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POSITIVE IMPLICATIVE HYPERBCK-ALGEBRAS

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Abstract. We introduce the concept of weak positive implicative and positive implicative hyperBCK-algebras, and investigate some related properties. We give a relation between a weak positive implicative hyperBCK-algebra and a positive implicative hyperBCK-algebra. We also introduce the notion of a positive implicative hyperBCK-ideal, and state its characterizations.

1. Introduction

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [7] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [6], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyperBCK-algebra which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a hyperBCK-ideal and a weak hyperBCK-ideal, and gave relations between hyperBCK-ideals and weak hyperBCK-ideals. Y. B. Jun et al. [5] gave a condition for a hyperBCK-algebra to be a BCK-algebra, and introduced the notion of a strong hyperBCK-ideal and a reflexive hyperBCK-ideal. They showed that every strong hyperBCK-ideal is a hypersubalgebra, a weak hyperBCK-ideal and a hyperBCK-ideal; and every reflexive hyperBCK-ideal is a strong hyperBCK-ideal. In [4] the present authors introduced the concept of (weak) scalar elements and hyperatoms, and gave relations between scalar elements and hyperatoms. In this paper we introduce the concept of weak positive implicative and positive implicative hyperBCK-algebras, and investigate some related properties. We give a relation between a weak positive implicative hyperBCK-algebra and a positive implicative hyperBCK-algebra. We also introduce the notion of a positive implicative hyperBCK-ideal, and state its characterizations.

2. Preliminaries

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An algebra $(X;*,0)$ of type $(2,0)$ is said to be a BCK-algebra if it satisfies: for all $x,y,z\in X$,

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0$ and $y*x=0$ imply $x=y$.

Note that an algebra $(X,*,0)$ of type (2,0) is a BCK-algebra if and only if

(i) $((y*z)*(x*z))*(y*x)=0$,
(ii) $((z*x)*y)*((z*y)*x)=0$,
(iii) $(x*y)*x=0$,
(iv) $x*y=0$ and $y*x=0$ imply that $x=y$,

for all $x,y\in X$ (see [8]). Note that the identity $x*(x*(x*y))=x*y$ holds in a BCK-algebra. A non-empty subset I of a BCK-algebra X is called an ideal of X if $0\in I$, and $x*y\in I$ and $y\in I$ imply $x\in I$ for all $x,y\in X$.

Let H be a non-empty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A\circ B$ the set $\bigcup\limits_{a\in A,b\in B} a\circ b$. We shall use $x\circ y$ instead of $x\circ\{y\}$, $\{x\}\circ y$, or $\{x\}\circ\{y\}$.

Definition 2.1 (Jun et al. [6]). By a *hyperBCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfing the following axioms:

(HK1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,
(HK2) $(x\circ y)\circ z=(x\circ z)\circ y$,
(HK3) $x\circ H\ll\{x\}$,
(HK4) $x\ll y$ and $y\ll x$ imply $x=y$,

for all $x,y,z\in H$, where $x\ll y$ is defined by $0\in x\circ y$ and for every $A,B\subseteq H$, $A\ll B$ is defined by $\forall a\in A$, $\exists b\in B$ such that $a\ll b$.

Example 2.2 (Jun et al. [6]). (1) Let $(H,*,0)$ be a BCK-algebra and define a hyperoperation "$\circ$" on H by $x\circ y=\{x*y\}$ for all $x,y\in H$. Then H is a hyperBCK-algebra.

(2) Define a hyperoperation "$\circ$" on $H:=[0,\infty)$ by

$$x\circ y:=\begin{cases}[0,x] & \text{if } x\leq y\\ (0,y] & \text{if } x>y\neq 0\\ \{x\} & \text{if } y=0\end{cases}$$

for all $x,y\in H$. Then H is a hyperBCK-algebra.

(3) Let $H=\{0,1,2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then H is a hyperBCK-algebra.

Proposition 2.3 (Jun et al. [6]). *In a hyperBCK-algebra H, the condition* (HK3) *is equivalent to the condition:*

(i) $x\circ y\ll\{x\}$ *for all* $x,y\in H$.

