SPECTRAL RELATIONS FOR ALUTHGE TRANSFORM

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Abstract. Let T be a bounded linear operator on a complex Hilbert space with the polar decomposition $T = U|T|$. Let $T(t) = |T|^t U|T|^{1-t}$ for $0 < t < 1, T(0) = U^*UU|T|$ and $T(1) = |T|U$. $T(t)$ is called Aluthge transform of T. In this paper, we investigate spectral relations between T and $T(t)$. For example, we prove that T and $T(t)$ have the same essential spectrum and Weyl spectrum, and prove that Weyl's theorem holds for T if and only if Weyl's theorem holds for $T(t)$.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B} = \mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. Let $|T| = (T^*T)^{1/2}$. Define $U \in \mathcal{B}(\mathcal{H})$ by

$$\begin{cases} U|T|x &= Tx, \quad \text{for} \quad |T|x \in \text{ran } |T|, \\ Ux &= 0, \quad \text{for} \quad x \in (\text{ran } T)^{\perp} = \ker T^*. \end{cases}$$

Then U is a partial isometry with the initial space the closure of the range of $|T|$ and the final space the closure of the range of T. Also, we have that $T = U|T|$, $U^*U|T| = |T| = |T|U^*U$ and $\ker T = \ker |T| = \ker U$. In this paper, we say that $T = U|T|$ is the polar decomposition of T. We remark that there may exist another partial isometry V such that $T = V|T|$. For example, if T is normal, then there exisits unitary V such that $T = V|T|$. But, in this paper, we consider only U in the above definition with the polar decomposition $T = U|T|$.

Let $T(t) = |T|^t U|T|^{1-t}$ for $0 < t < 1$. We think that it is natural to define $|T|^0 = U^*U$ since $|T|^t \to U^*U \ (t \to +0)$ strongly. Hence we define

$$T(0) = |T|^0 U|T|^1 = U^*UU|T|$$

and

$$T(1) = |T|^1 U|T|^0 = |T|UU^*U = |T|U$$

in this paper. We remark that if T is invertible, then $|T|$ is invertible and U is unitary. Hence $T = U|T|$ is similar to $T(t)$ if T is invertible.

$T(t)$ is called Aluthge transform of T. The idea of Aluthge transform is due to Aluthge [1], in which Aluthge proved that if $T = U|T|$ is p-hyponormal ($(T^*T)^p \leq (TT^*)^p, \ 0 < p < \frac{1}{2}$) and U is unitary, then $T(\frac{1}{2})$ is a $(p + \frac{1}{2})$-hyponormal operator. This idea is powerful to study p-hyponormal operators ([1, 4, 8, 14]).

Recently, Aluthge transform was considered for general operators by [4] and [9]. Let $\sigma(T), \sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, the point spectrum and the approximate point spectrum of T, respectively. Chō, Jeon, Jung, Lee and Tanahashi [4] proved the following results.

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Proposition 1.1. ([**4**]). *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Let $T(t) = |T|^t U |T|^{1-t}$ for $0 < t < 1$. Then*

$$\sigma(T) = \sigma(T(t)),$$
$$\sigma_p(T) = \sigma_p(T(t)),$$
$$\sigma_a(T) = \sigma_a(T(t)).$$

In this paper, we prove that $\sigma(T) = \sigma(T(t))$ and $\sigma_p(T) = \sigma_p(T(t))$ for $t = 0, 1$. Also, we prove that $\sigma_a(T) = \sigma_a(T(0))$, but $\sigma_a(T) \neq \sigma_a(T(1))$ in general.

