ON THE CATEGORIES OF LFNS AND QFTVS

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Abstract. The main results of this paper are the following: (1) the category of linear fuzzy neighborhood spaces (for short, **LFNS**) is isomorphic to that of co-towers of topological vector spaces and (2) the intersection of **LFNS** with the category of (QL)-type fuzzy topological vector spaces is exactly the category of induced fuzzy topological vector spaces.

1. Introduction

In 1982 Lowen [8] introduced a very important class of fuzzy topological spaces-fuzzy neighborhood spaces (according to the standardized terminology in [12], fuzzy topological spaces are also called [0,1]-topological space sometimes), since then, this kind of spaces has received wide attention in fuzzy topology. Combining this kind of fuzzy topological structure with vector structure, A.K.Katsaras [2] introduced the concept of linear fuzzy neighborhood spaces in 1985, and discussed many properties of this spaces in [3]. From then on, this method was generalized by T. M. G. Ahsanullah, he combined the fuzzy neighborhood structure with group structure, ring and modules, etc. The concepts of fuzzy neighborhood groups [9] and fuzzy neighborhood rings ([10], [11]) were introduced one after another. At the same time, Wu and Fang [13] introduced an important class of fuzzy topological vector spaces called (QL)-type fuzzy topological vector spaces in 1985. Based on this idea, fuzzy normble, locally bounded and locally convex of fuzzy topological vector spaces were studied ([14]-[16]). Moreover, this method was also generalized to the research of fuzzy topological groups and fuzzy topological algebras, etc. Here we must point out there are two different kind of neighborhood structures in these researches. It is a natural question to make clear the relationship between linear fuzzy neighborhood spaces and (QL)-type fuzzy topological vector spaces. The main purpose of this paper is to answer this question.

First, we fix some notations. In this paper, $I=[0,1]$, $I_0=(0,1]$, $I_1=[0,1)$, I^X will denote the family of all fuzzy sets of X, $\tilde{X}$ will denote the set of all fuzzy points. For all $r\in[0,1]$, r^* is the fuzzy set which takes the constant value r on X. A fuzzy point x_λ is said to be quasi-coincident with fuzzy set U, denoted by $x_\lambda \hat{q} U$, iff $U(x)>1-\lambda$. For $A\in I^X$, $\alpha\in I_1$, $\sigma_\alpha(A)=\{\ x\in X\mid \alpha<A(x)\}$, $A_{[\alpha]}=\{\ x\in X\mid \alpha\le A(x)\}$. Other symbols which is not mentioned here we refer to [4], [5].

Definition 1.1 [2] Let X be a vector space over $\mathbf{K}$ (where $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$). A fuzzy neighborhood system $\mathbf{N}$ on X is called linear if the functions

$$+: X\times X\to X,\ \ (x,y)\to x+y \quad \text{and} \quad \cdot\ :\mathbf{K}\times X\to X,\ \ (k,x)\to kx,$$

are n-continuous when $\mathbf{K}$ is equipped with the usual fuzzy neighborhood system N_K and $X\times X$ $\mathbf{K}\times X$ have the corresponding product fuzzy neighborhood systems.

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Definition 1.2 [5] Let (X, Δ) be a fuzzy topological space and $x_\lambda \in \tilde{X}$, A fuzzy set U in X is called a Q-neighborhood of x_λ iff there exists $G \in \Delta$ such that $G \subset U$ and $x_\lambda \hat{q} G$.

Definition 1.3 [13], [4] Let (X, Δ) be a fuzzy topological vector space. (X, Δ) is called a (QL)-type fuzzy topological vector space iff there exists a family of fuzzy sets $\mathcal{U}$ in X such that for each $\lambda \in I_0$

$$\mathcal{U}_\lambda = \{\ U \wedge \mu^* \mid U \in \mathcal{U},\ \mu \in (1-\lambda, 1]\ \}$$

is a Q-neighborhood base of θ_λ in (X, Δ). $\mathcal{U}$ is called a Q-prebase of (X, Δ).

The following definition is a special case of the definition of co-tower of topologies [7].