Proposition 2.4 (Jun et al. [6]). *Let H be a hyperBCK-algebra. Then*
(i) $x \circ 0 \ll \{x\}$, $0 \circ x \ll \{0\}$ *and* $0 \circ 0 \ll \{0\}$ *for all* $x, y \in H$,
(ii) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ *and* $0 \circ A \ll \{0\}$ *for every non-empty subsets* A, B *and* C *of* H.

Proposition 2.5 (Jun et al. [6]). *In any hyperBCK-algebra H, the following hold:*
(i) $0 \circ 0 = \{0\}$,
(ii) $0 \ll x$,
(iii) $x \ll x$,
(iv) $A \ll A$,
(v) $A \subseteq B$ *implies* $A \ll B$,
(vi) $0 \circ x = \{0\}$,
(vii) $0 \circ A = \{0\}$,
(viii) $A \ll \{0\}$ *implies* $A = \{0\}$,
(ix) $A \circ B \ll A$,
(x) $x \in x \circ 0$,
(xi) $x \circ 0 \ll \{y\}$ *implies* $x \ll y$,
(xii) $y \ll z$ *implies* $x \circ z \ll x \circ y$,
(xiii) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$ *and* $x \circ z \ll y \circ z$,
(xiv) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$,
for all $x, y, z \in H$ *and for all non-empty subsetes* A *and* B *of* H.

Definition 2.6 (Jun et al. [6]). Let $(H, \circ)$ be a hyperBCK-algebra and let S be a subset of H containing 0. If S is a hyperBCK-algebra with respect to the hyperoperation "$\circ$" on H, we say that S is a *hypersubalgebra* of H.

Theorem 2.7 (Jun et al. [6]). *Let S be a non-empty subset of a hyperBCK-algebra H. Then S is a hypersubalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Theorem 2.8 (Jun et al. [6]). *The set*

$$S(H) := \{x \in H | x \circ x = \{0\}\}$$

is a hypersubalgebra of H.

Theorem 2.9 (Jun et al. [6]). *Let H be a hyperBCK-algebra. Then $S(H)$ is a BCK-algebra.* We then call $S(H)$ the *BCK-part* of a hyperBCK-algebra H.

Corollary 2.10 (Jun et al. [6]). *A hyperBCK-algebra H is a BCK-algebra if and only if $H = S(H)$.*

Definition 2.11 (Jun et al. [6]). Let I be a non-empty subset of a hyperBCK-algebra H. Then I is said to be a *hyperBCK-ideal* of H if
(HI1) $0 \in I$,
(HI2) $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Definition 2.12 (Jun et al. [5]). A hyperBCK-ideal I of H is said to be *reflexive* if $x \circ x \subseteq I$ for all $x \in H$.

Definition 2.13 (Jun et al. [6]). Let I be a non-empty subset of a hyperBCK-algebra H. Then I is called a *weak hyperBCK-ideal* of H if
(HI1) $0 \in I$,
(WHI) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Note from [6] that every hyperBCK-ideal of H is a weak hyperBCK-ideal of H, but the converse may not be true.

Definition 2.14 (Jun and Xin [4]). Let H be a hyperBCK-algebra. An element $a \in H$ is said to be *left* (resp. *right*) *scalar* if $|a \circ x| = 1$ (resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar, we say that a is a *scalar* element.

Denote by $R(H)$ (resp. $L(H)$) the set of all right (resp. left) scalar elements of H. Note that $L(H) = S(H)$ (see [4, Theorem 3.4]).

Theorem 2.15 (Jun and Xin [4]). *Let H be a hyperBCK-algebra. Then* 0 *is a right scalar element of H, $a \circ 0 = \{a\}$ for all $a \in H$, and $A \circ 0 = A$ for every subset A of H.*

3. Positive implicative hyperBCK-algebras

We begin with the following proposition.

