ON MAXIMAL IDEALS OF ORDERED SEMIGROUPS

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Abstract. For a semigroup (or ordered semigroup) S, we denote by $\mathcal{I}(M)$ the ideal of S generated by M $(M \subseteq S)$. In this note we prove the following: If S is an ordered semigroup (or a semigroup), then a proper ideal M of S is a maximal ideal of S if and only if for every $a \in S \backslash M$, we have $\mathcal{I}(M \cup \{a\}) = S$. If S is a finitely generated ordered semigroup (or a semigroup), then each proper ideal of S is contained in a maximal ideal of S. If S is an ordered semigroup (or a semigroup) for which there exists an element a of S such that $\mathcal{I}(a) = S$, then each proper ideal of S is contained in a maximal ideal of S. Similar results hold if, in the results above, we replace the word "ideal" by "left ideal" or "right ideal".

If $(S, ., \leq)$ is an ordered semigroup, a non-empty subset A of S is called a left (resp. right) ideal of S if 1) $SA \subseteq A$ (resp. $AS \subseteq A$) and 2) $a \in A$, $S \ni b \leq a$ implies $b \in A$. The non-empty subset A of S is called an ideal of S if it is both a left and a right ideal of S [1]. An ideal I of a semigroup (resp. ordered semigroup) S is called prime if $a, b \in S$ such that $ab \in I$ implies $a \in I$ or $b \in I$. Equivalent Definition: $A, B \subseteq S$ such that $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ [1].
If S is an ordered semigroup (resp. semigroup) and $\emptyset \neq A \subseteq S$, we denote by $\mathcal{I}(A)$ the ideal of S generated by A i.e. the smallest -under inclusion relation- ideal of S containing A. We denote by $\mathcal{L}(A)$ (resp. $\mathcal{R}(A)$) the left (resp. right) ideal of S generated by A. If S is an ordered semigroup and $H \subseteq S$, we denote $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$. For an ordered semigroup S, we have $\mathcal{I}(A) = (A \cup SA \cup AS \cup SAS]$, $\mathcal{L}(A) = (A \cup SA]$, $\mathcal{R}(A) = (A \cup AS]$ (cf. [1]). For a semigroup S, we have $\mathcal{I}(A) = A \cup SA \cup AS \cup SAS$, $\mathcal{L}(A) = A \cup SA$, $\mathcal{R}(A) = A \cup AS$. For $A = \{a\}$, we write $\mathcal{I}(a)$ instead of $\mathcal{I}(\{a\})$. Similarly, we write $\mathcal{L}(a)$, $\mathcal{R}(a)$.
We denote by $S \backslash M$ the complement of S to M.

1. An ideal M of a semigroup (resp. ordered semigroup) S is called proper if $M \neq S$ [2]. A proper ideal M of a semigroup (resp. ordered semigroup) S is called maximal if there exists no ideal T of S such that $M \subset T \subset S$, equivalently, if for each ideal T of S such that $M \subseteq T$, we have $T = M$ or $T = S$ (cf. also [1]).

Proposition 1. *Let S be an ordered semigroup (resp. semigroup) and M a proper ideal of S. Then M is a maximal ideal of S if and only if for every $a \in S \backslash M$, we have $\mathcal{I}(M \cup \{a\}) = S$.*

Proof. $\Rightarrow$. Let $a \in S \backslash M$. Since $M \subseteq M \cup \{a\} \subseteq \mathcal{I}(M \cup \{a\})$, $\mathcal{I}(M \cup \{a\})$ is an ideal of S and M a maximal ideal of S, we have $\mathcal{I}(M \cup \{a\}) = M$ or $\mathcal{I}(M \cup \{a\}) = S$. Since

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$a \in \mathcal{I}(M \cup \{a\})$ and $a \notin M$, we have $\mathcal{I}(M \cup \{a\}) \neq M$. So $\mathcal{I}(M \cup \{a\}) = S$.

$\Leftarrow$. Let S be an ordered semigroup, T an ideal of S, $M \subseteq T$ and $M \neq T$. Let $a \in T$, $a \notin M$. Since $a \in S \backslash M$, we have $\mathcal{I}(M \cup \{a\}) = S$. Since $M \subseteq T$ and $a \in T$, we have $M \cup \{a\} \subseteq T$. Thus we have

$$\begin{aligned} S = \mathcal{I}(M \cup \{a\}) &= ((M \cup \{a\}) \cup S(M \cup \{a\}) \cup (M \cup \{a\})S \cup S(M \cup \{a\})S] \\ &\subseteq (T \cup ST \cup TS \cup STS] = (T] = T. \quad \square \end{aligned}$$

A chain is, by definition, a non-empty set.

