ERROR ANALYSIS OF THE SHORTLEY-WELLER FINITE DIFFERENCE METHOD APPLIED TO TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper the Shortley-Weller finite difference method applied to the two-point boundary value problems $-(p(x)u')' = f$ with the Dirichlet boundary condition is considered. We show several error bounds of the Shortley-Weller finite difference solutions in the case where p and f have certain regularity using Yamamoto's explicit inversion formula for tridiagonal matrices. We also consider the cases where p and f are discontinuous.

1 Introduction Let $I := (a, b) \subset \mathbb{R}$ be a one-dimensional bounded interval. Let $p \in L^\infty(I)$. We consider the following two-point boundary value problem: for a given $f(x)$ find $u(x)$ such that

$$-(p(x)u'(x))' = f(x) \text{ in } I, \quad u(a) = u(b) = 0. \tag{1.1}$$

It is well-known that if there exists a positive constant δ such that $p(x) \geq \delta > 0$, then (1.1) has a unique solution $u \in H_0^1(I)$ for any $f \in H^{-1}(I)$. Furthermore, it has been known (see [5]) that, if $1/p \in L^\infty(I)$ and $\int_I dx/p(x) \neq 0$, then the equation (1.1) has a unique solution $u \in W_0^{1,q}(I)$ for any $f \in W^{-1,q}(I)$ with any q, $1 \leq q \leq \infty$. In this paper we always suppose that (1.1) has a unique solution. In the above, $L^q(I)$, $W^{k,q}(I)$, $(1 \leq q \leq \infty$, k is an integer), $H_0^1(I)$, $H^{-1}(I)$ are usual Lebesgue and Sobolev spaces. For the exact definitions see, for example, [1]. Also $C^{k,\beta}(\bar{I})$ (k is a positive integer, and $0 < \beta \leq 1$) denotes usual Hölder space. Note that by Sobolev's imbedding theorem $W^{k,q}(I) \subset C^{k-1,1-(1/q)}(\bar{I})$ for a positive integer k. Note also that, in particular, $W^{k,\infty}(I) = C^{k-1,1}(\bar{I})$ since any Lipschitz functions are differentiable *a.e.* in I.

We approximate the solution of (1.1) by the Shortley-Weller finite difference method (see, for example, [3]). Let

$$a = x_0 < x_1 < \cdots < x_i < \cdots < x_{n+1} = b, \quad h_i := x_i - x_{i-1}, \quad h := \max_i h_i \tag{1.2}$$

be a triangulation (or partition) of the interval I. We also set $x_{i+\frac{1}{2}} := \frac{1}{2}(x_i + x_{i+1})$, $p_{i+\frac{1}{2}} := p(x_{i+\frac{1}{2}})$, and $f_i := f(x_i)$. Then the Shortley-Weller finite difference approximation of (1.1) is defined by

$$-\frac{p_{i+\frac{1}{2}}\frac{u_{i+1}-u_i}{h_{i+1}} - p_{i-\frac{1}{2}}\frac{u_i-u_{i-1}}{h_i}}{\frac{h_{i+1}+h_i}{2}} = f_i, \quad i = 1, 2, \cdots, n. \tag{1.3}$$

Let $U(x_i)$ be the solution of (1.3). By a usual interpolation U may be regarded as a piecewise linear function defined on I.

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In this paper we will develop a detailed analysis on the error

$$\max\{|u(x_i) - U(x_i)| : i = 1, \cdots, n\}.$$

First, in Section 3, we recall the recently obtained error estimates

$$\max_i |u(x_i) - U(x_i)| = \mathcal{O}(h^2)$$

by Fang-Tsuchiya-Yamamoto [2] on the non-uniform partition (1.2) under the assumption $p \in C^{1,1}(\bar{I})$. We will give a proof which is slightly different from theirs. As a direct consequence from our proof, we then show that the superconvergence phenomenon, which is originally observed and proven for uniform partition of two-dimensional domains by Matsunaga-Yamamoto [4], occurs on the non-uniform partition (1.2) as well.

Let us describe it more precisely. Let y be either $y = a$ or $y = b$, and K a positive constant. Then, there exists a positive constant C such that, for sufficiently small $h > 0$ and any nodal point x_i,

$$|x_i - y| \leq Kh \quad \Rightarrow \quad |u(x_i) - U(x_i)| \leq Ch^3.$$

In Section 4 we consider the cases where p and f are discontinuous at finitely many points in I. Our conclusion is that, if the set of nodal points includes discontinuous points, then the above estimates still hold. In Section 5 we give numerical examples which confirm the analysis in the previous sections.