Proposition 3.1. *Let H be a hyperBCK-algebra. Then we have*

$$(x \circ y) \circ z \ll (x \circ z) \circ (y \circ z) \text{ for all } x, y, z \in H.$$

Proof. Since $y \circ z \ll \{y\}$ by Proposition 2.3, we have $t \ll y$ for every $t \in y \circ z$. It follows from Proposition 2.5(xii) that $u \circ y \ll u \circ t \subseteq u \circ (y \circ z)$ for all $u \in H$. Thus $u \circ y \ll u \circ (y \circ z)$ for all $u \in H$, and so

$$(x \circ y) \circ z = (x \circ z) \circ y = \bigcup_{u \in x \circ z} u \circ y \ll \bigcup_{u \in x \circ z} u \circ (y \circ z) = (x \circ z) \circ (y \circ z).$$

This completes the proof. $\square$

The following example shows that the axioms

$$(x \circ z) \circ (y \circ z) \ll (x \circ y) \circ z \text{ and } (x \circ z) \circ (y \circ z) = (x \circ y) \circ z$$

do not hold.

Example 3.2. (1) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0, 1\}$

Then H is a hyperBCK-algebra. Note that

$$(2 \circ 1) \circ (1 \circ 1) = 1 \circ 0 = \{1\} \not\ll \{0\} = 1 \circ 1 = (2 \circ 1) \circ 1.$$

(2) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

Then H is a hyperBCK-algebra. Note that

$$(2 \circ 1) \circ (1 \circ 1) = \{1, 2\} \circ 0 = \{1, 2\} \neq \{0, 1, 2\} = \{1, 2\} \circ 1 = (2 \circ 1) \circ 1.$$

Definition 3.3. A hyperBCK-algebra H is said to be *weak positive implicative* (resp. *positive implicative*) if it satisfies the axiom

$$(x \circ z) \circ (y \circ z) \ll (x \circ y) \circ z \text{ (resp. } (x \circ z) \circ (y \circ z) = (x \circ y) \circ z)$$

for all $x, y, z \in H$.

Example 3.4. (1) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

Then H is a positive implicative hyperBCK-algebra.

(2) The set $H = \{0, 1, 2\}$ endowed with the hyperoperation "$\circ$" defined by the following table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

is a positive implicative hyperBCK-algebra.

Proposition 3.5. *Every positive implicative hyperBCK-algebra is a weak positive implicative hyperBCK-algebra.*

Proof. Straightforward. $\square$

The Example 3.2(2) shows that the converse of Proposition 3.5 may not be true.

Lemma 3.6. *Let H be a hyperBCK-algebra. For any $a \in L(H)$ and $x \in H$, we have*

(i) *every element of $a \circ x$ is left scalar.* In this case, we will use the notation $a \circ x \in L(H)$ because $a \circ x$ is a singleton set.

(ii) $(a \circ (a \circ x)) \circ x = \{0\}$.

(iii) $a \circ (a \circ (a \circ x)) = a \circ x$.

Proof. (i) Let $a \in L(H)$. Then $|a \circ x| = 1$ for all $x \in H$ and so

$$((a \circ x) \circ y) \circ ((a \circ x) \circ y) \ll (a \circ x) \circ (a \circ x) \ll a \circ a = \{0\}$$

for all $y \in H$. It follows from Proposition 2.5(viii) that

$$((a \circ x) \circ y) \circ ((a \circ x) \circ y) = \{0\}$$

for all $a \in L(H)$ and $x, y \in H$. Now let $s, t \in (a \circ x) \circ y$. Then $s \circ t \subseteq ((a \circ x) \circ y) \circ ((a \circ x) \circ y) = \{0\}$ and so $s \circ t = \{0\}$, i.e., $s \ll t$. Similarly we have $t \ll s$. Hence $s = t$, which implies

$$1 = |(a \circ x) \circ y| = |\bigcup_{t \in a \circ x} t \circ y| = |t \circ y|.$$

This proves (i).

(ii) Since $a \circ x \in L(H)$ for all $a \in L(H)$ and $x \in H$, we have $(a \circ (a \circ x)) \circ x = (a \circ x) \circ (a \circ x) = \{0\}$.