Lemma 1. *If $(P, \leq)$ is an ordered set and $\{a_1, a_2, ..., a_n\}$ a chain in P, then there exists an element $a_k \in \{a_1, a_2, ..., a_n\}$ such that $a_i \leq a_k$ for every $i = 1, 2, ..., n$.*

Proof. If $a_i \leq a_1$ for every $i = 1, 2, ..., n$, then we finish.
Suppose for the element $a_2 \in \{a_1, a_2, ..., a_n\}$ we have $a_2 \not\leq a_1$. (Otherwise, we change the order of the elements in $\{a_1, a_2, ..., a_n\}$). Then $a_1 \leq a_2$.
If $a_i \leq a_2$ for every $i = 3, 4, ..., n$, then we have
$a_1 \leq a_2, \quad a_3, a_4, ..., a_n \leq a_2$, and we finish.
Suppose for the element $a_3 \in \{a_1, a_2, ..., a_n\}$, we have $a_3 \not\leq a_2$. Then $a_2 \leq a_3$.
We continue this way.
Suppose for the element $a_{n-2} \in \{a_1, a_2, ..., a_n\}$, we have $a_{n-2} \not\leq a_{n-3}$. Then $a_{n-3} \leq a_{n-2}$.
If $a_i \leq a_{n-2}$ for every $i = n-1,\ n$, then we have
$a_1 \leq a_2 \leq ..... \leq a_{n-3} \leq a_{n-2}, \quad a_{n-1}, a_n \leq a_{n-2}$, and we finish .
Suppose for the element $a_{n-1} \in \{a_1, a_2, ..., a_n\}$, we have $a_{n-1} \not\leq a_{n-2}$. Then $a_{n-2} \leq a_{n-1}$.
If $a_n \leq a_{n-1}$, then we have $a_1 \leq a_2 \leq ..... \leq a_{n-2} \leq a_{n-1}, \ \ a_n \leq a_{n-1}$, and we finish.
If $a_n \not\leq a_{n-1}$, then $a_{n-1} \leq a_n$, and $a_1 \leq a_2 \leq ..... \leq a_{n-2} \leq a_{n-1} \leq a_n$. $\square$.

By Lemma 1, we have the Lemma 2 below:

Lemma 2. *If S is a non-empty set and $\{A_1, A_2, ..., A_n\}$ a chain in $(\mathcal{P}(S), \subseteq)$, then there exists a set $A_k \in \{A_1, A_2, ..., A_n\}$ such that $A_i \subseteq A_k$ for every $i = 1, 2, ..., n$* $\square$.

Let S be an ordered semigroup (resp. semigroup) and $\emptyset \neq A \subseteq S$. We say that S is generated by A, and write $S = < A >$, if
For each $x \in S$ there exist $a_1, a_2, ...a_n \in A$ (n natural number) such that $x = a_1, a_2, ...a_n$.
For $A = \{b_1, b_2, ...b_n\}$, we write $< b_1, b_2, ...b_n >$ instead of $< \{b_1, b_2, ...b_n\} >$.

Proposition 2. *Let S be a finitely generated ordered semigroup (resp. semigroup). Then, each proper ideal of S is contained in a maximal ideal of S.*

Proof. Let $a_1, a_2, ..., a_n \in S$ such tha $S = < a_1, a_2, ..., a_n >$, and I a proper ideal of S. We consider the set:

$$\mathcal{A} := \{J \mid J \text{ ideal of } S, I \subseteq J \subseteq S\}.$$

We have $\mathcal{A} \neq \emptyset$ (since $I \in \mathcal{A}$), thus $(\mathcal{A}, \subseteq)$ is an ordered set.
Let $\mathcal{B}$ be a chain in $\mathcal{A}$. The set $\bigcup\{B \mid B \in \mathcal{B}\}$ is an upper bound of $\mathcal{B}$ in $\mathcal{A}$. In fact:
Since $\mathcal{B} \neq \emptyset$, the set $\bigcup\{B \mid B \in \mathcal{B}\}$ is an ideal of S -the proof is easy.

