Scientiae mathematicae japonicae online, **vol. 5**, (2001), 87–91

REMARK ON ORDERED GROUPOIDS

Niovi Kehayopulu and Michael Tsingelis

Received March 28, 2001

Abstract. For an ordered groupoid G, we denote by G^0 the ordered groupoid arising from G by the adjunction of a zero element. We first prove the following: If S is an ordered groupoid, then there exists a proper prime ideal of S if and only if there exists and ordered group G and a mapping f of S onto G^0 which is a homomorphism. The following question arises: Given an ordered groupoid S under what conditions there exists and ordered group G such that S is isomorphic to G^0 ? We prove the following: If S is an ordered groupoid with a zero element θ such that $S\backslash\{\theta\}$ is a subgroup of S, then there exists an ordered group G such that S is isomorphic to G^0. The converse statement also holds: If S is an ordered groupoid and G an ordered group such that S is isomorphic to G^0, then there exists a zero element θ of S such that $S\backslash\{\theta\}$ is a subgroup of S.

For an ordered groupoid $(S, \circ, \leq)$ we denote by S^0 the ordered groupoid arising from S by the adjunction of a zero element, constructed as follows:
We consider an element 0 which does not belong to S $(0 \notin S)$, and the set $S^0 := S \cup \{0\}$ with the following multiplication-order.

$$* : S^0 \times S^0 \to S^0 \mid (a, b) \to \begin{cases} a \circ b & \text{if } a, b \in S \\ 0 & \text{otherwise} \end{cases}$$

$$\leq_0 : = \leq_S \cup \{(0, x) \mid x \in S^0\}.$$

In particular, if $(S, \circ, \leq)$ is an ordered semigroup, then $(S^0, *, \leq_0)$ is an ordered semigroup, as well, and 0 is the zero of S^0.
In the following, the multiplication and the order on S^0 are always denoted by " $*$ " and $\leq_0$ ".

If $(S, ., \leq)$ is an ordered groupoid, a zero element of S is an element of S, denoted by 0, such that $0x = x0 = 0$ and $0 \leq x$ for every $x \in S$ [1]. Let $(S, ., \leq)$, $(T, *, \preceq)$ be ordered groupoids, $f : S \to T$ a mapping of S into T. f is called isotone if $x \leq y$ implies $f(x) \preceq f(y)$. f is called reverse isotone if $x, y \in S$, $f(x) \preceq f(y)$ imply $x \leq y$. f is called a homomorphism if it is isotone and satisfies $f(xy) = f(x) * f(y)$ for all $x, y \in S$. f is called an isomorphism if it is a homomorphism, onto, an reverse isotone. S and T are called isomorphic, in symbol $S \cong T$, if there exists an isomorphism between them. Each reverse isotone mapping is (1-1).

2000 *Mathematics Subject Classification.* 06F05, 20N02.
Key words and phrases. Ordered groupoid (group, semigroup), ideal of an ordered groupoid, proper ideal, ordered group with zero, adjunction of a zero to an ordered groupoid.

Indeed: Let $x, y \in S$, $f(x) = f(y)$. Since $f(x) \preceq f(y)$, we have $x \leq y$. Since $f(y) \preceq f(x)$, we have $y \leq x$. Then $x = y$ [2].
If $(S, ., \leq)$ is an ordered groupoid, by an ideal of S, we mean a non-empty subset P of S such that 1) $SP \subseteq P$ and $PS \subseteq P$. 2) $a \in P$ and $S \ni b \leq a$ implies $b \in P$. An ideal P of S is called prime if $a, b \in S$, $ab \in T$ implies $a \in T$ or $b \in T$. Equivalent Definition: $A, B \subseteq S$, $AB \subseteq T$ implies $A \subseteq T$ or $B \subseteq T$ [3]. An ideal P of S is called proper if $P \neq S$ [4].