(iii) Using (ii), we know that $(a \circ (a \circ (a \circ x))) \circ (a \circ x) = \{0\}$ or equivalently $a \circ (a \circ (a \circ x)) \ll a \circ x$ for all $a \in L(H)$ and $x \in H$. On the other hand, since $a \circ x \in L(H) = S(H)$ and since $S(H)$ is a BCK-algebra, we get

$$\begin{aligned}(a \circ x) \circ (a \circ (a \circ (a \circ x))) &= (a \circ (a \circ (a \circ (a \circ x)))) \circ x \\ &= (a \circ (a \circ x)) \circ x = (a \circ x) \circ (a \circ x) = \{0\},\end{aligned}$$

i.e., $a \circ x \ll a \circ (a \circ (a \circ x))$. Hence $a \circ (a \circ (a \circ x)) = a \circ x$ for all $a \in L(H)$ and $x \in H$. $\square$

Proposition 3.7. *Let H be a positive implicative hyperBCK-algebra. For any $a \in L(H)$ and $x \in H$, we have*

(i) $x \circ a = (x \circ a) \circ a$,
(ii) $a \circ (a \circ x) = (a \circ (a \circ x)) \circ (a \circ x)$,
(iii) $a \circ x = (a \circ x) \circ x$,
(iv) $a \circ x = (a \circ x) \circ (a \circ (a \circ x))$,
(v) $a \circ (a \circ x) = (a \circ (a \circ x)) \circ (a \circ x)$.

Proof. Let $a \in L(H)$ and $x \in H$. We have that

$$(x \circ a) \circ a = (x \circ a) \circ (a \circ a) = (x \circ a) \circ 0 = x \circ a,$$

which proves (i).

(ii) is by (i) and Lemma 3.6(i).

(iii) Note that

$$\begin{aligned}
&((a \circ x) \circ x) \circ (a \circ x) \\
&= ((a \circ x) \circ (a \circ x)) \circ x \qquad [\text{by (HK2)}] \\
&= ((a \circ a) \circ x) \circ x \qquad [\because H \text{ is positive implicative}] \\
&= (0 \circ x) \circ x \qquad [\because a \in L(H) = S(H)] \\
&= 0 \circ x \qquad [\text{by Proposition 2.5(vi)}] \\
&= \{0\},
\end{aligned}$$

and by using the positive implicativity of H and Lemma 3.6(ii) we get

$$(a \circ x) \circ ((a \circ x) \circ x) = (a \circ (a \circ x)) \circ x = \{0\}.$$

Hence $a \circ x = (a \circ x) \circ x$.

(iv) is by (iii) and Lemma 3.6(iii).

(v) We have that

$$\begin{aligned}
&((a \circ (a \circ x)) \circ (a \circ x)) \circ (a \circ (a \circ x)) \\
&= ((a \circ (a \circ (a \circ x))) \circ (a \circ x)) \circ (a \circ x) \qquad [\text{by (HK2)}] \\
&= ((a \circ x) \circ (a \circ x)) \circ (a \circ x) \qquad [\text{by Lemma 3.6(iii)}] \\
&= 0 \circ (a \circ x) \qquad [\because a \circ x \in L(H) = S(H)] \\
&= \{0\} \qquad [\text{by Proposition 2.5(vii)}]
\end{aligned}$$

and

$$\begin{aligned}
&(a \circ (a \circ x)) \circ ((a \circ (a \circ x)) \circ (a \circ x)) \\
&= (a \circ (a \circ (a \circ x))) \circ (a \circ x) \qquad [\because H \text{ is positive implicative}] \\
&= (a \circ x) \circ (a \circ x) \qquad [\text{by Lemma 3.6(iii)}] \\
&= \{0\}.
\end{aligned}$$

Therefore $a \circ (a \circ x) = (a \circ (a \circ x)) \circ (a \circ x)$, ending the proof. □

Definition 3.8. Let H be a hyperBCK-algebra. A non-empty subset I of H is said to be a *positive implicative hyperBCK-ideal* of H if $0 \in I$ and it satisfies:

(PI) $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$ for all $x, y, z \in H$.

Example 3.9. In Example 3.2(2), $I_1 := \{0, 1\}$ is a positive implicative hyperBCK-ideal of H. But in Example 3.2(1), $I_2 := \{0, 2\}$ is not a positive implicative hyperBCK-ideal of H since $(2 \circ 1) \circ 1 = \{0\} \ll I_2$ and $1 \circ 1 = \{0\} \subseteq I_2$ but $2 \circ 1 = \{1\} \not\subseteq I_2$.

