ON FRATTINI EXTENSIONS

JAMSHID MOORI[1] AND B.G. RODRIGUES[2]

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ABSTRACT. Let G be an extension of N by H. We say that G is a Frattini extension if N is a subgroup of the Frattini subgroup of G. In this paper we establish some properties of the Frattini extensions.

1. INTRODUCTION

The motivation for this paper comes from an ongoing research by the authors on the theory of group extensions. In [2] B.Eick and H.Besche outlined an algorithm to determine, up to isomorphism, all finite soluble groups of a given order. This method which relays strongly on the theory of Frattini extensions is known as the Frattini Extension Method and it has been implemented in the computer algebra systems GAP and MAGMA. In the present paper we deal with theoretical aspects of Frattini extensions and our main results are outlined in Theorems [2.1, 2.3, 2.6, 2.11 and 2.14]. We also provide more detailed proofs to some known results and these are listed in Theorems [2.7, 2.8, 2.10 and 2.18]. In addition in Example 2.16 we determine the non-abelian groups of order 16 and 32 for which $\Phi(G)$ is a Frattini extension of G' by $\Phi(G/G')$. We encourage readers to consult [[7], [11], [12],[13]] for a background material on group extensions and on Frattini subgroups.

We would like to thank Derek Holt for helpful comments. In particular he pointed out an article by R. L. Griess and P. Schmid (see [8]) and the reference to Theorem 2.13. The authors would also like to thank the referee whose suggestions and comments led to significant improvement of the content and the presentation of this article.

2. FRATTINI EXTENSIONS

If N and H are two groups, then an **extension** of the group N by the group H is a group G having a normal subgroup $K \cong N$ and $G/K \cong H$. Equivalently an extension can be defined in terms of short exact sequences of groups and homomorphisms as follows: Let ϕ and ψ be the isomorphisms described above. Consider $N \xrightarrow{\phi} \mathrm{K} \xrightarrow{i} G$ and $G \xrightarrow{\pi} \mathrm{G/K} \xrightarrow{\psi} H$, where i is the inclusion map and π is the natural homomorphism. Let $\alpha = i \circ \phi$ and $\epsilon = \psi \circ \pi$. Then we have the following short exact sequence $\{1_N\} \longrightarrow \mathrm{N} \xrightarrow{\alpha} \mathrm{G} \xrightarrow{\epsilon} \mathrm{H} \longrightarrow \{1_H\}$, which we say it represents the extension (G, ϵ).

Given a (multiplicativelly written) group G, a G**-module** is an (additivelly written) abelian group A together with an action of G on A such that the following axioms hold for

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all $a, b \in A,\ g, h \in G$:
$(i)\ \ (a+b)g = ag + bg$, $(ii)\ \ a(gh) = (ag)h$ and $(iii)\ \ a1_G = a$.

An extension (G, ϵ) is called a **Frattini extension** if the Kernel of ϵ is contained in the Frattini subgroup $\Phi(G)$ of G. Theorem 2.1 gives a useful characterization of the Frattini extensions for finite groups.

Theorem 2.1. *Let G be an extension of N by H and assume that G is finite. Then G is a Frattini extension of N by H if and only if $G/\Phi(G) \cong H/\Phi(H)$.*

Proof. Assume that G is a Frattini extension of N by H. Since $N \leq \Phi(G)$and $G/N \cong H$, we have $\Phi(G)/N = \Phi(G/N) \cong \Phi(H)$. Thus $H/\Phi(H) \cong (G/N)/\Phi(G/N) \cong (G/N)/(\Phi(G)/N) \cong G/\Phi(G)$.