Let $\mathcal{B}_0 = \mathcal{B}_0(\mathcal{H})$ be the set of all compact operators. Let $\mathcal{B}/\mathcal{B}_0$ be the Calkin algebra and let $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{B}_0$ be the natural map. The essential spectrum $\sigma_e(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is defined by $\sigma_e(T) = \sigma(\pi(T))$. T is called a left (right) Fredholm operator if $\pi(T)$ is left (right) invertible. Let $\mathcal{F}_l$ ($\mathcal{F}_r$) denote the set of all left (right) Fredholm operators. T is called a semi-Fredholm operator if $T \in \mathcal{F}_l \cup \mathcal{F}_r$ and called a Fredholm operator if $T \in \mathcal{F}_l \cap \mathcal{F}_r = \mathcal{F}$. It is known that $T \in \mathcal{F}$ if and only if the range of T is closed, $\dim \ker T < \infty$ and $\dim \ker T^* < \infty$. For $T \in \mathcal{F}$, index of T is defined by $\operatorname{ind} T = \dim \ker T - \dim \ker T^*$. A Fredholm operator T with $\operatorname{ind} T = 0$ is called a Weyl operator. Let $\mathcal{F}_0$ denote the set of all Weyl operators. The Weyl spectrum $\sigma_w(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : \ T - \lambda \notin \mathcal{F}_0\}$. If $\mathcal{H}$ is infinite dimensional, then $\sigma_e(T)$ and $\sigma_w(T)$ are nonempty compact sets and $\sigma_e(T) \subset \sigma_w(T) \subset \sigma(T)$. In this paper, we show that T and $T(t)$ have the same essential spectrum and Weyl spectrum for $0 \leq t \leq 1$.

For $T \in \mathcal{B}(\mathcal{H})$, let $\pi_{00}(T)$ be isolated points of $\sigma(T)$ which are eigen values of finite multiplicity. We say that Weyl's theorem holds for T if $\sigma_w(T) = \sigma(T) \setminus \pi_{00}(T)$. We prove that $\pi_{00}(T) = \pi_{00}(T(t))$ for $0 \leq t \leq 1$. Also, we prove that Weyl's theorem holds for T if and only if Weyl's theorem holds for $T(t)$ where $0 \leq t \leq 1$.

2. Results

If $A, B \in \mathcal{B}(\mathcal{H})$, then it is well known that $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ (see [7]). The following results are due to B. A. Barnes [2] and play important roles in this paper.

Proposition 2.1. ([**2**]). *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\sigma_p(AB) \setminus \{0\} = \sigma_p(BA) \setminus \{0\},$$
$$\sigma_a(AB) \setminus \{0\} = \sigma_a(BA) \setminus \{0\},$$
$$\sigma_e(AB) \setminus \{0\} = \sigma_e(BA) \setminus \{0\},$$
$$\sigma_w(AB) \setminus \{0\} = \sigma_w(BA) \setminus \{0\}.$$

Let S be the unilateral shift on ℓ^2. Then $\sigma(S^*S) = \{1\}$ and $\sigma(SS^*) = \{0, 1\}$. Hence we cannot delete $\{0\}$ in these relations. In this paper, we prove that T and $T(t)$ have the same essential spectrum and Weyl spectrum for $0 \leq t \leq 1$.

See [6, IX Theorem 2.5] for the proof of Lemma 2.2.

Lemma 2.2. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Then following assertions are equivalent.*

$$T \in \mathcal{F}_l. \tag{2.1}$$

$$|T| \in \mathcal{F}. \tag{2.2}$$

$$|T|^s \in \mathcal{F} \text{ for some positive number } s. \tag{2.3}$$

$$|T|^s \in \mathcal{F} \text{ for any positive number } s. \tag{2.4}$$

And, in this case, we have that $\operatorname{ind} T = \operatorname{ind} U$.

First we prove that T and $T(t)$ have the same essential spectrum and Weyl spectrum for $0 \leq t \leq 1$.

Theorem 2.3. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Let $T(t) = |T|^t U|T|^{1-t}$ for $0 < t < 1$ and let $T(0) = U^*UU|T|, T(1) = |T|U$. Then, for $0 \leq t \leq 1$,*

$$\sigma_e(T) = \sigma_e(T(t)), \tag{2.5}$$

$$\sigma_w(T) = \sigma_w(T(t)). \tag{2.6}$$

Proof. (2.5) By Proposition 2.1, we have that

$$\begin{aligned}\sigma_e(T(t)) \setminus \{0\} &= \sigma_e(|T|^t U|T|^{1-t}) \setminus \{0\} = \sigma_e(U|T|^{1-t}|T|^t) \setminus \{0\} \\ &= \sigma_e(U|T|) \setminus \{0\} = \sigma_e(|T|U) \setminus \{0\} \\ &= \sigma_e(|T|U^*UU) \setminus \{0\} = \sigma_e(U^*UU|T|) \setminus \{0\}.\end{aligned}$$

Hence we have to prove that $0 \notin \sigma_e(T)$ if and only if $0 \notin \sigma_e(T(t))$.