Theorem 3.10. *In a hyperBCK-algebra H, every positive implicative hyperBCK-ideal is a hyperBCK-ideal.*

Proof. Let I be a positive implicative hyperBCK-ideal of a hyperBCK-algebra H and let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Putting $z = 0$ in (PI), we get $(x \circ y) \circ 0 = x \circ y \ll I$ and $y \circ 0 = \{y\} \subseteq I$. It follows from (PI) that $\{x\} = x \circ 0 \subseteq I$. Thus I is a hyperBCK-ideal of H. $\square$

The following example shows that the converse of Theorem 3.10 may not be true.

Example 3.11. In Example 3.2(1), $I := \{0\}$ is clearly a hyperBCK-ideal which is not a positive implicative hyperBCK-ideal of H.

Lemma 3.12 (Jun and Xin [4, Proposition 3.7])**.** *Let A be a subset of a hyperBCK-algebra H. If I is a hyperBCK-ideal of H such that $A \ll I$, then A is contained in I.*

Proposition 3.13. *In a hyperBCK-algebra H the following axiom holds:*

$$((x \circ z) \circ (y \circ z)) \circ u \ll (x \circ y) \circ u \text{ for all } x, y, z, u \in H.$$

Proof. For any $x, y, z, u \in H$ we have

$$((x \circ z) \circ (y \circ z)) \circ u = ((x \circ u) \circ z) \circ (y \circ z) = \bigcup_{t \in x \circ u} (t \circ z) \circ (y \circ z).$$

Using (HK1), it follows that

$$((x \circ z) \circ (y \circ z)) \circ u = \bigcup_{t \in x \circ u} (t \circ z) \circ (y \circ z) \ll \bigcup_{t \in x \circ u} t \circ y = (x \circ u) \circ y = (x \circ y) \circ u.$$

This completes the proof. $\square$

Theorem 3.14. *Let I be a positive implicative hyperBCK-ideal of a hyperBCK-algebra H and let $a \in H$. Then the set*

$$I_a := \{x \in H | x \circ a \subseteq I\}$$

is a weak hyperBCK-ideal of H.

Proof. Clearly $0 \in I_a$. Let $x, y \in H$ be such that $x \circ y \subseteq I_a$ and $y \in I_a$. Then $(x \circ y) \circ a \subseteq I$ and $y \circ a \subseteq I$, which imply that $(x \circ y) \circ a \ll I$ and $y \circ a \subseteq I$. It follows from (PI) that $x \circ a \subseteq I$ or equivalently $x \in I_a$. Hence I_a is a weak hyperBCK-ideal of H. $\square$

Theorem 3.15. *Let I be a hyperBCK-ideal of a hyperBCK-algebra H. If $I_a := \{x \in H | x \circ a \subseteq I\}$ is a weak hyperBCK-ideal of H for all $a \in H$, then I is a positive implicative hyperBCK-ideal of H.*

Proof. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$. Then $(x \circ y) \circ z \subseteq I$ by Lemma 3.12, and $y \in I_z$. Thus for each $t \in x \circ y$, we have $t \circ z \subseteq I$ or equivalently $t \in I_z$. Hence $x \circ y \subseteq I_z$. Since I_z is a weak hyperBCK-ideal of H, it follows that $x \in I_z$, i.e., $x \circ z \subseteq I$. Therefore I is a positive implicative hyperBCK-ideal of H. $\square$

Lemma 3.16 (Jun et. al [5]). *Let I be a reflexive hyperBCK-ideal of H. Then*

$$(x \circ y) \cap I \neq \emptyset \text{ implies } x \circ y \subseteq I \text{ for all } x, y \in H.$$

Lemma 3.17. *Let I be a hyperBCK-ideal of a hyperBCK-algebra H. Then $A \circ B \subseteq I$ and $B \subseteq I$ imply that $A \subseteq I$ for every subsets A and $B (\neq \emptyset)$ of H.*

Proof. Let $a \in A$ and $b \in B$. Then $a \circ b \subseteq A \circ B \subseteq I$, which implies that $a \circ b \ll I$. It follows from (HI2) that $a \in I$ so that $A \subseteq I$. □

Now we give a characterization of a positive implicative hyperBCK-ideal.