Theorem 1. *Let $(S, ., \leq)$ is an ordered groupoid. The following are equivalent:*
1) *There exists a proper prime ideal of S.*
2) *There exists an ordered group $(G, \circ, \leq_G)$ and a mapping*
   $f : (S, ., \leq) \to (G^0, *, \leq_0)$
   which is homomorphism and onto.

Proof. 1) $\Rightarrow$ 2). Let P be a proper prime ideal of S. We consider the set $G = \{e\}$ with the multiplication " $\circ$ " and the order " $\leq_G$ " on G defined by:

$$e \circ e := e, \quad \leq_G := \{(e, e)\}.$$

Then $(G, \circ, \leq_G)$ is an ordered group and e is the unit of G.
Let $0 \notin G$. We consider the set $G^0 := G \cup \{0\}$ with the multiplication and the order below:

.	e	0
e	e	0
0	0	0

$$\leq_0 = \{(e, e), (0, 0), (0, e)\}.$$

Since P is a proper ideal of S, we have $P \subseteq S$ and $P \neq S$. Then $P \neq \emptyset$ and $S \backslash P \neq \emptyset$.
We consider the mapping:

$$f : (S, \cdot, \leq) \to (G^0, *, \leq_0) \mid a \to f(a) := \begin{cases} e & \text{if } a \in S \backslash P \\ 0 & \text{if } a \in P \end{cases}$$

1) The mapping f is well defined.

2) f is a homomorphism.
Let $a, b \in S$ $\quad (\Rightarrow \quad f(ab) = f(a) * f(b) \quad ?\ )$
If $a \in P$, then $f(a) := 0$, $\ f(a) * f(b) = 0 * f(b) = 0$, $\ ab \in PS \subseteq P$, $\ f(ab) := 0$. Then $f(ab) = f(a) * f(b)$.

Let $a \notin P$. Then $f(a) := e$, $\ f(a) * f(b) = e * f(b)$. Since $f(b) \in G^0$, we have $f(b) = e$ or $f(b) = 0$. If $f(b) = e$, then $e * f(b) = e * e := e = f(b)$. If $f(b) = 0$, then $e * f(b) = e * 0 := 0 = f(b)$. Thus we have $f(a) * f(b) = f(b)$..........(*)
Then: If $b \in P$, then $f(b) := 0$, $\ ab \in PS \subseteq P$, $\ f(ab) := 0$. Then, by (*), $f(a) * f(b) = f(ab)$.
If $b \notin P$, then $f(b) := e$, $\ ab \notin P$ (since $ab \in P$ implies $a \in P$ or $b \in P$), $f(ab) := e$. Then, by (*), $f(a) * f(b) = f(ab)$.

Let $a, b \in S$, $\ a \leq b \quad (\Rightarrow \quad f(a) \leq_0 f(b) \quad ?\ )$
Let $b \in P$. Then $f(b) := 0$, $\ a \leq b \in P$, $\ a \in P$, $\ f(a) := 0$. Then $f(a) \leq_0 f(b)$.

Let $b \notin P$. Then $f(b) := e$. Since $f(a) \in G^0 := \{0, e\}$, by the definition of " $\leq_0$ ", we have $f(a) \leq_0 f(b)$.

3) f is onto.
Since $G^0 = \{0, e\}$, $P \neq \emptyset$ and $S \backslash P \neq \emptyset$.

We finally remark that $P = f^{-1}(0)$. Indeed: Let $a \in f^{-1}(0)$. Then $f(a) = 0$, and $a \in P$ (since $a \notin P$ implies $0 = f(a) := e \in G$. Impossible).