2 Yamamoto's formula and other useful lemmas In this section we recall the explicit formula of the solutions of the Shortley-Weller finite difference approximation (1.3) given by Yamamoto [6, 7]. The matrix form of (1.3) is written as $A = HA_0$ and $A\vec{U} = \vec{f}$, where $\vec{U} := (U(x_1), \cdots, U(x_n))^t$, $\vec{f} := (f(x_1), \cdots, f(x_n))^t$, $H := \mathrm{diag}(2(h_1 + h_2)^{-1}, \cdots, 2(h_n + h_{n+1})^{-1})$,

$$A_0 := \begin{pmatrix} a_1 + a_2 & -a_2 & & & \\ -a_2 & a_2 + a_3 & -a_3 & & \\ & -a_3 & \ddots & \ddots & \\ & & \ddots & \ddots & -a_n \\ & & & -a_n & a_n + a_{n+1} \end{pmatrix},$$

and $a_i := p_{i-\frac{1}{2}}/h_i$. By the GD-decomposition given in [7, Theorem 3.1], we know that $A^{-1} = GD$, where $D = H^{-1} = \mathrm{diag}((h_1 + h_2)/2, \cdots, (h_n + h_{n+1})/2)$, and $G = A_0^{-1} = (g_{ij})$ with

$$(2.1) \qquad g_{ij} := \begin{cases} \left(\sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right)^{-1} \left(\sum_{k=1}^{i} \frac{h_k}{p_{k-\frac{1}{2}}}\right) \left(\sum_{k=j+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right) & (i \leq j), \\ \left(\sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right)^{-1} \left(\sum_{k=1}^{j} \frac{h_k}{p_{k-\frac{1}{2}}}\right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right) & (i \geq j). \end{cases}$$

We immediately notice that each entry g_{ij} of the matrix G is an approximation of $G(x_i, x_j)$, where $G(x, y)$ is the Green function of the differential operator P, $Pu := -(pu')'$ defined on the functional space $\{u \in C^2[a,b] \mid u(a) = u(b) = 0\}$, whose explicit form is

$$G(x,y) := \begin{cases} \left(\int_a^b \frac{ds}{p(s)}\right)^{-1} \int_a^x \frac{ds}{p(s)} \int_y^b \frac{ds}{p(s)} & (x \le y), \\ \left(\int_a^b \frac{ds}{p(s)}\right)^{-1} \int_a^y \frac{ds}{p(s)} \int_x^b \frac{ds}{p(s)} & (x \ge y). \end{cases} \tag{2.2}$$

From the above equations we obtain the explicit formula of the solution U of the Shortley-Weller finite difference approximation (1.3) as

$$\begin{aligned} U(x_i) = \sum_{j=1}^{n} g_{ij} \frac{h_j + h_{j+1}}{2} f(x_j) &\sim \sum_{j=1}^{n} G(x_i, x_j) \frac{h_j + h_{j+1}}{2} f(x_j) \\ &= \sum_{j=1}^{n+1} \frac{h_j}{2} (G(x_i, x_{j-1}) f(x_{j-1}) + G(x_i, x_j) f(x_j)) \end{aligned} \tag{2.3}$$

which, as is pointed out in [6] and [2], is an approximation of

$$u(x_i) = \int_a^b G(x_i, y) f(y) dy \tag{2.4}$$

by the trapezoidal rule. Therefore, the error analysis of the Shortley-Weller finite difference method is reduced to error estimates of numerical integration by the mid-point and trapezoidal rules. We make use the following lemmas in our error analysis. Although they are well known, we here give their proofs for convenience of the readers.

Lemma 2.1 *Let $h > 0$ and $J := (0,h)$. Suppose that F is a function of $W^{2,q}(J)$, $(1 \le q \le \infty)$ class. Let $\alpha \in [0,1]$. Then there exists a positive constant C independent of α, q, and F such that*

$$\left| \int_0^h F(t)dt - hF(\alpha h) + \frac{h^2}{2}(2\alpha - 1)F'(\alpha h) \right| \le C h^{2+(1/q')} \|F''\|_{L^q(J)},$$

where $q' \in [1,\infty]$ such that $(1/q) + (1/q') = 1$.

Proof. Define $K \in L^\infty(J)$ by

$$K(t) := \begin{cases} t^2/2, & 0 \le t \le \alpha h, \\ (h-t)^2/2, & \alpha h < t \le h. \end{cases}$$

Then, integrating by parts, we have

$$\begin{aligned} \int_0^h K(t)F''(t)dt &= \int_0^{\alpha h} K(t)F''(t)dt + \int_{\alpha h}^h K(t)F''(t)dt \\ &= \frac{h^2}{2}(2\alpha - 1)F'(\alpha h) - hF(\alpha h) + \int_0^h F(t)dt. \end{aligned}$$

The Hölder inequality yields $\left|\int_0^h K(t)F''(t)dt\right| \le \|K\|_{L^{q'}(J)} \|F''\|_{L^q(J)}$. It is easy to verify that there exists a positive constant C independent of α and q such that

$$\|K\|_{L^{q'}(J)} = \frac{h^{2+(1/q')}}{2} \left(\frac{\alpha^{2q'+1} + (1-\alpha)^{2q'+1}}{2q'+1} \right)^{1/q'} \le C h^{2+(1/q')}$$

for any $\alpha \in [0,1]$ and $q \in [1,\infty]$. $\square$

Lemma 2.2 *Let $h > 0$ and $J := (0, h)$. Suppose that F is a function of $W^{1,q}(J)$, $(1 \le q \le \infty)$ class. Let $\alpha \in [0, 1]$. Then there exists a positive constant C independent of α, q, and F such that*

$$\left| \int_0^h F(t)dt - hF(\alpha h) \right| \le Ch^{1+(1/q')} \|F'\|_{L^q(J)},$$

where $q' \in [1, \infty]$ such that $(1/q) + (1/q') = 1$.

