U \end{pmatrix} \begin{pmatrix} U \\ 0 \end{pmatrix} \right] \subset [Rad(T)\,T\,T] \subset Rad(T) = \begin{pmatrix} R_1 \\ R_1 \end{pmatrix}.$$

This means $(R_1UU) \subset R_1$. Similarly we can obtain $(UR_1U) \subset R_1$ and $(UUR_1) \subset R_1$. Thus $R_1$ is an $\sigma$-invariant ideal of $U$. For large enough $m$, $Rad(T)^{(m)} = \{0\}$. Therefore by Lemma 2.4, $R_1^{(m)} \oplus R_1^{(m)} = (R_1 \oplus R_1)^{(m)} = Rad(T)^{(m)} = \{0\}$. Hence we have $R_1^{(m)} = \{0\}$, and $R_1 \subset Rad(U)$. Consequently $Rad(T) = R_1 \oplus R_1 \subset Rad(U) \oplus Rad(U)$. This completes the proof. □

**Theorem 2.6.** Let $U$ be a $U(\sigma)$-algebra, and let $T(U)$ and $\mathfrak{L}(U)$ be the LTS and the GLA associated with $U$ respectively. Then the following statements are equivalent each other:
(1) $U$ is semisimple.
(2) $T(U)$ is semisimple.
(3) $\mathfrak{L}(U)$ is semisimple.

**Proof.** From Lemma 2.5, we see that (1) and (2) are equivalent one another. From the corollary to Theorem 7 in [10] (p.55), (2) and (3) are equivalent one another. □

The non-degenerations of the trace form $\gamma$ of $U$ and the Killing form $\beta$ of $T(U)$ are equivalent one another. Moreover $\beta$ is non-degenerate if and only if $T(U)$ is semisimple ([11] Theorem 2.1). Hence, from Theorem 2.6, we have

**Corollary 2.7.** A $U(\sigma)$-algebra $U$ is semisimple if and only if its trace form is non-degenerate.

A $U(\sigma)$-algebra $U$ is said to be *$\sigma$-simple* if $(UUU) \neq \{0\}$ and $U$ has no non-trivial $\sigma$-invariant ideal. We note that $\sigma$-simplicity coincides with the usual simplicity if $U$ is a Freudenthal-Kantor triple system.

**Theorem 2.8.** A semisimple $U(\sigma)$-algebra $U$ is decomposed into a direct sum of $\sigma$-simple ideals of $U$.

**Proof.** Let $I_1(\neq \{0\})$ be an minimal $\sigma$-invariant ideal of $U$. We put

$$I_1^{\perp} = \{x \in U | \gamma(x, I_1) = 0\}.$$

We will prove that $I_1^{\perp}$ is an $\sigma$-invariant ideal of $U$. By (1.5),

$$\gamma((U\,U\,I_1^{\perp}), I_1) = \gamma(I_1^{\perp}, (U\,U\,I_1)) = 0$$

and therefore $(U\,U\,I_1^{\perp}) \subset I_1^{\perp}$. Similarly we have $(I_1^{\perp}\,U\,U) \subset I_1^{\perp}$ and $(U\,I_1^{\perp}\,U) \subset I_1^{\perp}$. Hence $I_1^{\perp}$ is an ideal of $U$. Using (1.7),

$$\gamma(\sigma(I_1^{\perp}), I_1) = \gamma(I_1^{\perp}, \sigma^{-1}(I_1)) = \gamma(I_1^{\perp}, I_1) = 0.$$

This means that $I_1^{\perp}$ is $\sigma$-invariant. Since $I_1 \cap I_1^{\perp}$ is an $\sigma$-invariant ideal of $U$, we have $I_1 \cap I_1^{\perp} = I_1$ or $I_1 \cap I_1^{\perp} = \{0\}$ by the assumption of minimality. If we suppose that $I_1 \cap I_1^{\perp} = I_1$, then $I_1 \subset I_1^{\perp}$ and therefore $\gamma(I_1, I_1) = 0$. For any element $y, w \in U$ and $x, z \in I_1$, using (1.5), we have