Theorem 3.18. *Let I be a subset of a hyperBCK-algebra H such that $x \circ x \subseteq I$ for all $x \in H$. Then the following are equivalent:*

(i) *I is a positive implicative hyperBCK-ideal of H.*

(ii) *I is a hyperBCK-ideal of H, and for every $x, y \in H$*

$$(x \circ y) \circ y \subseteq I \text{ implies } x \circ y \subseteq I.$$

(iii) *I is a hyperBCK-ideal of H, and for every $x, y, z \in H$*

$$(x \circ y) \circ z \subseteq I \text{ implies } (x \circ z) \circ (y \circ z) \subseteq I.$$

(iv) *$((x \circ y) \circ y) \circ z \ll I$ and $z \in I$ imply $x \circ y \subseteq I$ for all $x, y, z \in H$.*

(v) *I and $I_a := \{x \in H : x \circ a \subseteq I\}$ are hyperBCK-ideals of H for all $a \in H$.*

Proof. (i) $\Rightarrow$ (ii) Let I be a positive implicative hyperBCK-ideal of H. Then I is a hyperBCK-ideal of H (see Theorem 3.10). Let $x, y \in H$ be such that $(x \circ y) \circ y \subseteq I$ and hence $(x \circ y) \circ y \ll I$. Since $y \circ y \subseteq I$ by hypothesis, it follows from (PI) that $x \circ y \subseteq I$.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds and let $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq I$. Using Proposition 3.13, we have

$$((x \circ (y \circ z)) \circ z) \circ z = ((x \circ z) \circ (y \circ z)) \circ z \ll (x \circ y) \circ z \subseteq I$$

and so $((x \circ (y \circ z)) \circ z) \circ z \ll I$. It follows from Lemma 3.12 that $((x \circ (y \circ z)) \circ z) \circ z \subseteq I$. Therefore $(t \circ z) \circ z \subseteq I$ for every $t \in x \circ (y \circ z)$. Applying (ii), then $t \circ z \subseteq I$ for every $t \in x \circ (y \circ z)$. This shows that

$$(x \circ z) \circ (y \circ z) = (x \circ (y \circ z)) \circ z = \bigcup_{t \in x \circ (y \circ z)} t \circ z \subseteq I.$$

(iii) $\Rightarrow$ (iv) Assume (iii) holds and let $x, y, z \in H$ be such that $((x \circ y) \circ y) \circ z \ll I$ and $z \in I$. Since I is a hyperBCK-ideal, we get $((x \circ y) \circ y) \circ z \subseteq I$ by Lemma 3.12 and hence $((x \circ z) \circ y) \circ y \subseteq I$. For any $t \in x \circ z$, we obtain $(t \circ y) \circ y \subseteq I$ which implies $(t \circ y) \circ (y \circ y) \subseteq I$ by (iii). Hence $((x \circ z) \circ y) \circ (y \circ y) \subseteq I$. Since I is a hyperBCK-ideal and $y \circ y \subseteq I$, it follows from Lemma 3.17 that $(x \circ y) \circ z = (x \circ z) \circ y \subseteq I$. Noticing that $\{z\} \subseteq I$, we get $x \circ y \subseteq I$ by Lemma 3.17.

(iv) $\Rightarrow$ (i) Assume that (iv) is true. We first show that I is a hyperBCK-ideal. Note that $0 \in x \circ x \subseteq I$ for all $x \in H$. Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then $((x \circ 0) \circ 0) \circ y = x \circ y \ll I$ and $y \in I$. Using (iv), we obtain $\{x\} = x \circ 0 \subseteq I$. Hence I is a hyperBCK-ideal of H. Now let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$. Since I is a hyperBCK-ideal, therefore $(x \circ y) \circ z \subseteq I$ by Lemma 3.12. By using Proposition 3.13 we get

$$((x \circ z) \circ z) \circ (y \circ z) = ((x \circ z) \circ (y \circ z)) \circ z \ll (x \circ y) \circ z \subseteq I$$

and hence $((x \circ z) \circ z) \circ (y \circ z) \ll I$. Using Lemma 3.12 again, then $((x \circ z) \circ z) \circ (y \circ z) \subseteq I$. Let $t \in y \circ z$. Then $((x \circ z) \circ z) \circ t \subseteq I$ and $t \in I$. It follows that $((x \circ z) \circ z) \circ t \ll I$ and $t \in I$ for all $t \in y \circ z$ so from (iv) that $x \circ z \subseteq I$. This proves that I is positive implicative.