Proof. Define $G \in L^\infty(J)$ by

$$G(t) := \begin{cases} t, & 0 \le t \le \alpha h, \\ t - h, & \alpha h < t \le h. \end{cases}$$

Then Lemma 2.2 is proved similarly as above by considering $\int_0^h G(t)F'(t)dt$. □

Lemma 2.3 *Let $h > 0$ and $J := (0, h)$. Suppose that $F \in W^{2,q}(J)$, $(1 \le q \le \infty)$ and $F(0) = F(h) = 0$. Then there exists a positive constant C independent of q and F such that*

$$\left| \int_0^h F(t)dt \right| \le Ch^{2+(1/q')} \|F''\|_{L^q(J)},$$

where $q' \in [1, \infty]$ such that $(1/q) + (1/q') = 1$. If $F \in W^{1,q}(J)$, $(1 \le q \le \infty)$ and $F(0) = F(h) = 0$, then we have

$$\left| \int_0^h F(t)dt \right| \le Ch^{1+(1/q')} \|F'\|_{L^q(J)}.$$

Proof. Firstly, suppose that $F \in W^{2,q}(J)$. Using Lemma 2.1 with $\alpha = 0, 1$ and $F(0) = F(h) = 0$, we obtain

$$\int_0^h F(t)dt - \frac{h^2}{2}F'(0) = \mathcal{O}(h^{2+(1/q')}), \qquad \int_0^h F(t)dt + \frac{h^2}{2}F'(h) = \mathcal{O}(h^{2+(1/q')}).$$

Since

$$|F'(h) - F'(0)| \le \int_0^h |F''(t)|dt \le h^{1/q'} \|F''\|_{L^q(J)},$$

the first inequality of Lemma 2.3 follows. The second inequality is obtained immediately from Lemma 2.2. □

Lemma 2.4 *Let $h > 0$ and $J := (0, h)$. Suppose that $F \in W^{2,q}(J)$, $(1 \le q \le \infty)$. Then there exists a positive constant C independent of q and F such that*

$$\left| \int_0^h F(t)dt - \frac{h}{2}\big(F(0) + F(h)\big) \right| \le Ch^{2+(1/q')} \|F''\|_{L^q(J)},$$

where $q' \in [1, \infty]$ such that $(1/q) + (1/q') = 1$. If $F \in W^{1,q}(J)$, $(1 \le q \le \infty)$ then we have

$$\left| \int_0^h F(t)dt - \frac{h}{2}\big(F(0) + F(h)\big) \right| \le Ch^{1+(1/q')} \|F'\|_{L^q(J)}.$$

Proof. Apply Lemma 2.3 to the function

$$F(t) - \left(\frac{F(h) - F(0)}{h} t + F(0) \right). \quad \square$$

3 Error bounds of Shortley-Weller finite difference solutions In this section we give several error bounds of Shortley-Weller finite difference solution U. First, we recall the error bound given in [2].

Theorem 3.1 (Fang-Tsuchiya-Yamamoto) *Let $I := (a, b)$ be a bounded interval. Let u be the exact solution of (1.1) and U the corresponding Shortley-Weller finite difference solution on the non-uniform partition (1.2). Suppose that $f \in C^{1,1}(\bar{I})$. Then, we have the following estimates for the Shortley-Weller finite difference solution U:*

$$u(x_i) - U(x_i) = \begin{cases} o(h) & \text{if } p \in C^1(\bar{I}), \\ \mathcal{O}(h^2) & \text{if } p \in C^{1,1}(\bar{I}). \end{cases}$$

In this paper, we show the following theorem which is a slight generalization of Theorem 3.1.

Theorem 3.2 *Let u be the exact solution of (1.1) and U the corresponding Shortley-Weller finite difference solution on the non-uniform partition (1.2). Let $p,\ f \in W^{2,q}(I)$ with $1 \le q \le \infty$. Suppose that $p(x) \ge \delta > 0$ for any $x \in I$ with a positive constant δ. Then there exists a positive constant C depends only on I, δ, and $\|1/p\|_{W^{2,q}(I)}$ such that*

$$|u(x_i) - U(x_i)| \le C(b - x_i)(x_i - a)h^{1+(1/q')}\|f\|_{W^{2,q}(I)} \tag{3.1}$$

for any x_i, $i = 1, \cdots, n$, where $q' \in [1, \infty]$ with $(1/q) + (1/q') = 1$. If $p,\ f \in W^{1,q}(I)$ with $1 \le q \le \infty$, then we have

$$|u(x_i) - U(x_i)| \le C(b - x_i)(x_i - a)h^{1/q'}\|f\|_{W^{1,q}(I)}, \tag{3.2}$$

where the constant C depends only on I, δ, and $\|1/p\|_{W^{1,q}(I)}$.