Conversely assume that $G/\Phi(G) \cong H/\Phi(H)$. Then $G/\Phi(G) \cong (G/N)/\Phi(G/N)$, since $H \cong G/N$. Since $\Phi(G/N) \supseteq \Phi(G)N/N \cong \Phi(G)/N \cap \Phi(G)$, we have

$$\frac{|G|}{|\Phi(G)|} = \frac{|G|}{|N|} \times \frac{1}{|\Phi(G/N)|} \leq \frac{|G|}{|N|} \times \frac{|N \cap \Phi(G)|}{|\Phi(G)|}.$$

We deduce that $1 \leq \frac{|N \cap \Phi(G)|}{|N|}$. Thus $|N| \leq |N \cap \Phi(G)|$ and hence $|N| = |N \cap \Phi(G)|$. Therefore $N = N \cap \Phi(G)$, which implies that $N \leq \Phi(G)$. $\square$

The following Lemma [Lemma 2.2] is a stated problem by D.Holt and W.Plesken in [9]. We give a detailed proof for this Lemma and derive some new properties of the Frattini extensions.

Lemma 2.2. [9] *Let (G, ϵ) be an extension by H in which $Ker(\epsilon)$ is a finite H-module. If L is a maximal subgroup of G which does not contain $Ker(\epsilon)$, then (L, α) is an extension with $\alpha = \epsilon|_L$ and $Ker(\alpha)$ a maximal H-submodule of $Ker(\epsilon)$.*

Proof. Let $\{1_N\} \to N \to \mathrm{G} \xrightarrow{\epsilon} \mathrm{H} \to \{1_H\}$ be the given extension where $N = Ker(\epsilon)$. We need to show that:
(i) (L, α) is an extension of $L \cap N$ by H.
(ii)$Ker(\alpha)$ is a maximal H-submodule of $Ker(\epsilon)$.

(i) Since $N \not\leq L$ and L is a maximal subgroup of G, $L < LN \leq G$. But L is maximal in G implies that $LN = G$. Now, since (G, ϵ) is an extension we have that $G/N \cong H$, so $LN/N = G/N \cong H$. But $LN/N \cong L/N \cap L \cong H$ implies that L is an extension of $N \cap L$ by H. Also note that

$$\begin{aligned} Ker(\alpha) &= \{l \mid l \in L,\ \alpha(l) = 1_H\} = \{l \mid l \in L,\ \epsilon(l) = 1_H\} \\ &= \{l \mid l \in L,\ l \in Ker(\epsilon)\} = L \cap Ker(\epsilon) = L \cap N. \end{aligned}$$

(ii) Suppose to the contrary, that is, $Ker(\alpha) = N \cap L$ is not a maximal H-submodule of $Ker(\epsilon)$. Then there exists an H-submodule K of N such that $N \cap L < K < N$. Since $N \not\leq L$, we deduce that $L < LK < G$. This contradicts the maximality of L in G. Hence $N \cap L$ is a maximal H-submodule of $Ker(\epsilon)$. $\square$

Now if $Ker(\epsilon)$ is an irreducible H-module, we use Lemma 2.2 to prove the following result.

Theorem 2.3. *Let (G, ϵ) be an extension by H. Suppose that $Ker(\epsilon)$ is a non-trivial irreducible H-module. Then (G, ϵ) is a Frattini extension if and only if it is non-split.*

Proof. Suppose that $\{1_N\} \to N \to G \xrightarrow{\epsilon} H \to \{1_H\}$ $(*)$ is a non-split extension. We need to show that $N = Ker(\epsilon)$ is a subgroup of $\Phi(G)$. If $N \not\leq \Phi(G)$, then there exists L a maximal subgroup of G such that $N \not\leq L$, and so by Lemma 2.2 we have that (L, α) is an extension, with $\alpha = \epsilon|_L$. Moreover, $Ker(\alpha)$ is a maximal H-submodule of $Ker(\epsilon)$ and since $Ker(\epsilon)$ is irreducible as an H-module we have that $Ker(\alpha) = N \cap L = \{1_G\}$ or $Ker(\alpha) = N$. If $Ker(\alpha) = N$ we have that $N \leq L$ which is a contradiction. Hence $Ker(\alpha) = \{1_G\}$ and therefore α is a monomorphism. But α is an epimorphism, since (L, α) is an extension. Hence α is an isomorphism, and so $\alpha^{-1} : H \longrightarrow L$ is also an isomorphism. Since $L \leq G$, we have that $\alpha^{-1} : H \longrightarrow G$ is a monomorphism. Now for all $h \in H$ we have

$$\begin{aligned}(\epsilon\alpha^{-1})(h) = \epsilon(\alpha^{-1}(h)) &= \epsilon(l) \text{ where } \alpha^{-1}(h) = l \text{ for some } l \in L \\ &= \alpha(l) = h.\end{aligned}$$

Hence $(\epsilon\alpha^{-1}) = I_H$, and so (*) splits, which is a contradiction.