Definition 1.4 A co-tower of topologies on a set X (indexed by I_1) is a family of topologies $\Gamma = \{\ \tau_a \mid a \in I_1\}$ such that τ_a is generated by $\bigcup_{a<b} \tau_b$ as a subbasis. τ_a is called the a-level topology of Γ.

Particularly, if each τ_a is a vector topology for $a \in I_1$, then we say Γ is a co-tower of vector topologies and (X, Γ) is called a co-tower vector space for short. A linear function between two co-tower vector spaces is called continuous if it is continuous with respect to every level topology. The category of co-towers vector spaces is denoted $\mathbf{TVS}^c$.

Proposition 1 [2] *A fuzzy neighborhood system* $\mathbf{N}$ *on a vector space* X *is linear iff* $(X, t(\mathbf{N}))$ *is a fuzzy topological vector space.*

Proposition 2 [7] *Let* $\{\ \tau_a \mid a \in I_1\}$ *be a co-tower of topologies on a set* X*, then the operator* $^\circ : I^X \to I^X$ *defined by*

$$A^\circ = \bigvee_{a \in I_1} a \wedge int_a(\sigma_a(A)) = \bigvee_{a \in I_1} a \wedge int_a A_{[a]},$$

where int_a *is the interior operator with respect to* τ_a*, is a fuzzy interior operator on* X*. Thus, it induces a fuzzy topology on* X*, denoted* $\delta(\Gamma)$*. And a fuzzy set* A *is open in* $\delta(\Gamma)$ *iff* $\sigma_a(A) \in \tau_a$ *for all* $a \in I_1$*.*

Proposition 3 [7] *Let* (X, Δ) *be a fuzzy topological space, then the following are equivalent:*

(1) *For all* $A \in \Delta,\ a \in I_1,\ a\sigma_a(A) \in \Delta$,
(2) Δ *has a basis consisting of leveled characteristic functions,*
(3) Δ *has a subbasis consisting of leveled characteristic functions.*
(4) *There exists a co-tower of topologies on* X*,* $\Gamma = \{\iota_a \mid a \in I_1\ \}$ *such that* $\Delta = \delta(\Gamma)$.

Proposition 3 is a special case of Theorem 3.1 [7] with $L = [0, 1]$. By the results in [6], [7], [18] and [20], we know that a fuzzy topological space (X, Δ) satisfying one of the equivalent conditions in the above Proposition is just a fuzzy neighborhood space in the sense of R. Lowen [8].

Proposition 4. [19] *Let* $(X,\ \Delta)$ *be a ftvs, then the following conclusions hold:*

(1) $(X,\ \Delta)$ *is a Hausdorff fuzzy topological space;*
(2) *For each* $\lambda \in I_0$*, the* θ_λ *is a closed fuzzy set;*
(3) *For each* $\lambda \in I_0, x \in X$ *with* $x \neq \theta$*, there exists a* Q*-neighborhood* U *of* θ_λ *such that* $U(x) = 0$.

Proposition 5. [4] *Let* (X, Δ) *be a ftvs. Then*

(1) The mapping f in Definition 1.1 (addition) is continuous iff for every fuzzy point $(x,y)_\lambda$ in $X \times X$ and any Q-neighborhood W of $(x+y)_\lambda$, there exist Q-neighborhoods U of x_λ and V of y_λ such that $U+V \subset W$;

(2) The mapping g in Definition 1.1 (scalar multiplication) is continuous iff for every fuzzy point $(k,x)_\lambda$ in $\mathbf{K}\times X$ and any Q-neighborhood W of kx_λ, there exists a Q-neighborhood V of x_λ and $\varepsilon > 0$ such that $tV \subset W$ for all $t \in \mathbf{K}$ with $|t-k| < \varepsilon$.

2. The categories of LFNS and TVS^c

Theorem 1. *Let (X,Δ) be a Hausdorff [5] fuzzy topological vector space, then for each $\lambda \in I_1$, $(X,\ \tau_\lambda(\Delta))$ is a crisp Hausdorff topological vector space, where $\tau_\lambda(\Delta) = \{\sigma_\lambda(A) \mid A \in \Delta\}$, called level topology of Δ.*

Proof. For each $\lambda \in I_1$, clearly $(X, \tau_\lambda(\Delta))$ is a crisp topological space. Next we prove this topological structure is compatible with the vector structure on X.