$I \subseteq \bigcup\{B \mid B \in \mathcal{B}\} \subset S$. In fact:
Since $I \subseteq B \subset S \ \forall \ B \in \mathcal{A} \supseteq \mathcal{B}$, we have $I \subseteq \bigcup\{B \mid B \in \mathcal{B}\} \subseteq S$.
Let $I \subseteq \bigcup\{B \mid B \in \mathcal{B}\} = S$. Since $a_1, a_2, ..., a_n \in S$, we have

$$a_1, a_2, ..., a_n \in \bigcup\{B \mid B \in \mathcal{B}\}.$$

Let $a_1 \in B_1,\ a_2 \in B_2, ..., a_n \in B_n$ for some $B_1, B_2, ..., B_n \in \mathcal{B}$.
Since $\mathcal{B}$ is a chain in $(\mathcal{A}, \subseteq)$ and $\mathcal{A} \subseteq \mathcal{P}(S)$, $\mathcal{B}$ is a chain in $\mathcal{P}(S)$, $\mathcal{B}$ is a chain in $(\mathcal{P}(S), \subseteq)$.
Since $B_1, B_2, ..., B_n \in \mathcal{B}$, $\{B_1, B_2, ..., B_n\}$ is a chain in $(\mathcal{P}(S), \subseteq)$. By Lemma 2, there exists $B_k \in \{B_1, B_2, ..., B_n\}$ such that $B_1, B_2, ..., B_n \subseteq B_k$. Then $a_i \in B_k$ for every $i = 1, 2, ..., n$.
We have $B_k = S$. Indeed: Let $x \in S$. Since $S =< a_1, a_2, ...a_n >$, we have

$$x = b_1.....b_m \text{ for some } b_1, ..., b_m \in \{a_1, a_2, ..., a_n\}.$$

Since $b_m = a_1 (\in B_1)$ or $b_m = a_2 (\in B_2)$ or ..... or $b_m = a_n (\in B_n)$, we have $b_m \in B_k$. Then we have $x \in SB_k \subseteq B_k$.
On the other hand, since $B_k \in \mathcal{B} \subseteq \mathcal{A}$, we have $B_k \subset S$. Contradiction.

By Zorn's Lemma, there exists a maximal element in $\mathcal{A}$, say M.
Since $M \in \mathcal{A}$, we have $I \subseteq M$, and $M \subset S$ (that is, M is a proper ideal of S).
Let T be an ideal of S such that $M \subseteq T$ and $T \neq M$ $\ (\Rightarrow \quad T = S \ \ ?\ )$
Let $T \neq S$. Since $I \subseteq M$, we have $I \subseteq T \subset S$, and $T \in \mathcal{A}$.
Since $M \subseteq T \in \mathcal{A}$ and M maximal in $\mathcal{A}$, we have $M = T$. Impossible. □

If S is an ordered semigroup (resp. semigroup), a unit of S is an element $e \in S$ such that $ex = xe = x$ for all $x \in S$.

Proposition 3. *Let S be an ordered semigroup (resp. semigroup) for which there exists an element $a \in S$ such that $\mathcal{I}(a) = S$. Then, each proper ideal of S is contained in a maximal ideal of S.*

Proof. Let I be a proper ideal of S. We consider the set:

$$\mathcal{A} \ := \{J \mid J \text{ ideal of } S,\ I \subseteq J \subset S\}.$$

Since $I \in \mathcal{A}$, we have $\mathcal{A} \neq \emptyset$, then the set $M := \bigcup\{J \mid J \in \mathcal{A}\}$ is an ideal of S, and $I \subseteq M$.
The set M is a maximal ideal of S. In fact:
If $M = S$, then $a \in \bigcup\{J \mid J \in \mathcal{A}\}$. Then there exists $J \in \mathcal{A}$ such that $a \in J$. Since J is an ideal of S containing a, by hypothesis, we have $S = \mathcal{I}(a) \subseteq J$. On the other hand, since $J \in \mathcal{A}$, we have $J \subset S$. Impossible. Thus M is a proper ideal of S.
Let now T be an ideal of S such that $M \subseteq T$ and $T \neq S$. Then we have $I \subseteq M \subseteq T \subset S$, $T \in \mathcal{A}$, and $T \subseteq M$. Then $T = M$.

Remark 1. If S is an ordered semigroup (resp. semigroup) and if there is a unit element $e \in S$, then $\mathcal{I}(e) = S$. In fact: Let $x \in S$. Then

$$x = xe \in Se \subseteq (e \cup Se \cup eS \cup SeS] = \mathcal{I}(e). \quad \square$$

By Proposition 3 and Remark 1, we have the following:

Proposition 4. *Let S be an ordered semigroup (resp. semigroup) with unit. Then each proper ideal of S is contained in a maximal ideal of S.*

Remark 2. Let S be an ordered semigroup. If $a \in S$ such that $(Sa] = S$ (or (aS]=S or (SaS]=S), then $\mathcal{I}(a) = S$. In fact, we have

$$S = (Sa] \subseteq (a \cup Sa \cup aS \cup SaS] = \mathcal{I}(a) \subseteq S.$$

If $a \in S$ such that $Sa = S$, then $(Sa] = (S] = S$.
If $a \in S$ such that $aS = S$ (resp. SaS=S), then $(aS] = S$ (resp. $(SaS] = S$).