$$\gamma((xyz), w) = \gamma(z, (y\,\sigma x\,w)) = 0.$$

Since $\gamma$ is non-degenerate from Corollary 2.7, we have $(xyz) = 0$, hence $(I_1\, U\, I_1) = \{0\}$. Similarly, using the identities in Lemma 1.2, we can obtain that $(U\, I_1\, I_1) = \{0\}$ and $(I_1\, I_1\, U) = \{0\}$. Thus we have $I_1^{(1)} = \{0\}$, which contradicts the assumption that $U$ is semisimple. Consequently we get $I_1 \cap I_1^{\perp} = \{0\}$, and $U = I_1 \oplus I_1^{\perp}$. Next we will prove that $I_1^{\perp}$ is also semisimple. Let $I$ be an arbitrary $\sigma$-invariant ideal of $I_1^{\perp}$. Since $I_1 \cap I_1^{\perp} = \{0\}$, we have $(IUU) = (I\, I_1^{\perp}\, I_1^{\perp}) \subset I$. Similarly we get $(UIU) \subset I$ and $(UUI) \subset I$. Therefore $I$ is also an $\sigma$-invariant ideal of $U$. Moreover it is easily seen that $I^{(n)}$ in $I_1^{\perp}$ coincides with $I^{(n)}$ in $U$. Hence $I_1^{\perp}$ is also semisimple, and the proof of the theorem is completed by induction on the dimension of $U$. □

## §3. The Killing form of $\mathfrak{L}(U)$

In this section, we will concretely write down the Killing form of the GLA $\mathfrak{L}(U)$ associated with a semisimple $U(\sigma)$-algebra $U$.

Let $T = T(U)$ and $\mathfrak{L} = \mathfrak{L}(U) = \sum_{i=-2}^{2} \mathfrak{L}_i$ be the LTS and the GLA associated with $U$, respectively. Since the subspace $\mathfrak{L}_{-1}$ (identified with $U$) is invariant under an element $D \in \mathfrak{L}_0$, we denote $\mathrm{Tr}(D|_U)$ by $\mathrm{Tr}_U D$. For $E \in \mathfrak{L}_{-2}$ and $F \in \mathfrak{L}_2$, we also denote $\mathrm{Tr}(EF|_U)$ by $\mathrm{Tr}_U(EF)$.

**Lemma 3.1.** For

$$D_i = L(\begin{pmatrix} a_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y_i \end{pmatrix})\ (i = 1, 2), \quad E = L(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}), \quad F = L(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}),$$

we have

(3.1) $\mathrm{Tr}_T \mathrm{ad} D_1 \mathrm{ad} D_2 = 2\mathrm{Tr}_U(D_1 D_2)$,

(3.2) $\mathrm{Tr}_T \mathrm{ad} E \mathrm{ad} F = \mathrm{Tr}_U(EF)$.

**Proof.** From (1.1), we have

$$\mathrm{ad}_T D_i = \begin{pmatrix} L(a_i, y_i) & 0 \\ 0 & -L(y_i, \sigma a_i) \end{pmatrix}, \quad \mathrm{ad}_T E = E, \quad \mathrm{ad}_T F = F.$$

Hence it follows that

$$\mathrm{Tr}_T \mathrm{ad} D_1 \mathrm{ad} D_2 = \mathrm{Tr}_U\{L(a_1, y_1)L(a_2, y_2) + L(y_1, \sigma a_1)L(y_2, \sigma a_2)\}. \tag{3.3}$$

Since the trace form $\gamma$ is non-degenerate, we denote by $\varphi^*$ the right adjoint operator of an endomorphism $\varphi$ on $U$ with respect to $\gamma$, $\gamma(\varphi x, y) = \gamma(x, \varphi^* y)$. From (1.5), we get

$$\begin{aligned} (3.4)\quad \mathrm{Tr}_U L(y_1, \sigma a_1)L(y_2, \sigma a_2) &= \mathrm{Tr}_U L(a_1, y_1)^* L(a_2, y_2)^* = \mathrm{Tr}_U L(a_2, y_2)^* L(a_1, y_1)^* \\ &= \mathrm{Tr}_U\{L(a_1, y_1)L(a_2, y_2)\}^* = \mathrm{Tr}_U L(a_1, y_1)L(a_2, y_2) = \mathrm{Tr}_U(D_1 D_2). \end{aligned}$$

From (3.3) and (3.4), equation (3.1) follows. Since

$$\mathrm{ad}_T E \mathrm{ad}_T F = \begin{pmatrix} K(a, b)K(x, y)\sigma & 0 \\ 0 & 0 \end{pmatrix},$$

it follows that

$$\mathrm{Tr}_T \mathrm{ad} E \mathrm{ad} F = \mathrm{Tr}_U(K(a, b)K(x, y)\sigma) = \mathrm{Tr}_U(EF).$$