For each open neighborhood $\sigma_\lambda(A)$ of $x+y$, i.e., $A(x+y) > \lambda = 1-(1-\lambda)$, then A is an open Q-neighborhood of $(x+y)_{1-\lambda}$. By Proposition 5, there exists open Q-neighborhoods U of $x_{1-\lambda}$ and V of $y_{1-\lambda}$ such that $U+V \subset A$. So $x \in \sigma_\lambda(U) \in \sigma_\lambda(\Delta)$, $y \in \sigma_\lambda(V) \in \sigma_\lambda(\Delta)$ and $\sigma_\lambda(U)+\sigma_\lambda(V) = \sigma_\lambda(U+V) \subset \sigma_\lambda(A)$. Since the $\sigma_\lambda(U)$ and $\sigma_\lambda(V)$ are open neighborhoods of x and y respectively, the continuity of addition operator holds. Similarly we may prove the continuity of the scalar multiplication. Hence $(X,\ \tau_\lambda(\Delta))$ is a crisp topological vector space.

Now we check that $(X, \tau_\lambda(\Delta))$ is a Hausdorff space. For each $x \neq \theta$ and $1-\lambda \in I_0$, by Proposition 4, we have an open Q-neighborhood U of $\theta_{1-\lambda}$ such that $U(x) = 0$. Obviously $\sigma_\lambda(U)$ is an open neighborhood of θ and $x \notin \sigma_\lambda(U)$, this shows that $(X, \tau_\lambda(\Delta))$ is Hausdorff. □

For each fuzzy topology Δ, denote $\tau(\Delta) = \{\ \tau_\alpha \mid \alpha \in I_1\}$, then we have the following:

Corollary 1. *The correspondence $\tau : \mathbf{LFNS} \to \mathbf{TVS}^c,\ \Delta \to \tau(\Delta)$, is a functor.*

Proof. Let $(X,\Delta))$ be a linear fuzzy neighborhood space [2], from the results in [17] and Theorem 1, $\tau(\Delta) = \{\ \tau_\alpha \mid \alpha \in I_1\}$ is a co-tower of vector topologies. We easily prove the following fact: a linear functional $f : (X, \Delta^X) \to (Y, \Delta^Y)$ is fuzzy continuous iff $f : (X, \tau(\Delta^X)) \to (Y, \tau(\Delta^Y))$ is continuous, thus the conclusion holds. □

Theorem 2. *Let (X,Δ) be a (QL)-type fuzzy topological vector space and $\mathcal{U}$ a Q-prebase of it. Then for each $\lambda \in I_1$, $\tau_\lambda(\mathcal{U}) = \{\ \sigma_\lambda(U) \mid U \in \mathcal{U}\}$ is a neighborhood base of θ in $(X, \tau_\lambda(\Delta))$. Specially, if (X,Δ) is locally convex ftvs, then $(X, \tau_\lambda(\Delta))$ is locally convex topological vector space for all $\lambda \in I_1$.*

Proof. Suppose $U \in \mathcal{U}$, first we prove $\sigma_\lambda(U)$ is a neighborhood of θ with respect to $\tau_\lambda(\Delta)$. Since $\mathcal{U}$ is Q-prebase, for $\lambda \in I_1$, $\mathcal{U}_{1-\lambda} = \{\ U \wedge r^* \mid U \in \mathcal{U},\ r \in (\lambda, 1]\}$ is a Q-neighborhood base of $\theta_{1-\lambda}$. Notice that $U \wedge r^* \subset U$, we have U is a Q-neighborhood of $\theta_{1-\lambda}$, then $\sigma_\lambda(U)$ is a neighborhood of θ with respect to $\tau_\lambda(\Delta)$.