If now S is a semigroup such that $Sa = S$ or $aS = S$ or $SaS = S$, then $\mathcal{I}(a) = S$.

Remark 3. If for an ordered semigroup (or a semigroup) S there exists an element $a \in S$ such that $\mathcal{I}(a) = S$, then is S finitely generated ? It is not, in general:
We notice first that for each $n \in N$ (N the set of natural numbers) there exists a prime p such that $p > n$. Moreover, if $a_1, a_2, \ldots.., a_k \in N$ and p prime such that $p/a_1a_2.....a_k$, then there exists $a_j \in \{a_1, a_2, \ldots.., a_k\}$ such that p/a_j.
We consider now the ordered semigroup $(N, ., \leq)$ of natural numbers with the usual multiplication-order. Since 1 is the unit of N, we have $\mathcal{I}(1) = N$. Let N be finitely generated, that is, let $a_1, a_2, \ldots.., a_k \in N$ such that $N = < a_1, a_2, \ldots.., a_k >$.
Let p be a prime such that $p > max\{a_1, a_2, \ldots.., a_k\}$. Then $p > a_1, a_2, \ldots.., a_k$. Since $p \in N$, there exist $b_1, b_2, \ldots.., b_m \in \{a_1, a_2, \ldots.., a_k\}$ such that $p = b_1b_2.....b_m$. Then $p/b_1b_2.....b_m$, and p/b_j for some $b_j \in \{b_1, b_2, \ldots.., b_m\}$. Then $b_j = \mu p$ for some $\mu \in N$. Since $\mu \geq 1$, we have $b_j = \mu p \geq p$. Impossible.

2. Similar results hold if, in the results above, we replace the word "ideal" by "left ideal" or "right ideal".
A left ideal L (resp. right ideal R) of an ordered semigroup (resp. semigroup) S is called proper if $L \neq S$ (resp. $R \neq S$). A proper left ideal M of an ordered semigroup (resp. semigroup) S is called maximal if for every left ideal I of S such that $M \subseteq L$, we have $M = L$ or $M = S$. The definition of maximal right ideals is similar.
We have the following:

Proposition 5. *Let S be an ordered semigroup (resp. semigroup) and M a proper left (resp. right) ideal of S. Then M is a maximal ideal of S if and only if for every $a \in S \backslash M$, we have $\mathcal{L}(M \cup \{a\}) = S$.*

Proposition 6. *Let S be a finitely generated ordered semigroup (resp. semigroup). Then, each proper left (resp. right) ideal of S is contained in a maximal left (resp. right) ideal of S.*

Proposition 7. *Let S be an ordered semigroup (resp. semigroup) and let $a \in S$ such that $\mathcal{L}(a) = S$ (resp. $\mathcal{R}(a) = S$). Then, each proper left (resp. right) ideal of S is contained in a maximal left (resp. right) ideal of S.*

Remark 4. If S is an ordered semigroup (resp. semigroup) and if there is a unit element $e \in S$, then $\mathcal{L}(e) = S$ and $\mathcal{R}(e) = S$.

Proposition 8. *Let S be an ordered semigroup (resp. semigroup) with unit. Then each proper left (resp. right) ideal of S is contained in a maximal left (resp. right) ideal of S.*

Remark 5. Let S be an ordered semigroup. If $a \in S$ such that $(Sa] = S$, then $\mathcal{L}(a) = S$. If $a \in S$ such that $(aS] = S$, then $\mathcal{R}(a) = S$.

Remark 6. Let S be an ordered semigroup (resp. semigroup) and let $a \in S$ such that $\mathcal{L}(a) = S$. Then, for each left ideal L of S containing a, we have $L = S$. In fact: Let L be a left ideal of S such that $a \in L$. Then $S = \mathcal{L}(a) \subseteq L \subseteq S$.
Similar results hold if we replace the word "left ideal" by "right ideal" or "ideal'.

Remark 7. If S is an ordered semigroup (resp. semigroup) and $\emptyset \neq A \subseteq S$ such that $S =< A >$ then, for each left ideal L of S such that $A \subseteq L$, we have $L = S$. In fact:
Let $x \in S$. Since $S =< A >$, there exists $a_1, a_2, ..., a_n \in A$ such that $x = a_1 a_2 ..... a_n$.
If $n = 1$, then $x = a_1 \in A \subseteq L$, and $x \in L$.
If $n \geq 2$, then $x = a_1 a_2 ..... a_n = (a_1 a_2 ..... a_{n-1}) a_n \in SA \subseteq SL \subseteq L$, and $x \in L$.

Similarly, if S is an ordered semigroup (resp. semigroup) and $\emptyset \neq A \subseteq S$ such that $S =< A >$ then, for each right ideal R of S such that $A \subseteq R$, we have $R = S$. □

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