Let $0 \notin \sigma_e(T)$. Then $T = U|T| \in \mathcal{F}$. Hence $|T| \in \mathcal{F}$ by Lemma 2.2. Hence $U, U^*, |T|^t, |T|^{1-t} \in \mathcal{F}$. Hence $T(t) \in \mathcal{F}$ and $0 \notin \sigma_e(T(t))$.

Conversely, let $0 < t < 1$ and let $0 \notin \sigma_e(T(t))$. Then $T(t) = |T|^t U|T|^{1-t} \in \mathcal{F}$ and $\pi(|T|^t U|T|^{1-t})$ is invertible. Hence $\pi(|T|^{1-t})$ is left invertible and $|T|^{1-t} \in \mathcal{F}_l \cap \mathcal{F}_r = \mathcal{F}$. Hence $|T| \in \mathcal{F}$ by Lemma 2.2. Hence $U \in \mathcal{F}$ and $T = U|T| \in \mathcal{F}$. The proofs of cases $t = 0, 1$ are similar.

(2.6) By Proposition 2.1, we have to prove that $0 \notin \sigma_w(T)$ if and only if $0 \notin \sigma_w(T(t))$.

Let $0 \notin \sigma_w(T)$. Then $T = U|T| \in \mathcal{F}$ and ind $T = 0$. By (2.1), $T(t) \in \mathcal{F}$ and

$$\text{ind } T(t) = \text{ind } |T|^t U|T|^{1-t} = \text{ind } U = 0.$$

Thus $0 \notin \sigma_w(T(t))$. The rest of the proof is similar. □

Theorem 2.4. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Let $T(t) = |T|^t U|T|^{1-t}$ for $0 < t < 1$ and let $T(0) = U^*UU|T|, T(1) = |T|U$. Then, for $0 \leq t \leq 1$,*

$$\sigma(T) = \sigma(T(t)), \tag{2.7}$$

$$\sigma_p(T) = \sigma_p(T(t)). \tag{2.8}$$

Proof. (2.7) We have to prove that $0 \notin \sigma(T)$ if and only if $0 \notin \sigma(T(t))$ for $t = 0, 1$ by Propositions 1.1 and 2.1.

First we prove the case $t = 1$. Let $0 \notin \sigma(T)$. Then $T = U|T|$ is invertible. Hence $|T|$ is invertible and U is unitary. Hence $T(1) = |T|U$ and $T(0) = U^*UU|T|$ are invertible.

Conversely, let $0 \notin \sigma(T(1)) = \sigma(|T|U)$. Then $|T|U$ is invertible and $U^*|T|$ is invertible. Hence $|T|$ is bijective. Hence $|T|$ is invertible and $U = |T|^{-1}T(1)$. This implies that U and T are invertible.

The proof of the case $t = 0$ is similar.

(2.8) We have to prove that $0 \in \sigma_p(T)$ if and only if $0 \in \sigma_p(T(t))$ for $t = 0, 1$ by Propositions 1.1 and 2.1.

Let $0 \in \sigma_p(T)$. Then there exists non-zero vector $x \in \mathcal{H}$ such that $Tx = 0$. Then $x \in \ker T = \ker |T| = \ker U$. Hence $T(1)x = |T|Ux = 0$ and $T(0)x = U^*UU|T|x = 0$.

Conversely, let $0 \in \sigma_p(T(1))$. Then there exists a non-zero vector $x \in \mathcal{H}$ such that $T(1)x = |T|Ux = 0$. If $Ux = 0$, then $x \in \ker U = \ker T$. If $Ux \neq 0$, then $TUx = U|T|Ux = 0$. Hence $0 \in \sigma_p(T)$.

Let $0 \in \sigma_p(T(0))$. Then there exists a non-zero vector $x \in \mathcal{H}$ such that $T(0)x = U^*UU|T|x = 0$. Then

$$\langle UU|T|x, UU|T|x\rangle = \langle U^*UU|T|x, U|T|x\rangle = 0.$$

Hence $UU|T|x = UTx = 0$. If $Tx \neq 0$, then $Tx \in \ker U = \ker T$. Hence $0 \in \sigma_p(T)$.