Proof. We suppose firstly that $p,\ f \in W^{2,q}(I)$, $1 \le q \le \infty$. From (2.1) and (2.3) we have an explicit formula for $U(x_i)$:

$$\begin{aligned} U(x_i) = {} & \beta_h^{-1}\left(\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac{1}{2}}}\right)\sum_{j=1}^{i-1}\left(\sum_{k=1}^{j}\frac{h_k}{p_{k-\frac{1}{2}}}\right)\frac{h_j + h_{j+1}}{2}f(x_j) \\ & + \beta_h^{-1}\left(\sum_{k=1}^{i}\frac{h_k}{p_{k-\frac{1}{2}}}\right)\left(\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac{1}{2}}}\right)\frac{h_i + h_{i+1}}{2}f(x_i) \\ & + \beta_h^{-1}\left(\sum_{k=1}^{i}\frac{h_k}{p_{k-\frac{1}{2}}}\right)\sum_{j=i+1}^{n}\left(\sum_{k=j+1}^{n+1}\frac{h_k}{p_{k-\frac{1}{2}}}\right)\frac{h_j + h_{j+1}}{2}f(x_j), \end{aligned} \tag{3.3}$$

where $\beta_h := \sum_{k=1}^{n+1}\frac{h_k}{p_{k-\frac{1}{2}}}$. From (2.2) and (2.4) we have

$$\begin{aligned} u(x_i) = {} & \beta^{-1}\left(\int_{x_i}^{b}\frac{ds}{p(s)}\right)\int_a^{x_i}\left(\int_a^{y}\frac{ds}{p(s)}\right)f(y)dy \\ & + \beta^{-1}\left(\int_a^{x_i}\frac{ds}{p(s)}\right)\int_{x_i}^{b}\left(\int_y^{b}\frac{ds}{p(s)}\right)f(y)dy, \end{aligned} \tag{3.4}$$

where $\beta := \int_a^b \frac{ds}{p(s)}$. We write

(3.5)

$$\begin{aligned}
&\beta^{-1}\left(\int_{x_i}^b \frac{ds}{p(s)}\right)\int_a^{x_i}\left(\int_a^y \frac{ds}{p(s)}\right)f(y)dy \\
&\qquad -\beta_h^{-1}\left(\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac12}}\right)\left[\sum_{j=1}^{i-1}\left(\sum_{k=1}^{j}\frac{h_k}{p_{k-\frac12}}\right)\frac{h_j+h_{j+1}}{2}f(x_j)+\left(\sum_{k=1}^{i}\frac{h_k}{p_{k-\frac12}}\right)\frac{h_i}{2}f(x_i)\right] \\
&= \left(\beta^{-1}-\beta_h^{-1}\right)\left(\int_{x_i}^b \frac{ds}{p(s)}\right)\int_a^{x_i}\left(\int_a^y \frac{ds}{p(s)}\right)f(y)dy \\
&\quad +\beta_h^{-1}\left(\int_{x_i}^b \frac{ds}{p(s)}\right)\left[\int_a^{x_i}\left(\int_a^y \frac{ds}{p(s)}\right)f(y)dy-\sum_{j=1}^{i-1}\left(\int_a^{x_j}\frac{ds}{p(s)}\right)\frac{h_j+h_{j+1}}{2}f(x_j)\right. \\
&\qquad\qquad\qquad\qquad \left.-\left(\int_a^{x_i}\frac{ds}{p(s)}\right)\frac{h_i}{2}f(x_i)\right] \\
&\quad +\beta_h^{-1}\left\{\int_{x_i}^b \frac{ds}{p(s)}-\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac12}}\right\}\left[\sum_{j=1}^{i-1}\left(\int_a^{x_j}\frac{ds}{p(s)}\right)\frac{h_j+h_{j+1}}{2}f(x_j)\right. \\
&\qquad\qquad\qquad\qquad \left.+\left(\int_a^{x_i}\frac{ds}{p(s)}\right)\frac{h_i}{2}f(x_i)\right] \\
&\quad +\beta_h^{-1}\left(\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac12}}\right)\left[\sum_{j=1}^{i-1}\left\{\int_a^{x_j}\frac{ds}{p(s)}-\sum_{k=1}^{j}\frac{h_k}{p_{k-\frac12}}\right\}\frac{h_j+h_{j+1}}{2}f(x_j)\right. \\
&\qquad\qquad\qquad\qquad \left.+\left\{\int_a^{x_i}\frac{ds}{p(s)}-\sum_{k=1}^{i}\frac{h_k}{p_{k-\frac12}}\right\}\frac{h_i}{2}f(x_i)\right].
\end{aligned}$$

From Lemma 2.1 with $\alpha = 1/2$ we have $\int_{x_{k-1}}^{x_k}\frac{ds}{p(s)} - \frac{h_k}{p_{k-\frac12}} = \|1/p\|_{W^{2,q}(I)}\mathcal{O}(h_k^{2+(1/q')})$. Combining this estimate we see that

(3.6)
$$\begin{aligned}
\beta-\beta_h &= (b-a)\|1/p\|_{W^{2,q}(I)}\mathcal{O}(h^{1+(1/q')}), \\
\int_{x_i}^b \frac{ds}{p(s)}-\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac12}} &= (b-x_i)\|1/p\|_{W^{2,q}(I)}\mathcal{O}(h^{1+(1/q')}), \\
\int_a^{x_j}\frac{ds}{p(s)}-\sum_{k=1}^{j}\frac{h_k}{p_{k-\frac12}} &= (x_j-a)\|1/p\|_{W^{2,q}(I)}\mathcal{O}(h^{1+(1/q')}), \\
\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac12}} &= (b-x_i)\mathcal{O}(1).
\end{aligned}$$