Conversely, suppose that (G, ϵ) is a Frattini extension. We need to show that $(*)$ is non-split. Suppose that $(*)$ splits. Then there exists a homomorphism say ϕ from H into G such that $\epsilon\phi = I_H$, and so ϕ is a monomorphism and therefore $H \cong Im(\phi)$. Hence $\phi : H \longrightarrow Im(\phi)$ is an isomorphism. Also, we have that $Im(\phi) \leq G$ and $Ker(\epsilon) \leq G$. It is easy to show that $G = Ker(\epsilon).Im(\phi)$ and $Ker(\epsilon) \cap Im(\phi) = \{1_G\}$. Thus $G = Ker(\epsilon){:}Im(\phi)$ is a split extension. Since $Im(\phi) \not\supseteq Ker(\epsilon)$ then $Im(\phi)$ is not a maximal subgroup of G. Thus there exists L a maximal subgroup of G such that $Im(\phi) < L < G$. But maximality of L implies that $L \supseteq \Phi(G) \supseteq Ker(\epsilon)$. Now, $Ker(\epsilon) \leq L$ and $Im(\phi) < L$ implies that $L \supseteq Ker(\epsilon)Im(\phi) = G$, which is contradiction. □

Remark 2.4. A simplest example of a Frattini extension is a non-split extension with $Ker(\epsilon)$ an irreducible H-module.

Corollary 2.5. *If (G, ϵ) is a Frattini extension by H with $Ker(\epsilon) \neq \{1_G\}$, then (G, ϵ) is non-split.*

Proof. The proof follows from the argument used in the second part of the proof of Theorem 2.3 or alternatively we can use Lemma 11.4 on p. 269 of [12]. □

Theorem 2.6. *If (G, ϵ) is a Frattini extension by H then $(\Phi(G), \epsilon)$ is an extension of $Ker(\epsilon)$ by $\Phi(H)$.*

Proof. Since (G, ϵ) is an extension, we have $G/Ker(\epsilon) \cong H$, and so $\Phi(G/Ker(\epsilon)) \cong \Phi(H)$. Since $Ker(\epsilon) \leq \Phi(G)$, we have that $\Phi(G/Ker(\epsilon)) = \Phi(G)/Ker(\epsilon)$. □

Theorem 2.7. [9] *Let G be a finite group and let (G, ϵ) be a Frattini extension of N by H with H a perfect group. Then G is perfect.*

Proof. Suppose that G is not perfect. Set $N = Ker(\epsilon)$. Thus we have that $G/N \cong H$. Now $(G/N)' \cong H' = H$ imply that $(G/N)' \cong G/N$ $(*)$. Since $G'N \leq G$ and $G'N \supseteq N$, we have $(G/N)' = G'N/N \leq G/N$. So by (*) we have that $G'N/N = G/N$. Since G is finite and $G'N \leq G$, we have that $|G'N| = |G|$. Hence $G'N = G$. Since $N \leq \Phi(G)$, there is no proper subgroup K of G such that $KN = G$. (See Lemma 11.4 on p. 269 of [12].) Hence $G = G'$ and the result follows. □

In [3] J.Cossey, O. Këgel and L. Kovács stated the following results (Theorem 2.8, Proposition 2.9 and Theorem 2.10) about Frattini extensions to which we give detailed proofs.