Hence we have (3.2). □

**Theorem 3.2.** Let $U$ be a semisimple $U(\sigma)$-algebra and $\gamma$ its trace form. Let $\mathfrak{L}(U) = \sum_{i=-2}^{2} \mathfrak{L}_i$ be the GLA associated with $U$ and $\alpha$ its Killing form. Let $\alpha_0$ be the Killing form of the subalgebra $L(T,T)$ of $\mathfrak{L}(U)$. For $X_i = E_i + a_i + D_i + x_i + F_i \in \mathfrak{L}(U)$ $(i = 1,2)$, where $E_i \in \mathfrak{L}_{-2},\ a_i \in \mathfrak{L}_{-1}(= U),\ D_i \in \mathfrak{L}_0,\ x_i \in \mathfrak{L}_1(= U),\ F_i \in \mathfrak{L}_2$, we have

$$\begin{aligned}(3.5)\quad \alpha(X_1, X_2) = {} & \alpha_0(E_1, F_2) + \alpha_0(D_1, D_2) + \alpha_0(F_1, E_2) + \mathrm{Tr}_U(E_1F_2 + 2D_1D_2 + F_1E_2) \\ & +2\{\gamma(a_1, x_2) + \gamma(a_2, x_1)\}.\end{aligned}$$

**Proof.** Since $\alpha(\mathfrak{L}_i, \mathfrak{L}_j) = 0$ for $i$ and $j$ such that $i + j \neq 0$, we have

$$(3.6)\qquad \alpha(X_1, X_2) = \alpha(E_1, F_2) + \alpha(D_1, D_2) + \alpha(F_1, E_2) + \alpha(a_1, x_2) + \alpha(x_1, a_2).$$

From (1.4), $\alpha(a_1, x_2) = 2\gamma(a_1, x_2),\ \alpha(x_1, a_2) = 2\gamma(a_2, x_1)$.
Now let $Y, Z \in L(T,T)$. Since the subspaces $L(T,T)$ and $T$ are invariant under the mapping $\mathrm{ad}Y\mathrm{ad}Z$, we have

$$\alpha(Y, Z) = \mathrm{Tr}_{L(T,T)}(\mathrm{ad}Y\mathrm{ad}Z) + \mathrm{Tr}_T(\mathrm{ad}Y\mathrm{ad}Z) = \alpha_0(Y, Z) + \mathrm{Tr}_T(\mathrm{ad}Y\mathrm{ad}Z).$$

Hence, from Lemma 3.1, we have

$$(3.7)\qquad \begin{aligned}\alpha(E_1, F_2) &= \alpha_0(E_1, F_2) + \mathrm{Tr}_U(E_1F_2),\\ \alpha(D_1, D_2) &= \alpha_0(D_1, D_2) + 2\mathrm{Tr}_U(D_1D_2),\\ \alpha(F_1, E_2) &= \alpha_0(F_1, E_2) + \mathrm{Tr}_U(F_1E_2).\end{aligned}$$

From (3.6) and (3.7), (3.5) follows. □

## References


[1] B. N. Allison, A construction of Lie algebras from J-ternary algebras, Amer. J. Math., 98(1976), 285-294.

[2] H. Asano, Classification of non-compact real simple generalized Jordan triple systems of the second kind, Hiroshima Math. J., 21(1991), 463-489.

[3] H. Asano and S. Kaneyuki, On compact generalized Jordan triple systems of the second kind, Tokyo J. Math., 11(1988), 105-118.

[4] K. Atsuyama and Y. Taniguchi, On generalized Jordan triple systems and their modifications, Yokohama Math. J., 47(2000), 165-175.

[5] H. Freudenthal, Beziehungen der $E_7$ und $E_8$ zur Oktavenebene. I, Indag. Math., 16(1954), 218-230.

[6] W. Hein, A construction of Lie algebras by triple sysytems, Trans. Amer. Math. Soc., 205(1975), 79-95.

[7] W. Hein, Innere Lie-Tripelsysteme und J-ternäre Algebren, Math. Ann., 213(1975), 195-202.

[8] N. Kamiya, A structure theory of Freudenthal-Kantor triple systems, J. Algebra 110(1987), 108-123.

[9] I. L. Kantor, Models of exceptional Lie algebras, Soviet Math. Dokl., 14(1973), 254-258.

[10] K. Meyberg, Lectures on algebras and triple systems, Univ. Virginia,Charlottesville, 1972.

[11] T. S. Ravisankar, Some remarks on Lie triple systems, Kumamoto J. Sci. (Math.), 11(1974), 1-8.

[12] K. Yamaguti, On the metasymplectic geometry and triple systems, Kokyuroku RIMS, Kyoto Univ., 308(1977), 55-92(in Japanese).

Nishinippon Institute of Technology, Kanda, Fukuoka 800-0394, Japan
and
Sojo University, Ikeda, Kumamoto 860-0082, Japan