□

Theorem 2.5. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Let $T(0) = U^*UU|T|$. Then*

$$\sigma_a(T) = \sigma_a(T(0)). \tag{2.9}$$

Proof. We have to prove that $0 \in \sigma_a(T)$ if and only if $0 \in \sigma_a(T(0))$ by Proposition 2.1. Let $0 \in \sigma_a(T(0))$. Then there exist unit vectors x_n such that $T(0)x_n = U^*UU|T|x_n \to 0$. If $\ker |T| = \{0\}$, then $U^*U = I$. Hence $Tx_n = U|T|x_n = U^*UU|T|x_n \to 0$. If $\ker |T| = \ker(|T|) \neq \{0\}$, then there exists a non-zero vector $y \in \mathcal{H}$ such that $|T|y = 0$. Hence $Ty = U|T|y = 0$. Thus $0 \in \sigma_a(T)$. The converse implication is clear.

□

Remark 2.6. *We prove that $\sigma_a(T) \neq \sigma_a(T(1))$ in general.*

Let $\mathcal{H} = L^2([0,1])$. Define $S \in \mathcal{B}(\mathcal{H})$ by

$$(Sf)(t) = tf(t), \ t \in [0,1].$$

We can write $\mathcal{H} = L^2([0,1]) = L^2([0,\frac{1}{2}]) \oplus L^2([\frac{1}{2},1])$.
Let $\{e_1, e_3, e_5, \cdots\} \subset L^2([0,\frac{1}{2}])$ be a complete orthonormal basis of $L^2([0,\frac{1}{2}])$ and $\{e_2, e_4, e_6, \cdots\} \subset L^2([\frac{1}{2},1])$ be a complete orthonormal basis of $L^2([\frac{1}{2},1])$. Define $U \in B(\mathcal{H})$ by

$$Ue_n = e_{2n}, \ n = 1, 2, \cdots.$$

Then U is isometry and $U\mathcal{H} = L^2([\frac{1}{2},1])$. Let $T = US$. Then $|T|^2 = T^*T = S^*U^*US = S^2$ and $|T| = S$. Since $|T|\mathcal{H} = S\mathcal{H}$ is dense, T has the polar decomposition $T = US = U|T|$.

Since $0 \in \sigma_a(S)$, there exist unit vectors $f_n \in \mathcal{H}$ such that $Sf_n \to 0$. Hence $U|T|f_n = USf_n \to 0$ and $0 \in \sigma_a(U|T|) = \sigma_a(T)$.

Let $f \in \mathcal{H}$ be any unit vector. Let $g = Uf$. Since U is isometry, we have that $\|g\| = \|Uf\| = \|f\| = 1$. Since $g = Uf \in L^2([\frac{1}{2},1])$, we have that

$$\begin{aligned} \||T|Uf\|^2 = \langle |T|g, |T|g\rangle &= \int_0^1 t^2|g(t)|^2 dt \\ &= \int_{\frac{1}{2}}^1 t^2|g(t)|^2 dt \geq \frac{1}{4}\int_{\frac{1}{2}}^1 |g(t)|^2 dt = \frac{1}{4}\|g\|^2 = \frac{1}{4}\|f\|^2. \end{aligned}$$

Hence

$$\||T|Uf\| \geq \frac{1}{2}\|f\|.$$

This implies that $0 \notin \sigma_a(|T|U) = \sigma_a(T(1))$. (We remark that $0 \in \sigma_r(|T|U)$.)

B. A. Barnes [2] proved that

$$\sigma_c(AB) \setminus \{0\} = \sigma_c(BA) \setminus \{0\},$$
$$\sigma_r(AB) \setminus \{0\} = \sigma_r(BA) \setminus \{0\}$$

for $A, B \in \mathcal{B}(\mathcal{H})$, where $\sigma_c(T)$ and $\sigma_r(T)$ denote the continuous spectrum and the residual spectrum of T, respectively. The above example shows that we cannot delete $\{0\}$ in these relations.

$T \in \mathcal{B}(\mathcal{H})$ is called isoloid if any isolated point of $\sigma(T)$ is an eigen value of T. If T is hyponormal ($TT^* \leq T^*T$), then T is isoloid by [11].