Theorem 2.8. [3] *If $\epsilon : L \longrightarrow M$ is an epimorphism of finite groups and G is minimal among the subgroups of L, with $\epsilon(G) = M$, then $(G, \epsilon|_G)$ is a Frattini extension.*

Proof. Let K be a maximal subgroup of G such that $Ker(\epsilon|_G) \nsubseteq K$ then we have $K < K.Ker(\epsilon|_G) \leq G$. Now maximality of K implies that $K.Ker(\epsilon|_G) = G$. Also we have that $Ker(\epsilon) \trianglelefteq L$, $Ker(\epsilon) \supseteq Ker(\epsilon|_G)$ and $K \leq L$. Thus $K \leq G = K.Ker(\epsilon|_G) \leq K.Ker(\epsilon) \leq L$ (1). Let $W = K.Ker(\epsilon)$, then

$$\begin{aligned} \epsilon(W) &= \{\epsilon(kx) \mid k \in K \text{ and } x \in Ker(\epsilon)\} = \{\epsilon(k)\epsilon(x) \mid k \in K \text{ and } x \in Ker(\epsilon)\} \\ &= \{\epsilon(k) \mid k \in K\} = \epsilon(K). \end{aligned}$$

Now by (1) we have that $\epsilon(G) \leq \epsilon(W) \leq \epsilon(L)$, that is $M \leq \epsilon(W) \leq M$. Thus $\epsilon(W) = M$ and hence $\epsilon(K) = M$. Since $K < G$ and $\epsilon(K) = M$, minimality of G produces a contradiction. It follows that $Ker(\epsilon|_G)$ is contained in every maximal subgroup of G and hence in the Frattini subgroup of G. Therefore $(G, \epsilon|_G)$ is a Frattini extension. □

Proposition 2.9. [3] *If (G, α) is a Frattini extension by F, and $\beta : F \longrightarrow G$ is a homomorphism, such that $\alpha\beta$ is surjective, then β is surjective.*

Proof. Let $K = Im(\beta) = \beta(F)$. Since $\alpha\beta$ is surjective, we have that $\alpha(G) = F = (\alpha\beta)(F) = \alpha(\beta(F)) = \alpha(K)$. So for any $g \in G$, there exists $k \in K$ such that $\alpha(g) = \alpha(k)$. We have,

$$\begin{aligned} \alpha(g) = \alpha(k) &\Rightarrow \alpha(gk^{-1}) = 1_F \Rightarrow gk^{-1} \in Ker(\alpha) \\ &\Rightarrow g \in Ker(\alpha)K, \forall g \in G \Rightarrow G \subseteq Ker(\alpha)K. \quad (1) \end{aligned}$$

Now, $K \leq G$ and $Ker(\alpha) \trianglelefteq G$ imply that $Ker(\alpha)K \leq G$. Hence by (1) we have that $G = Ker(\alpha)K$. Since $Ker(\alpha) \leq \Phi(G)$, as in the proof of Theorem 2.7 we deduce that $K = G$. Hence β is surjective. □

Theorem 2.10. [3] *Composites of Frattini extensions are Frattini extensions.*

Proof. Consider $\{1\} \rightarrow Ker(\alpha) \rightarrow \mathrm{G} \xrightarrow{\alpha} \mathrm{H} \rightarrow \{1\}$ with α an epimorphism and $Ker(\alpha) \leq \Phi(G)$ and let $\{1\} \rightarrow Ker(\beta) \rightarrow \mathrm{H} \xrightarrow{\beta} \mathrm{M} \rightarrow \{1\}$ with β an epimorphism and $Ker(\beta) \leq \Phi(H)$. We need to show that $\beta\alpha$ is an epimorphism and that $Ker(\beta\alpha) \leq \Phi(G)$.

(i) $\beta\alpha$ is an epimorphism: Since $h \in H$, then there exists $g \in G$ such that $\alpha(g) = h$. If $m \in M$, then there exists $h \in H$ such that $\beta(h) = m$. Hence $(\beta\alpha)(g) = \beta(\alpha(g)) = \beta(h) = m$.