2) $\Rightarrow$ 1). Let $(G, \circ, \leq_G)$ be an ordered group and $f: (S, ., \leq) \rightarrow (G^0, *, \leq_0)$ a mapping which is homomorphism and onto.
Since $0 \in G^0$ and f onto, there exists $x \in S$ such that $f(x) = 0$. Then $x \in f^{-1}(0)$, and $\emptyset \neq f^{-1}(0) \subseteq S$.
The set $f^{-1}(0)$ is a proper ideal of S. In fact: Let $a \in S$, $b \in f^{-1}(0)$. Then $f(b) = 0$. Since f is a homomorphism, $f(ab) = f(a) * f(b) = 0$. Then $ab \in f^{-1}(0)$.
Similarly $f^{-1}(0)S \subseteq f^{-1}(0)$.
Let $b \in f^{-1}(0)$ and $S \ni a \leq b$. Then $f(a) \leq_0 f(b) = 0$. Since 0 is the zero of G^0, we have $f(a) = 0$. Then $a \in f^{-1}(0)$.
Let $a, b \in S$, $ab \in f^{-1}(0)$. Then $f(a) * f(b) = f(ab) = 0$. We have $f(a), f(b) \in G^0$. If $f(a) \neq 0$ and $f(b) \neq 0$, then $f(a), f(b) \in G$ and
$0 = f(a) * f(b) = f(a) \circ f(b) \in G$. Impossible.
Thus $f(a) = 0$ or $f(b) = 0$. Then $a \in f^{-1}(0)$ and $b \in f^{-1}(0)$.
$f^{-1}(0) \neq S$. Indeed: Let e be the unit of G. Then $e \neq 0$. Since $e \in G^0$ and f onto, there exists $y \in S$ such that $f(y) = e$. If $y \in f^{-1}(0)$, then $f(y) = 0$, and $0 = e \in G$. Impossible. Thus $y \notin f^{-1}(0)$.

Corollary. *Let $(S, ., \leq)$ be an ordered groupoid and $P \subseteq S$. The following are equivalent:*
1) *The set P is a proper prime ideal of S.*
2) *There exists an ordered group G and a mapping $f: S \rightarrow G^0$ which is homomorphism and onto such that $P = f^{-1}(0)$.*

Definition. Let $(S, ., \leq)$ be an ordered groupoid. S is called an ordered group with zero if there exists an ordered group $(G, \circ, \leq_G)$ such that $S \cong G^0$.

Theorem 2. *Let $(S, ., \leq)$ be an ordered groupoid. The following are equivalent:*
1) *S is an ordered group with zero.*
2) *There exists a zero element θ of S such that $S \backslash \{\theta\}$ is a subgroup of S.*

Proof. 1) $\Rightarrow$ 2). Let $(G, \circ, \leq_G)$ be an ordered group, and
$f: (S, ., \leq) \rightarrow (G^0, *, \leq_G)$ be an isomorphism (0 is the zero of G^0).
Since $0 \in G^0$, there exists $\theta \in S$ such that $f(\theta) = 0$ .... (*)

The element θ is the zero of S. Indeed: Let $x \in S$. We have

$$f(x\theta) = f(x) * f(\theta) = f(x) * 0 = 0 = f(\theta) \text{ (by (*))} .$$

Since f is (1-1), we have $x\theta = \theta$. Similarly, $\theta x = \theta$.
Since $f(x) \in G^0$ and 0 is the zero of G^0, we have $0 \leq f(x)$. Then $f(\theta) \leq f(x)$, and $\theta \leq x$.

Let $a,b \in S\backslash\{\theta\}$. Then $ab \in S$. If $ab=\theta$, then $f(ab)=f(\theta)=0$. Since f is a homomorphism, $f(a)*f(b)=0$. If $f(a), f(b) \in G$, then

$$0 = f(a)*f(b) := f(a)\circ f(b) \in G.$$

Impossible. Thus $f(a)=0$ or $f(b)=0$. Then $f(a)=f(\theta)$ or $f(b)=f(\theta)$, then $a=\theta$ or $b=\theta$. Impossible. Then $ab \in S\backslash\{\theta\}$.