Theorem 2.7. *Let* $T = U|T|$ *be the polar decomposition of* $T \in \mathcal{B}(\mathcal{H})$. *Let* $T(t) = |T|^t U|T|^{1-t}$ *for* $0 < t < 1$ *and let* $T(0) = U^*UU|T|, T(1) = |T|U$. *Then* T *is isoloid if and only if* $T(t)$ *is isoloid where* $0 \leq t \leq 1$.

Proof. Let T be isoloid. Let $\lambda \in \sigma(T(t))$ be an isolated point of T. Since $\sigma(T(t)) = \sigma(T)$ by Proposition 1.1 and Theorem 2.4, λ is an isolated point of $\sigma(T)$. Since T is isoloid, we have that $\lambda \in \sigma_p(T) = \sigma_p(T(t))$ by Proposition 1.1 and Theorem 2.4. Hence $T(t)$ is isoloid. The proof of the converse is similar. □

$T \in \mathcal{B}(\mathcal{H})$ is called a log-hyponormal operator, if T is invertible and $\log(TT^*) \leq \log(T^*T)$. It is known that invertible p-hyponormal operators are log-hyponormal and that there exists a log-hyponormal operator which is not p-hyponormal for any $p > 0$ ([12]). The authors [4] proved that log-hyponormal operators are isoloid. Also, Chō, Itoh and Ōshiro [3] proved that p-hyponormal operators are isoloid.

Corollary 2.8. *If* $T \in \mathcal{B}(\mathcal{H})$ *is* p*-hyponormal or log-hyponormal operator, then* T *is isoloid.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Then, at least, $T(\frac{1}{2})$ is $\frac{1}{2}$-hyponormal by [12] or [14]. Let $T(\frac{1}{2}) = V|T(\frac{1}{2})|$ be the polar decomposition of $T(\frac{1}{2})$ and $S = |T(\frac{1}{2})|^{\frac{1}{2}} V |T(\frac{1}{2})|^{\frac{1}{2}}$. Then S is hyponormal by [14]. Since S is isoloid by [11], $T(\frac{1}{2})$ and T are isoloid by Theorem 2.7. □

Theorem 2.9. *Let* $T = U|T|$ *be the polar decomposition of* $T \in \mathcal{B}(\mathcal{H})$. *Let* $T(t) = |T|^t U|T|^{1-t}$ *for* $0 < t < 1$ *and let* $T(0) = U^*UU|T|, T(1) = |T|U$. *Then, for* $0 \leq t \leq 1$,

$$\pi_{00}(T) = \pi_{00}(T(t)). \tag{2.10}$$

Proof. We can prove that

$$\pi_{00}(T) \setminus \{0\} = \pi_{00}(T(t)) \setminus \{0\}$$

for $0 \leq t \leq 1$ by similar arguments of the proof of Proposition 2.1. Hence we have to prove that $0 \in \pi_{00}(T)$ if and only if $0 \in \pi_{00}(T(t))$.

(Case $0 < t < 1$) Let $0 \in \pi_{00}(T) = \pi_{00}(U|T|)$. Then $\dim \ker T = \dim \ker |T| < \infty$. Since $\sigma(T) = \sigma(T(t))$ and $\sigma_p(T) = \sigma_p(T(t))$ by Proposition 1.1, we have that 0 is an isolated point of $\sigma(T(t))$ which is an eigen value of T. Assume $0 \notin \pi_{00}(T(t))$ and let $\mathcal{M} = \ker(T(t))$. Then $\dim \mathcal{M} = \infty$. Let $x \in \mathcal{M}$. Then we can write $x = x_1 \oplus x_2 \in \ker |T| \oplus (\ker |T|)^{\perp}$. Since

$$x_1 \in \ker |T| = \ker |T|^{1-t} \subset \mathcal{M},$$

we have that $x_2 = x - x_1 \in \mathcal{M}$. This implies that

$$\mathcal{M} = (\mathcal{M} \cap \ker |T|) \oplus \left(\mathcal{M} \cap (\ker |T|)^{\perp}\right).$$

Since $\dim(\mathcal{M} \cap \ker |T|) \leq \dim \ker |T| < \infty$, we have that $\dim \mathcal{M} \cap (\ker |T|)^{\perp} = \infty$. Hence there exist orthogonal unit vectors $x_n \in (\ker |T|)^{\perp}$ such that $T(t)x_n = |T|^t U|T|^{1-t} x_n = 0$.