(ii) $Ker(\beta\alpha) \leq \Phi(G)$: Since $\alpha(G) = H$ and α is a homomorphism, we have $\alpha(\Phi(G)) = \Phi(\alpha(G)) = \Phi(H)$. Thus $\Phi(G) = \alpha^{-1}(\Phi(H))$, the inverse image of $\Phi(H)$. But,

$$\begin{aligned} Ker(\beta\alpha) &= \{g \mid (\beta\alpha)(g) = 1_M \ , \ g \in G\} = \{g \mid \beta(\alpha(g)) = 1_M \ , \ g \in G\} \\ &= \{g \mid \alpha(g) \in Ker(\beta) \ , \ g \in G\} = \{g \mid g \in \alpha^{-1}(Ker(\beta))\}. \end{aligned}$$

Since $Ker(\beta) \leq \Phi(H)$, we have that $\alpha^{-1}(Ker(\beta)) \leq \alpha^{-1}(\Phi(H)) = \Phi(G)$. Hence $Ker(\beta\alpha) \leq \Phi(G)$. □

The following theorem is a straightforward consequence of a theorem of Gaschutz, which states that if N is a finite group with $Inn(N) \leq \Phi(Aut(N))$, then there exists a finite group G such that $N \trianglelefteq G$ and $N \leq \Phi(G)$. The converse of this statement has been recently proved in [6].

Theorem 2.11. *Let N be a finite abelian group. Then there exists a Frattini extension (G, ϵ) with $Ker(\epsilon) = N$.*

Proof. Since N is abelian, $Inn(N) = \{1_{Aut(N)}\}$ and hence $Inn(N) \leq \Phi(Aut(N))$. Thus there exists a finite group G with $N \trianglelefteq G$ and $N \leq \Phi(G)$. If ϵ is the natural homomorphism from G into G/N, then $Ker(\epsilon) = N$. Hence (G, ϵ) is a Frattini extension with $Ker(\epsilon) = N$. □

Remark 2.12. If N is an elementary abelian group, then the following theorem (see [[5] or [8]]), due to Gaschutz, describes the existence of a Frattini extension G of N by H.

Theorem 2.13. *Let p be a prime, and let H be a group whose order is divisible by p. Then there exists a group G with a non-trivial normal elementary abelian p-subgroup N such that*
(i) $N \leq \Phi(G)$, *and*
(ii) $G/N \cong H$.

Proof. See Theorem 11.8 of [5]. □

Now if G is a p-group we will show that $\Phi(G)$ is an extension of G' by $\Phi(G/G')$.

Theorem 2.14. *If G is a p-group, then $\Phi(G)$ is an extension of G' by $\Phi(G/G')$.*

Proof. Since G is a p-group, G is nilpotent and hence $G' \leq \Phi(G)$. Also $G' \trianglelefteq G$ implies that $G' \trianglelefteq \Phi(G)$. Since $G' \leq \Phi(G)$, we have that $\Phi(G/G') = \Phi(G)/G'$. We deduce that $\Phi(G)$ is an extension of G' by $\Phi(G/G')$. □

Remark 2.15. The extension given in Theorem 2.14 is not a Frattini extension in general. For example take $G = Q$, the group of quaternions or $G = D_8$. It can be easily checked that $G' = \Phi(G) \cong C_2$ and $\Phi(\Phi(G)) = \{1_G\}$. Thus $G' \not\leq \Phi(\Phi(G))$ and hence the extension $\Phi(G)$ of G' by $\Phi(G/G')$ is not a Frattini extension.

In the following we determine all the non-abelian groups of order 16 and 32 for which $G' \leq \Phi(\Phi(G))$, respectively.

Example 2.16. From the library of small groups of GAP4 [14] we found that there are 14 non-isomorphic groups of order 16. Only 9 of these groups are non-abelian. We consider these non-abelian groups and label them as G_i with $1 \leq i \leq 9$ as listed in the first row of Table 1. The remaining rows of Table 1 have been computed by using GAP. It can be observed from Table 1 that the group G_3 is the only non-abelian group of order 16 for which $\Phi(G)$ is a Frattini extension of G' by $\Phi(G/G')$.