Note that constants hidden in "$\mathcal{O}$" depend only on the constant C in Lemma 2.1 and δ. It follows immediately from (3.5) and (3.6) that

(3.7)
$$|\text{the first term of the right-hand side of (3.5)}| \le C_1(b-x_i)(x_i-a)h^{1+(1/q')}\|f\|_{L^\infty(I)}.$$

Setting $F_1(y) := \left(\int_a^y ds/p(s)\right) f(y)$ we estimate the second term of the right-hand side of (3.5). Combining

$$\begin{aligned}\int_a^{x_i}\left(\int_a^y \frac{ds}{p(s)}\right) f(y)dy - \sum_{j=1}^{i-1}\left(\int_a^{x_j}\frac{ds}{p(s)}\right)\frac{h_j+h_{j+1}}{2}f(x_j) - \left(\int_a^{x_i}\frac{ds}{p(s)}\right)\frac{h_i}{2}f(x_i) \\ = \int_a^{x_i} F_1(y)dy - \sum_{j=1}^{i-1} F_1(x_j)\frac{h_j+h_{j+1}}{2} - F_1(x_i)\frac{h_i}{2} \\ = \sum_{j=1}^{i}\left(\int_{x_{j-1}}^{x_j} F_1(y)dy - \frac{h_j}{2}\left(F_1(x_{j-1})+F_1(x_j)\right)\right) \\ = (x_i-a)\|F_1\|_{W^{2,q}(I)}\mathcal{O}(h^{1+(1/q')}) \qquad \text{(by Lemma 2.4)},\end{aligned}$$

with (3.6) we obtain

(3.8)
$$|\text{the second term of the right-hand side of (3.5)}| \le C_2(b-x_i)(x_i-a)h^{1+(1/q')}\|f\|_{W^{2,q}(I)}.$$

Similarly, combining

$$\begin{aligned}\sum_{j=1}^{i-1}\left(\int_a^{x_j}\frac{ds}{p(s)}\right)\frac{h_j+h_{j+1}}{2}f(x_j) + \left(\int_a^{x_i}\frac{ds}{p(s)}\right)\frac{h_i}{2}f(x_i) \\ = \sum_{j=1}^{i}\frac{h_j}{2}\left(F_1(x_{j-1})+F_1(x_j)\right) = (x_i-a)\|f\|_{L^\infty(I)}\mathcal{O}(1)\end{aligned}$$

with (3.6) we obtain

(3.9)
$$|\text{the third term of the right-hand side of (3.5)}| \le C_3(b-x_i)(x_i-a)h^{1+(1/q')}\|f\|_{L^\infty(I)}.$$

Finally, setting $F_2(y) := (y-a)f(y)\|1/p\|_{W^{2,q}(I)}$, and applying (3.6) to

$$\begin{aligned}\sum_{j=1}^{i-1}\left\{\int_a^{x_j}\frac{ds}{p(s)} - \sum_{k=1}^{j}\frac{h_k}{p_{k-\frac12}}\right\}\frac{h_j+h_{j+1}}{2}f(x_j) + \left\{\int_a^{x_i}\frac{ds}{p(s)} - \sum_{k=1}^{i}\frac{h_k}{p_{k-\frac12}}\right\}\frac{h_i}{2}f(x_i) \\ = \sum_{j=1}^{i}\frac{h_j}{2}\left(F_2(x_{j-1})+F_2(x_j)\right)\mathcal{O}(h^{1+(1/q')}),\end{aligned}$$

we obtain

(3.10)
$$|\text{the fourth term of the right-hand side of (3.5)}| \le C_4(b-x_i)(x_i-a)h^{1+(1/q')}\|f\|_{L^\infty(I)}.$$

Note that the positive constants C_1, C_2, C_3 and C_4 depend only on I, δ, and $\|1/p\|_{W^{2,q}(I)}$.

Gathering the inequalities (3.7)–(3.10) we conclude that

(3.11)
$$\begin{aligned}\Bigg|&\beta^{-1}\left(\int_{x_i}^{b}\frac{ds}{p(s)}\right)\int_a^{x_i}\left(\int_a^y\frac{ds}{p(s)}\right)f(y)dy\\&-\beta_h^{-1}\left(\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac12}}\right)\left[\sum_{j=1}^{i-1}\left(\sum_{k=1}^{j}\frac{h_k}{p_{k-\frac12}}\right)\frac{h_j+h_{j+1}}{2}f(x_j)+\left(\sum_{k=1}^{i}\frac{h_k}{p_{k-\frac12}}\right)\frac{h_i}{2}f(x_i)\right]\Bigg|\\&\le C_5(b-x_i)(x_i-a)h^{1+(1/q')}\|f\|_{W^{2,q}(I)}.\end{aligned}$$