On the other hand, for each neighborhood A of θ with respect to $\tau_\lambda(\Delta)$, then there exists a $\sigma_\lambda(G) \in \tau_\lambda(\Delta)$ such that $\theta \in \sigma_\lambda(G) \subset A$. Since $G \in \Delta$ and $G(\theta) > \lambda$, we have G is an open Q-neighborhood of $\theta_{1-\lambda}$, so there are $U \in \mathcal{U}$ and $r > \lambda$ such that $U \wedge r^* \subset G$, thus

$$\theta \in \sigma_\lambda(U \wedge r^*) = \sigma_\lambda(U) \subset \sigma_\lambda(G) \subset A.$$

Hence the first part holds. As for the second part, since (X,Δ) is locally convex ftvs, we may assume $\mathcal{U}$ is a convex Q-prebase of it, each element in $\tau_\lambda(\mathcal{U}) = \{\ \sigma_\lambda(U) \mid U \in \mathcal{U}\}$ is convex, $\forall \lambda \in\ I_1$. Therefore the conclusion holds. □

By Proposition 2, the proof of the next Lemma is trivial.

Lemma 1. *Let $(X, \delta(\Gamma))$ be a fuzzy topological space induced by a co-tower of topologies $\Gamma = \{\tau_\alpha\}_{\alpha \in I_1}$. Then for each $\alpha \in I_1$ and $U \in \tau_\alpha$, $\alpha^* \wedge U \in \delta(\Gamma)$.*

Theorem 3. *Suppose that $\Gamma = \{\iota_\alpha\}_{\alpha \in I_1}$ be a co-tower of vector topologies on X. Then there exists a fuzzy topology $\delta(\Gamma)$ on X such that $(X, \delta(\Gamma))$ is a fuzzy topological vector space and for each $\alpha \in I_1$, $\tau_\alpha(\delta(\Gamma)) = \iota_\alpha$.*

Proof. By Proposition 2, there exists a fuzzy topology $\delta(\Gamma)$ on X such that for each $A \in I^X$, $A \in \delta(\Gamma)$ iff for each $\alpha \in I_1, \sigma_\alpha(A) \in \iota_\alpha$. First we prove that $\delta(\Gamma)$ is a fuzzy vector topology. for each $(x, y)_\alpha$ in $X \times X$ and any open Q-neighborhood W of $(x+y)_\alpha$, we have $W(x+y) > 1-\alpha$. Then there exists a $\mu > 0$ such that $W(x+y) > 1-\alpha+\mu > 1-\alpha$, so $(x+y) \in \sigma_{1-\alpha+\mu}(W) \in \tau_{1-\alpha+\mu}$. Since $(X, \iota_{1-\alpha+\mu})$ is a crisp topological vector space, there exist open neighborhoods U of x and V of y such that $U+V \subset \sigma_{1-\alpha+\mu}(W)$, thus $(1-\alpha+\mu)^* \wedge U + (1-\alpha+\mu)^* \wedge V \subset (1-\alpha+\mu)^* \wedge (U+V) \subset (1-\alpha+\mu)^* \wedge \sigma_{1-\alpha+\mu}(W) \subset W$. By Lemma 1, $(1-\alpha+\mu)^* \wedge U \in \delta(\Gamma)$ and $(1-\alpha+\mu)^* \wedge V \in \delta(\Gamma)$, it is clear $x_\alpha \hat{q} (1-\alpha+\mu)^* \wedge U$ and $y_\alpha \hat{q} (1-\alpha+\mu)^* \wedge V$. This shows $(1-\alpha+\mu)^* \wedge U$ and $(1-\alpha+\mu)^* \wedge V$ are Q-neighborhood of x_α and y_α respectively, from Proposition 5, the mapping f in Definition 1.1 is continuous.

On the other hand, Let $(k, x)_\alpha$ in $\mathbf{K} \times X$ and W is an open Q-neighborhood of $(kx)_\alpha$. Similarly the above proof, there exists $\mu > 0$ such that $\sigma_{1-\alpha+\mu}(W)$ is an open neighborhood of kx in $(X, \iota_{1-\alpha+\mu})$, then we have $\varepsilon > 0$ and an open neighborhood V of x such that $tV \subset \sigma_{1-\alpha+\mu}(W)$ for all t with $|t-k| < \varepsilon$. Thus $t \wedge (1-\alpha+\mu)^* \wedge V \subset (1-\alpha+\mu)^* \wedge \sigma_{1-\alpha}(W) \subset W$. Clearly $(1-\alpha+\mu)^* \wedge V$ is an open Q-neighborhood of x_α, then g is continuous. Hence $(X, \delta(\Gamma))$ is a ftvs.