(i) $\Rightarrow$ (v) Let I be a positive implicative hyperBCK-ideal of H. Then I is a hyperBCK-ideal of H (see Theorem 3.10) and I_a is a weak hyperBCK-ideal of H for each $a \in H$ (see Theorem 3.14). Let $x, y \in H$ be such that $x \circ y \ll I_a$ and $y \in I_a$, and let $t \in x \circ y$. Then there exists $s \in I_a$ such that $t \ll s$, i.e., $0 \in t \circ s$. Hence $(t \circ s) \cap I \neq \emptyset$. Since I is a reflexive hyperBCK-ideal of H, it follows from (HK1) and Lemma 3.16 that $(t \circ a) \circ (s \circ a) \ll t \circ s \subseteq I$ so that $t \circ a \subseteq I$ since $s \circ a \subseteq I$ and I is a hyperBCK-ideal. Thus $t \in I_a$ and so $x \circ y \subseteq I_a$. Since I_a is a weak hyperBCK-ideal, it follows from (WHI) that $x \in I_a$. Therefore I_a is a hyperBCK-ideal of H.

(v) $\Rightarrow$ (i) is by Theorem 3.15. This completes the proof. $\square$

Theorem 3.19. *Let I and A be reflexive hyperBCK-ideals of H such that $I \subseteq A$. If I is positive implicative, then so is A.*

Proof. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \in A$. Since $(x \circ y) \circ (x \circ y) \ll x \circ x \subseteq I$, we have $(x \circ y) \circ (x \circ y) \ll I$ and so $(x \circ y) \circ (x \circ y) \subseteq I$. Let $t, s \in x \circ y$. Then $(t \circ z) \circ (s \circ z) \ll t \circ s \subseteq I$ and hence $(t \circ z) \circ (s \circ z) \ll I$, which implies from Lemma 3.12 that $(t \circ z) \circ (s \circ z) \subseteq I$. Thus $((x \circ y) \circ z) \circ ((x \circ y) \circ z) \subseteq I$ and consequently $((x \circ y) \circ z) \circ u \subseteq I$ for all $u \in (x \circ y) \circ z$. It follows from (HK2) that

$$((x \circ u) \circ y) \circ z \subseteq I \text{ for all } u \in (x \circ y) \circ z.$$

Therefore $(v \circ y) \circ z \subseteq I$ for all $v \in x \circ u$. Using Theorem 3.18(iii), we get

$$((v \circ z) \circ (y \circ z) \subseteq I \text{ for all } v \in x \circ u.$$

Hence $((x \circ u) \circ z) \circ (y \circ z) = ((x \circ z) \circ (y \circ z)) \circ u \subseteq I$ for all $u \in (x \circ y) \circ z$, and thus $((x \circ z) \circ (y \circ z)) \circ ((x \circ y) \circ z) \subseteq I \subseteq A$. This implies that $a \circ b \subseteq A$ for all $a \in (x \circ z) \circ (y \circ z)$ and $b \in (x \circ y) \circ z$. Since $b \in A$, it follows that $a \in A$ which shows that $(x \circ z) \circ (y \circ z) \subseteq A$. Applying Theorem 3.18(iii), we know that A is a positive implicative hyperBCK-ideal of H. $\square$

Theorem 3.20. *Let H be a positive implicative hyperBCK-algebra. Then*

(i) *every hyperBCK-ideal is positive implicative.*

(ii) *if I is a reflexive hyperBCK-ideal of H, then I_a is a positive implicative hyperBCK-ideal of H for each $a \in H$.*

Proof. (i) Let I be a hyperBCK-ideal of H and let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$. Then $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z \ll I$ which implies that $x \circ z \subseteq I$. Thus I is positive implicative.

(ii) is by (i) and Theorem 3.18. $\square$

The following example shows that the converse of Theorem 3.20 may not be true.

Example 3.21. Let H be given in Example 3.2(2). Then $\{0\}$, $\{0, 1\}$ and H are all hyperBCK-ideals of H. We can see that they are positive implicative and that only H is reflexive. Hence the conditions (i) and (ii) of Theorem 3.20 hold, but H is not positive implicative because $(2 \circ 1) \circ (1 \circ 1) \neq (2 \circ 1) \circ 1$.

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