By the exactly same manner we obtain

(3.12)
$$\begin{aligned}\Bigg|&\beta^{-1}\left(\int_a^{x_i}\frac{ds}{p(s)}\right)\int_{x_i}^{b}\left(\int_y^b\frac{ds}{p(s)}\right)f(y)dy\\&-\beta_h^{-1}\left(\sum_{k=1}^{i}\frac{h_k}{p_{k-\frac12}}\right)\left[\sum_{j=i+1}^{n}\left(\sum_{k=j+1}^{n+1}\frac{h_k}{p_{k-\frac12}}\right)\frac{h_j+h_{j+1}}{2}f(x_j)+\left(\sum_{k=i+1}^{n+1}\frac{h_k}{p_{k-\frac12}}\right)\frac{h_i}{2}f(x_i)\right]\Bigg|\\&\le C_6(b-x_i)(x_i-a)h^{1+(1/q')}\|f\|_{W^{2,q}(I)}.\end{aligned}$$

Therefore, from (3.3), (3.4), (3.11) and (3.12) we finally obtain (3.1). Supposing p, $f\in W^{1,q}(I)$, $1\le q\le\infty$, (3.2) is obtained by the exactly same manner. □

From Theorem 3.2 we immediately obtain the following superconvergence estimate near the end-points.

Corollary 3.3 *Let u be the exact solution of* (1.1) *and U the corresponding Shortley-Weller finite difference solution on the non-uniform partition* (1.2). *Let p, $f\in W^{k,q}(I)$ with $k=1,2$, $1\le q\le\infty$. Suppose that $p(x)\ge\delta>0$ for any $x\in I$ with a positive constant δ. Let y be either $y=a$ or $y=b$ and K a positive constant. Then there exists a positive constant C depends only on I, δ, K, and $\|1/p\|_{W^{k,q}(I)}$ such that*

$$|x_i-y|\le Kh\quad\Longrightarrow\quad|u(x_i)-U(x_i)|\le Ch^{k+(1/q')}\|f\|_{W^{k,q}(I)}$$

for any x_i, $i=1,\cdots,n$, where $q'\in[1,\infty]$ with $(1/q)+(1/q')=1$.

4 Error analysis with non-smooth data In this section we consider the cases where p and f are discontinuous at finitely many points in $I=(a,b)$. More precisely, let I be divided into finitely many subintervals $J_l:=(y_{l-1},y_l)$, $l=1,\cdots,m+1$, that is, $J_l\cap J_s=\emptyset$, $(l\ne s)$ and $I=\left(\bigcup_{l=1}^{m+1}\overline{J_l}\right)^{\circ}$. We consider the cases where p, $f\in L^\infty(I)$ and $p|_{J_l}$, $f|_{J_l}\in W^{k,q}(J_l)$, where $k=1,2$ and $l=1,\cdots,m+1$. In the following theorem we claim that even if p and f are discontinuous at y_l the error estimates obtained in the previous section remain valid with appropriate modifications.

Theorem 4.1 *Let $J_l:=(y_{l-1},y_l)$, $y_0=a$ and $y_{m+1}=b$ be such that $J_l\cap J_s=\emptyset$, $(l\ne s)$ and $I=\left(\bigcup_{l=1}^{m+1}\overline{J_l}\right)^{\circ}$. Suppose that*

(4.1)
$$\{y_l\}_{l=0}^{m+1}\subset\{x_i\}_{i=0}^{n+1},$$

where x_i, $(i = 1, \cdots, n+1)$ are nodal points of the non-uniform partition (1.2). Let p, $f \in L^\infty(I)$ and $p|_{J_l}$, $f|_{J_l} \in W^{k,q}(J_l)$, where $k = 1, 2$, $l = 1, \cdots, m+1$ and $1 \le q \le \infty$. Suppose that $|p(x)| \ge \delta > 0$ and $\int_a^b dx/p(x) \ne 0$. Then, the equation (1.1) has a unique solution u for any given f. Let U be the Shortley-Weller finite difference solution of (1.3) with

$$f_i := \frac{h_i}{h_i + h_{i+1}} f(y_l - 0) + \frac{h_{i+1}}{h_i + h_{i+1}} f(y_l + 0) \tag{4.2}$$

at the discontinuous point $x_i = y_l$.

Then there exists a positive constant C depends only on I, δ and p such that

$$|u(x_i) - U(x_i)| \le C(b - x_i)(x_i - a) h^{k-1+(1/q')} \sum_{l=1}^{m+1} \|f|_{J_l}\|_{W^{k,q}(J_l)}$$

for any x_i, $i = 1, \cdots, n$, where $q' \in [1, \infty]$ with $(1/q) + (1/q') = 1$.

Proof. Obviously the proof should be very similar to that of Theorem 3.2 except handling the discontinuity of f (and p) at y_l. We modify the proof of Theorem 3.2 according to our present situation. First, we suppose that $k = 2$.