Finally, we prove that for each $\alpha \in I_1$, $\iota_\alpha = \tau_\alpha(\delta(\Gamma))$, if $A \in \iota_\alpha$, then $A = \bigcup_{\lambda > \alpha} A^{\lambda, \alpha}$, here $A^{\lambda,\alpha} \in \iota_\lambda$. So for each $x \in A$, there exists $\lambda > \alpha$ such that $x \in A^{\lambda,\alpha}$, by Lemma 1, $\lambda^* \wedge A^{\lambda,\alpha} \in \delta(\Gamma)$, thus $x \in A^{\lambda,\alpha} = \tau_\alpha(\lambda^* \wedge A^{\lambda,\alpha}) \in \tau_\alpha(\delta(\Gamma))$. This implies A is a neighborhood of x with respect to $\tau_\alpha(\delta(\Gamma))$, hence $A \in \tau_\alpha(\delta(\Gamma))$. Therefore the conclusion holds. □

Remark. By Proposition 2 and the results in [6], [7], [18], $(X, \delta(\Gamma))$ in Theorem 3 is a special kind of fuzzy topological vector spaces–linear fuzzy neighborhood space. So we may define a mapping $\delta : \mathbf{TVS}^c \to \mathbf{LFNS}$, $\Gamma \to \delta(\Gamma)$. Moreover, we have the following:

Theorem 4. *The category* **LFNS** *is isomorphic to the category* $\mathbf{TVS}^c$.

Proof. By Corollary 1 and the above Remark, it suffices to show the following: $\delta \circ \tau = 1_{LFNS}$ and $\tau \circ \delta = 1_{TVS^c}$.

For each $\Delta \in \mathbf{LFNS}$, $\delta(\tau(\Delta)) \supseteq \Delta$ is obvious. On the contrary, if $A \in \delta(\tau(\Delta))$ and each $x_\alpha \hat{q} A$, then there exists a $\varepsilon > 0$ such that $x \in \sigma_{1-\alpha+\varepsilon}(A) \in \tau_{1-\alpha+\varepsilon}(\Delta)$. Thus we have a $B \in \Delta$ such that $\sigma_{1-\alpha+\varepsilon}(A) = \sigma_{1-\alpha+\varepsilon}(B)$, since (X, Δ) is a fuzzy neighborhood space, from Proposition 3, $(1-\alpha+\varepsilon)^* \wedge \sigma_{1-\alpha+\varepsilon}(B) \in \Delta$. Clearly $(1-\alpha+\varepsilon)^* \wedge \sigma_{1-\alpha+\varepsilon}(B)$ is a Q-neighborhood of x_α with respect to Δ and $(1-\alpha+\varepsilon)^* \wedge \sigma_{1-\alpha+\varepsilon}(B) \subseteq (1-\alpha+\varepsilon)^* \wedge \sigma_{1-\alpha+\varepsilon}(A) \subset A$, this shows A is a Q-neighborhood of x_α with respect to Δ, so $A \in \Delta$. Hence $\delta \circ \tau = 1_{LFNS}$, the second part holds by Theorem 3. □

Therefore, a linear fuzzy neighborhood space can be characterized completely by a co-tower of vector topologies.

Theorem 5. *Suppose that $(X, \delta(\Gamma))$ be a fuzzy topological vector space determined by Γ, if (X, τ_α) is a separated topological vector space for each $\alpha \in I_1$. Then $(X, \delta(\Gamma))$ is also a separated fuzzy topological vector space.*

Proof. For each $\lambda \in I_0$ and $x \neq \theta$, then $1-\lambda \in I_1$, further we may choose $\varepsilon > 0$ such that $1-\lambda+\varepsilon \in (0,1)$. Since $(X, \tau_{1-\lambda+\varepsilon})$ is a separated topological vector space. Thus there exists an open neighborhood U of θ such that $x \notin U$. $\Big((1-\lambda+\varepsilon)^* \wedge U\Big)(x) = 0$. By Lemma 1, we know $(1-\lambda+\varepsilon)^* \wedge U$ is a Q-neighborhood of θ_λ. From Proposition 3, $(X, \delta(\Gamma))$ is a Hausdorff fuzzy topological space. □

3. The categories of LFNS and QFTVS

In this section, we will give two Examples at first.