We prove that $U|T|^{1-t}x_1, \cdots, U|T|^{1-t}x_n$ are linearly independent. Let

$$c_1 U|T|^{1-t} x_1 + \cdots + c_n U|T|^{1-t} x_n = U|T|^{1-t}(c_1 x_1 + \cdots + c_n x_n) = 0.$$

Then

$$|T|^{1-t}(c_1 x_1 + \cdots + c_n x_n) = U^*U|T|^{1-t}(c_1 x_1 + \cdots + c_n x_n) = 0.$$

Hence $c_1 x_1 + \cdots + c_n x_n \in \ker |T|^{1-t} = \ker |T|$. Hence $c_1 x_1 + \cdots + c_n x_n = 0$ and $c_1 = \cdots = c_n = 0$.

Since

$$TU|T|^{1-t}x_j = U|T|^{1-t}|T|^tU|T|^{1-t}x_j = 0,$$

we have that $U|T|^{1-t}x_j \in \ker T$. Hence $\dim\ker T = \infty$. This is a contradiction.

Conversely, let $0 \in \pi_{00}(T(t))$. Then 0 is an isolated point of $\sigma(T)$ which is an eigen value of T by Proposition 1.1. Since $\ker T = \ker|T| = \ker|T|^{1-t} \subset \ker T(t)$, we have that $\dim\ker T \leq \dim\ker T(t) < \infty$. Hence $0 \in \pi_{00}(T)$.

(Case $t = 1$) Let $0 \in \pi_{00}(T) = \pi_{00}(U|T|)$. Then $\dim\ker T = \dim\ker|T| < \infty$ and 0 is an isolated point of $\sigma(T(t))$ which is an eigen value of T by Theorem 2.4. Assume $0 \notin \pi_{00}(T(1)) = \pi_{00}(|T|U)$ and let $\mathcal{M} = \ker|T|U$. Then $\dim\mathcal{M} = \infty$. Let $x \in \mathcal{M}$. Then we can write $x = x_1 \oplus x_2 \in \ker|T| \oplus (\ker|T|)^{\perp}$. Since

$$x_1 \in \ker|T| = \ker U \subset \ker|T|U = \mathcal{M},$$

we have that $x_2 = x - x_1 \in \mathcal{M}$. This implies that

$$\mathcal{M} = (\mathcal{M} \cap \ker|T|) \oplus \left(\mathcal{M} \cap (\ker|T|)^{\perp}\right).$$

Since $\dim\mathcal{M}\cap\ker|T| \leq \dim\ker|T| < \infty$, we have that $\dim\mathcal{M}\cap(\ker|T|)^{\perp} = \infty$. Hence there exist orthogonal unit vectors $x_n \in (\ker|T|)^{\perp}$ such that $|T|Ux_n = 0$. Since $Ux_1, \cdots, Ux_n$ are linearly independent and $TUx_j = U|T|Ux_j = 0$, we have that $\dim\ker T = \infty$. This is a contradiction.

Conversely, let $0 \in \pi_{00}(T(t))$. Then 0 is an isolated point of $\sigma(T)$ which is an eigen value of T by Theorem 2.4. Since $\ker T = \ker|T| = \ker U \subset \ker|T|U$, we have that $\dim\ker T \leq \dim\ker|T|U < \infty$. Hence $0 \in \pi_{00}(T)$.

We can prove the case $t = 0$ similarly. This completes the proof.

□

Theorem 2.10. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Let $T(t) = |T|^tU|T|^{1-t}$ for $0 < t < 1$ and let $T(0) = U^*UU|T|, T(1) = |T|U$. Then Weyl's theorem holds for T if and only if Weyl's theorem holds for $T(t)$ where $0 \leq t \leq 1$.*

Proof. Let Weyl's theorem hold for T. Then

$$\begin{aligned}\sigma_w(T) &= \sigma(T) \setminus \pi_{00}(T)\\ &= \sigma(T(t)) \setminus \pi_{00}(T(t))\end{aligned}$$

by Theorem 2.9, Proposition 1.1 and Theorem 2.4. Since $\sigma_w(T) = \sigma_w(T(t))$ by Theorem 2.3, we have that

$$\sigma_w(T(t)) = \sigma(T(t)) \setminus \pi_{00}(T(t)).$$

Hence Weyl's theorem holds for $T(t)$. The proof of the converse is similar.

□

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