We note that, with the assumption (4.1) and the definition (4.2), the equation (3.3) is rewritten as

$$\begin{aligned} U(x_i) = {} & \beta_h^{-1} \left(\sum_{k=i+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right) \sum_{j=1}^{i-1} \left(\sum_{k=1}^{j} \frac{h_k}{p_{k-\frac{1}{2}}} \right) \left(\frac{h_j}{2} f(x_j - 0) + \frac{h_{j+1}}{2} f(x_j + 0) \right) \\ & + \beta_h^{-1} \left(\sum_{k=1}^{i} \frac{h_k}{p_{k-\frac{1}{2}}} \right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right) \left(\frac{h_i}{2} f(x_i - 0) + \frac{h_{i+1}}{2} f(x_i + 0) \right) \\ & + \beta_h^{-1} \left(\sum_{k=1}^{i} \frac{h_k}{p_{k-\frac{1}{2}}} \right) \sum_{j=i+1}^{n} \left(\sum_{k=j+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right) \left(\frac{h_j}{2} f(x_j - 0) + \frac{h_{j+1}}{2} f(x_j + 0) \right). \end{aligned}$$

To rewrite (3.5), therefore, we only need to replace

$$\frac{h_j + h_{j+1}}{2} f(x_j) \qquad \text{by} \qquad \frac{h_j}{2} f(x_j - 0) + \frac{h_{j+1}}{2} f(x_j + 0)$$

and

$$\frac{h_i}{2} f(x_i) \qquad \text{by} \qquad \frac{h_i}{2} f(x_i - 0)$$

in (3.5). By the assumption (4.1), p is continuous at $x_{i+\frac{1}{2}}$, and thus (3.6) and (3.7) hold in our present situation. To show that (3.8) holds, we write that

$$\begin{aligned} & \int_a^{x_i} \left(\int_a^y \frac{ds}{p(s)} \right) f(y) dy - \sum_{j=1}^{i-1} \left(\int_a^{x_j} \frac{ds}{p(s)} \right) \left(\frac{h_j}{2} f(x_j - 0) + \frac{h_{j+1}}{2} f(x_j + 0) \right) \\ & \quad - \left(\int_a^{x_i} \frac{ds}{p(s)} \right) \frac{h_i}{2} f(x_i - 0) = \sum_{j=1}^{i} \left(\int_{x_{j-1}}^{x_j} F_1(y) dy - \frac{h_j}{2} \big(F_1(x_{j-1} + 0) + F_1(x_j - 0) \big) \right) \\ & \qquad = (x_i - a) \sum_k \|F_1\|_{W^{2,q}(J_k)} \mathcal{O}(h^{1+(1/q')}), \end{aligned}$$

where $F_1(y) := \left(\int_a^y ds/p(s)\right) f(y)$. Hence, (3.8) holds. Also, combining

$$\sum_{j=1}^{i-1}\left(\int_a^{x_j}\frac{ds}{p(s)}\right)\left(\frac{h_j}{2}f(x_j-0)+\frac{h_{j+1}}{2}f(x_j+0)\right)+\left(\int_a^{x_i}\frac{ds}{p(s)}\right)\frac{h_i}{2}f(x_i-0)$$
$$=\sum_{j=1}^{i}\frac{h_j}{2}\left(F_1(x_{j-1}+0)+F_1(x_j-0)\right)=(x_i-a)\|f\|_{L^\infty(I)}\mathcal{O}(1)$$

with (3.6) we show that (3.9) holds. It is now easy to show that (3.10) holds. Gathering the above consideration, we conclude that the (modified) estimates (3.11) and (3.12) hold. Therefore, Theorem 4.1 is proved in the case $k = 2$. The case of $k = 1$ is shown by the exactly same manner. □

Corollary 4.2 *Suppose that we have the same situation as in Theorem 4.1. Let y be either $y = a$ or $y = b$ and K a positive constant. Then there exists a positive constant C depends only on I, δ, K, and p such that*

$$|x_i - y| \le Kh \quad\Longrightarrow\quad |u(x_i)-U(x_i)| \le Ch^{k+(1/q')}\sum_{l=1}^{m+1}\left\|f|_{J_l}\right\|_{W^{k,q}(J_l)}$$

for any x_i, $i = 1,\cdots,n$, where $q' \in [1,\infty]$ with $(1/q)+(1/q')=1$.

5 Numerical examples In this section, we give numerical examples which confirm our error analysis done in Sections 3 and 4. We consider the following examples.

Example 5.1 *Set $I := (-1,1)$, $p(x) := 2-x^2$ and $f(x) := 2(2x^4-7x^2+2)\,e^{1-x^2}$. Then the exact solution of (1.1) is $u(x) = e^{1-x^2}-1$.*

Example 5.2 *Set $I := (-1,1)$, $p(x) := 4-x^2$ and*

$$f(x) := \begin{cases} 1 & (x>0), \\ 0 & (x<0). \end{cases}$$

Then the exact solution of (1.1) is

$$u(x) = \begin{cases} -\dfrac{1}{4}\left(1-\dfrac{\log 4}{\log 3}\right)\log\dfrac{2+x}{2-x}+\dfrac{1}{2}\log(4-x^2)-\dfrac{1}{4}\log 12 & (x>0), \\ -\dfrac{1}{4}\left(1-\dfrac{\log 4}{\log 3}\right)\log\dfrac{2+x}{2-x}-\dfrac{1}{4}\log\dfrac{3}{4} & (x<0). \end{cases}$$

Example 5.3 *Set $I := (-1,1)$, $f(x) := 1$ and*

$$p(x) := \begin{cases} 4-x^2 & (x>0), \\ 9-x^2 & (x<0). \end{cases}$$

Then the exact solution of (1.1) is

$$u(x) = \begin{cases} \dfrac{3}{2}\left(\dfrac{3\log 3-5\log 2}{2\log 2+3\log 3}\right)\log\dfrac{3(2-x)}{2+x}+\dfrac{1}{2}\log(4-x^2)-\dfrac{1}{2}\log 3 & (x>0), \\ -\left(\dfrac{3\log 3-5\log 2}{2\log 2+3\log 3}\right)\log\dfrac{2(3+x)}{3-x}+\dfrac{1}{2}\log(9-x^2)-\dfrac{3}{2}\log 2 & (x<0). \end{cases}$$