Example 1. *Let $X = \mathbf{R}$ and $\mathcal{U} = \{\ tA \mid t > 0\}$ a family of fuzzy sets on X, where $A(x) = 1$ for $x \in (-1,1)$, and $A(x) = \frac{1}{2}$ if $x \notin (-1,1)$. Then there exists a unique fuzzy topology Δ on X such that (X, Δ) is a (QL)-type fuzzy topological vector space and $\mathcal{U}$ is a Q-prebase of it. In addition, (X, Δ) is not a linear fuzzy neighborhood space.*

In order to verify the above Example, at first we show that $\mathcal{U}$ satisfies (a)-(d) of Theorem 5.1 [4]. (a) and (c) follow directly from that the fuzzy set A is balanced.

(b) If $tA \in \mathcal{U}$ and $\lambda \in I_0$, then $\frac{t}{2}A \in \mathcal{U}$. If $(\frac{t}{2}A + \frac{t}{2}A)(x) = a > 0$, then for each $b \in (0,a)$, there exist $y,\ z \in \mathbf{R}$ such that $\frac{t}{2}A(y) > b$ and $\frac{t}{2}A(z) > b$ with $y+z = x$, so $\frac{2y}{t} \in \sigma_b(A)$ and $\frac{2z}{t} \in \sigma_b(A)$, thus $y \in \frac{t}{2}\sigma_b(A)$ and $z \in \frac{t}{2}\sigma_b(A)$. Since $t\sigma_b(A)$ is a crisp convex set in $\mathbf{R}$, hence $x = y+z \in \Big(\frac{1}{2}(t\sigma_b(A)) + \frac{1}{2}(t\sigma_b(A))\Big) \subset t\sigma_b(A)$, i.e. $tA(x) > b$, from the arbitrariness of $b \in (0,a)$, we get $tA(x) \geq a$. This shows $\frac{t}{2}A + \frac{t}{2}A \subseteq tA$, thus for each $r \in (1-\lambda, 1]$, we have $(\frac{t}{2}A + \frac{t}{2}A) \wedge r^* \subseteq tA$.

(d) If $tA \in \mathcal{U}$, then for any $x_\lambda \in \tilde{X}$, then there exist a positive number $s > 0$ such that $x \in (-t,t)$, so $x_\lambda \hat{q} s(tA)$.

Therefore there exists a unique fuzzy topology Δ on X such that (X, Δ) is a (QL)-type fuzzy topological vector space and $\mathcal{U}$ is a Q-prebase of it. We must point out this space is not linear fuzzy neighborhood space. Otherwise, if (X, Δ) is a linear fuzzy neighborhood space, then its level topologies $\tau(\Delta) = \{\tau_\alpha(\Delta) \mid \alpha \in I_1\}$ is a co-tower of vector topologies. Let's consider two levels $(X, \tau_{\frac{3}{4}}(\Delta))$ and $(X, \tau_{\frac{1}{3}})$. From Theorem 2, $\tau_{\frac{3}{4}}(\mathcal{U}) = \{(-t,t) \mid t > 0\}$ is a neighborhood base of θ with respect to $\tau_{\frac{3}{4}}(\Delta)$ and $\tau_{\frac{1}{3}}(\mathcal{U}) = \{X\}$ is a neighborhood base of θ with respect to $\tau_{\frac{1}{3}}(\Delta)$. This contradicts with the co-tower of vector topologies $\tau(\Delta)$. Hence (X, Δ) is not a linear fuzzy neighborhood space.

Example 1 indicates there exists a (QL)-type fuzzy topological vector space which is not linear fuzzy neighborhood space.