Let $a \in S\backslash\{\theta\}$ $\quad$ ($\Rightarrow$ $\exists\, 1_S \in S\backslash\{\theta\}$ such that $1_S a = a 1_S = a$ $\quad$?)
Let e be the unit of G. Since $e \in G^0$, there exists $1_S \in S$ such that

$$f(1_S) = e..........(**)$$

If $1_S=\theta$, then $f(1_S)=f(\theta)=0$, and $G \ni e = 0$. Impossible. Thus $1_S \in S\backslash\{\theta\}$. We have $f(1_S a) = f(1_S)*f(a) = e*f(a) = f(a)$ (by (**)). Since f is (1-1), $1_S a = a$. Similarly, $a1_S = a$.

Let $a \in S\backslash\{\theta\}$ $\quad$ ($\Rightarrow$ $\exists\ a^{-1} \in S\backslash\{\theta\}$ such that $a^{-1}a = aa^{-1} = 1_S$ $\quad$?)
Since $a \in S$, we have $f(a) \in G^0$. If $f(a)=0$, then $f(a)=f(\theta)$, and $S \ni a = \theta$. Impossible. Thus $f(a) \in G$. Since G is a group, there exists $g \in G$ such that

$$f(a)\circ g = g \circ f(a) = e,$$

where e is the unit of G. Since $g \in G \subseteq G^0$, there exists $a^{-1} \in S$ such that $f(a^{-1}) = g$. Then we have

$$f(a)\circ f(a^{-1}) = f(a^{-1})\circ f(a) = f(1_S)$$

by (**). Since $f(a), f(a^{-1}) \in G$, we have $f(a)*f(a^{-1}) = f(a)\circ f(a^{-1})$ and $f(a^{-1})*f(a) = f(a^{-1})\circ f(a)$. Since f is a homomorphism, $f(a^{-1}a) = f(aa^{-1}) = f(1_S)$. Since f is (1-1), $a^{-1}a = aa^{-1} = 1_S$.

2) $\Rightarrow$ 1). By hypothesis, the set $G := S\backslash\{\theta\}$ with the multiplication " $\circ$ " and the order " $\leq_G$ " on G defined by

$\circ \ :\ G\times G \ \rightarrow\ G \ \mid\ (a,b) \rightarrow\ ab$

$\leq_G := \leq \cap (G\times G)$

is an ordered group. We consider the set $G^0 := G\cup\{0\}$, where $0 \notin G$ with the multiplication and the order on G^0 defined by

$$a*b := \begin{cases} a\circ b & \text{if } a,b \in G \\ 0 & \text{otherwise} \end{cases} \qquad (a,b \in G^0)$$

$$\leq_0 \,:=\, \leq_G \cup \{(0,x) \mid x \in G^0\}.$$

We have already seen in [5] that $(G^0,*,\leq_0)$ is an ordered groupoid and 0 is the zero of G^0. The mapping

$$f : (S,\cdot,\leq) \rightarrow (G^0,*,\leq_0) \mid x \rightarrow \begin{cases} x & \text{if } x \in S\backslash\{\theta\} \\ 0 & \text{if } x = \theta \end{cases}$$

is an isomorphism. The proof is easy.

This research was supported by the Special Research Account (No. 70/4/4205) of the University of Athens.

References

[1] G. Birkhoff, *Lattice Theory,* Amer. Math. Soc. Coll. Publ. Vol. **XXV**, Providence, Rh. Island, 1967.

[2] N. Kehayopulu and M. Tsingelis, *On subdirectly irreducible ordered semigroups,* Semigroup Forum, **50** (1995), 161–177.

[3] N. Kehayopulu, *On weakly prime ideals of ordered semigroups,* Mathematica Japonica, **35** (No. 6) (1990), 1051–1056.

[4] N. Kehayopulu, *Note on Green's relations in ordered semigroups,* Mathematica Japonica, **36** (No. 2) (1991), 211–214.

[5] N. Kehayopulu and M. Tsingelis, *A note on ordered groupoids, semigroups,* Scientiae Mathematicae, **3** (No. 2) (2000), 315–319.

University of Athens, Department of Mathematics
Mailinig (home) address: Nikomidias 18, 161 22 Kesariani, Greece
e-mail: nkehayop@cc.uoa.gr