Example 5.4 *Set* $I := (-1, 1)$,

$$p(x) := \begin{cases} x^2 - 2x + 3 & (x > 0), \\ x^2 + 2x + 2 & (x < 0), \end{cases}$$

$$f(x) := \begin{cases} -2(2x^4 - 4x^3 + 3x^2 + 4x - 3)\, e^{1-x^2} & (x > 0), \\ -2(2x^4 + 4x^3 + x^2 - 4x - 2)\, e^{1-x^2} & (x < 0). \end{cases}$$

Then the exact solution of (1.1) *is* $u(x) = e^{1-x^2} - 1$.

We use the following partition of the interval $I := (-1, 1)$ for computing numerical solutions for Examples 5.1–5.4. First, we divide $(-1, 1)$ into $2n$ equal subintervals, where n is a given positive number. Then, we divide each small interval (whose length is $1/n$) into three subintervals whose lengths are $0.3/n$, $0.5/n$, and $0.2/n$, respectively.

In the following tables we give the numerical results. In tables, "n" stands for the positive number used to make partition, "node#" stands for number of nodes, "max-error" means $\max_{1 \le i \le n} |u(x_i) - U(x_i)|$, and "n.b.max-error" means

$$\max\{|u(x_i) - U(x_i)| : x_i - a \le Kh \text{ or } b - x_i \le Kh\},$$

where $K := 5$.

On the above partition we compute Example 5.1 by the Shortley-Weller finite difference method and the finite element method with piecewise linear elements. These results are shown in Tables 5.1 and 5.2, respectively. The numerical result given in Table 5.1 shows that the Shortley-Weller finite difference solutions are superconvergent of $\mathcal{O}(h^3)$ near the end-points, which corresponds to the claim of Corollary 3.3. On the other hand, Table 5.2 shows that the finite element solutions are superconvergent of $\mathcal{O}(h^{3-\epsilon})$ near the end-points.

In Tables 5.3–5.5, we give the numerical results of the Shortley-Weller finite difference method applied to Examples 5.2–5.4. Although the given functions $f(x)$ and $p(x)$ have discontinuity at $x = 0$, we obtain good results which confirm the claims of Theorem 4.1 and Corollary 4.2.

Table 5.1: The errors of the Shortley-Weller finite difference solutions for Example 5.1.

n	node#	h	max-error	max-error/h^2	n.b.max-error	n.b.max-error/h^3
50	301	1.00E-2	4.63E-5	0.463	6.21E-07	0.621
250	1501	2.00E-3	1.85E-6	0.463	3.96E-09	0.495
500	3001	1.00E-3	4.63E-7	0.463	4.81E-10	0.481
1000	6001	5.00E-4	1.16E-7	0.465	6.09E-11	0.487

Table 5.2: The errors of the Finite element solutions for Example 5.1.

n	node#	h	max-error	max-error/h^2	n.b.max-error	n.b.max-error/h^3
50	301	1.00E-2	2.71E-6	0.0271	8.29E-07	0.829
250	1501	2.00E-3	1.09E-7	0.0271	7.24E-09	0.905
500	3001	1.00E-3	2.71E-8	0.0271	9.15E-10	0.915
1000	6001	5.00E-4	6.77E-9	0.0271	1.15E-10	0.919

Table 5.3: The errors of the Shortley-Weller finite difference solutions for Example 5.2.

n	node#	h	max-error	max-error/h^2	n.b.max-error	n.b.max-error/h^3
50	301	1.00E-2	0.458E-6	0.00458	0.715E-07	0.0715
250	1501	2.00E-3	0.183E-7	0.00458	0.607E-09	0.0758
500	3001	1.00E-3	0.457E-8	0.00457	0.765E-10	0.0765
1000	6001	5.00E-4	0.111E-8	0.00446	0.931E-11	0.0744

Table 5.4: The errors of the Shortley-Weller finite difference solutions for Example 5.3.

n	node#	h	max-error	max-error/h^2	n.b.max-error	n.b.max-error/h^3
50	301	1.00E-2	0.389E-6	0.00389	0.669E-07	0.0669
250	1501	2.00E-3	0.155E-7	0.00388	0.570E-09	0.0712
500	3001	1.00E-3	0.387E-8	0.00387	0.718E-10	0.0718
1000	6001	5.00E-4	0.979E-9	0.00392	0.919E-11	0.0735

Table 5.5: The errors of the Shortley-Weller finite difference solutions for Example 5.4.

n	node#	h	max-error	max-error/h^2	n.b.max-error	n.b.max-error/h^3
50	301	1.00E-2	0.108E-3	1.08	0.430E-05	4.30
250	1501	2.00E-3	0.431E-5	1.08	0.340E-07	4.25
500	3001	1.00E-3	0.108E-5	1.08	0.425E-08	4.25
1000	6001	5.00E-4	0.270E-6	1.08	0.532E-09	4.25

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