Example 2. *Let (X, J) be a crisp topological vector space and denote $\Delta^J = \{r^* \mid r \in (\frac{1}{2}, 1]\} \bigcup \{U \bigcap r^* \mid U \in J,\ r \in (0, \frac{1}{2}]\}$. Then (X, Δ^J) is a linear fuzzy neighborhood space but it is not a (QL)-type fuzzy topological vector space.*

It is easy to see Δ^J is a fuzzy topology. Moreover, for each $\alpha \geq \frac{1}{2}$, $\tau_\alpha(\Delta^J) = \{\ X,\ \emptyset\}$ and if $\alpha \in [0, \frac{1}{2})$, $\tau_\alpha(\Delta^J) = J$. Clearly $\Gamma = \tau(\Delta) = \{\ \tau_\alpha\}_{\alpha \in I_1}$ is a co-tower of vector topologies. By Theorem 3, there exists a fuzzy topology $\delta(\Gamma)$ such that $(X, \delta(\Gamma))$ is a linear fuzzy neighborhood space. Next we prove $\delta(\Gamma) = \Delta^J$, the relation $\delta(\Gamma) \supseteq \Delta^J$ is trivial. For each $A \in \delta(\Gamma)$, here $A \neq \emptyset$, then for each $x_\lambda \hat{q} A$, we have $\varepsilon > 0$ such that $x \in \sigma_{1-\lambda+\varepsilon}(A) \in \tau_{1-\lambda+\varepsilon}(\Delta^J)$. If $1-\lambda+\varepsilon \geq \frac{1}{2}$, then $\sigma_{1-\lambda+\varepsilon}(A) = X$, so $x_\lambda \hat{q}(1-\lambda+\varepsilon)^* \wedge X = (1-\lambda+\varepsilon)^* \wedge \sigma_{1-\lambda+\varepsilon}(A) \subset A$, this shows A is a Q-neighborhood of x_λ with respect to Δ^J. If $1-\lambda+\varepsilon < \frac{1}{2}$, thus there exists a $U \in J$ such that $\sigma_{1-\lambda+\varepsilon}(A) = U$, then $x_\lambda \hat{q}(1-\lambda+\varepsilon)^* \wedge U = (1-\lambda+\varepsilon)^* \wedge \sigma_{1-\lambda+\varepsilon}(A) \subset A$, this implies A is a Q-neighborhood of x_λ with respect to Δ^J. Hence $A \in \Delta^J$, then $\delta(\Gamma) \subseteq \Delta^J$. Therefore (X, Δ^J) is a linear fuzzy neighborhood space.

On the other hand, Wu and Fang proved that (X, Δ^J) is not a (QL)-type fuzzy topological vector space in [13].

The category of induced fuzzy topological vector space is denoted by $\omega(\mathbf{TVS})$. Then we have the following:

Theorem 6. $\mathbf{LFNS} \bigcap \mathbf{QFTVS} = \omega(\mathbf{TVS})$

Proof. From the results in [2] and [13], $\mathbf{LFNS} \bigcap \mathbf{QFTVS} \supseteq \omega(\mathbf{TVS})$ holds. Suppose that $(X, \Delta) \in \mathbf{LFNS} \bigcap \mathbf{QFTVS}$, then $\tau(\Delta)$ is a co-tower of vector topologies and there exists a family of fuzzy sets $\mathcal{U}$ such that $\mathcal{U}$ is a Q-prebase of Δ. By Theorem 2, for each $\alpha \in I_1$, $\{\sigma_\alpha(U) \mid U \in \mathcal{U}\}$ is a neighborhood base of θ in $(X, \tau_\alpha(\Delta))$. For each $\alpha \in I_1$ and $\mu > \alpha$, $\mu \in I_1$, if $A \in \tau_\alpha(\Delta)$ and any $x \in A$, then there is a $U \in \mathcal{U}$ such that $x + \sigma_\alpha(U) \subseteq A$. So $x + \sigma_\mu(U) \subseteq x + \sigma_\alpha(U) \subseteq A$, this shows A is a neighborhood of x with respect to $\tau_\mu(\Delta)$. Thus $A \in \tau_\mu(\Delta)$. Therefore $\tau_\alpha(\Delta) = \tau_\mu(\Delta)$ for all α, $\mu \in I_1$, i.e., all the levels of $\tau(\Delta)$ are equal. So (X, Δ) is an induced fuzzy topological vector space, this shows $\mathbf{LFNS} \bigcap \mathbf{QFTVS} \subseteq \omega(\mathbf{TVS})$, the conclusion holds. □

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