Table 1 : Non-abelian groups of order 16

G	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
$\Phi(G)$	V_4	V_4	C_4	C_4	C_4	C_4	C_2	C_2	C_2
G'	C_2	C_2	C_2	C_4	C_4	C_4	C_2	C_2	C_2
$\Phi(\Phi(G))$	$\{1_G\}$	$\{1_G\}$	C_2	C_2	C_2	C_2	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$

Similarly there are 51 non-isomorphic groups of order 32. Only 44 of these are non-abelian and are labeled as G_i with $1 \leq i \leq 44$. These groups are listed in the first row of Table 2. As in Table 1, we completed the remaining rows of Table 2. We observe that there are 6 possible candidates among G_i, $1 \leq i \leq 44$ for which we may have $G' \leq \Phi(\Phi(G))$, namely $G_2, G_3, G_{10}, G_{14}, G_{32}$ and G_{33}. It is clear that G_{14}, G_{32} and G_{33} satisfy the condition $G' \leq \Phi(\Phi(G))$. Further investigation eliminates G_3 and G_{10}. Hence G_2, G_{14}, G_{32} and G_{33} are the only non-abelian groups of order 32 for which $\Phi(G)$ is a Frattini extension of G' by $\Phi(G/G')$.

Table 2 : Non-abelian groups of order 32

G	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
$\Phi(G)$	C_{2^3}	$C_4 \times C_2$	$C_4 \times C_2$	C_{2^3}	$C_4 \times C_2$	$C_4 \times C_2$	$C_4 \times C_2$	$C_4 \times C_2$	$C_4 \times C_2$
G'	C_2	C_2	C_2	V_4	V_4	V_4	C_4	C_4	C_4
$\Phi(\Phi(G))$	$\{1_G\}$	C_2	C_2	$\{1_G\}$	C_2	C_2	C_2	C_2	C_2

Table 2 : Non-abelian groups of order 32(continued)

G	G_{10}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}	G_{17}	G_{18}	G_{19}	G_{20}
$\Phi(G)$	$C_4 \times C_2$	$C_4 \times C_2$	$C_4 \times C_2$	$C_4 \times C_2$	C_8	C_8	C_8	C_8	V_4	V_4	V_4
G'	C_2	C_4	C_4	C_4	C_2	C_8	C_8	C_8	C_2	C_2	C_2
$\Phi(\Phi(G))$	C_2	C_2	C_2	C_2	C_4	C_4	C_4	C_4	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$

Table 2 : Non-abelian groups of order 32(continued)

G	G_{21}	G_{22}	G_{23}	G_{24}	G_{25}	G_{26}	G_{27}	G_{28}	G_{29}	G_{30}	G_{31}	G_{32}
$\Phi(G)$	V_4	V_4	V_4	V_4	V_4	V_4	V_4	V_4	V_4	V_4	V_4	C_4
G'	C_2	C_2	V_4	V_4	V_4	V_4	V_4	V_4	V_4	V_4	V_4	C_2
$\Phi(\Phi(G))$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	C_2

Table 2 : Non-abelian groups of order 32(continued)

G	G_{33}	G_{34}	G_{35}	G_{36}	G_{37}	G_{38}	G_{39}	G_{40}	G_{41}	G_{42}	G_{43}	G_{44}
$\Phi(G)$	C_4	C_4	C_4	C_4	C_4	C_4	C_4	C_2	C_2	C_2	C_2	C_2
G'	C_2	C_4	C_4	C_4	C_4	C_4	C_4	C_2	C_2	C_2	C_2	C_2
$\Phi(\Phi(G))$	C_2	C_2	C_2	C_2	C_2	C_2	C_2	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$	$\{1_G\}$

As a concluding remark we would like to prove a result on minimal Frattini extensions. For further properties of the minimal Frattini extensions, readers are referred to [2].

A group G is called a **minimal Frattini extension** of N by H, if it is a Frattini extension of N by H and N is a minimal normal subgroup of G.

Lemma 2.17. *Let N be a minimal normal subgroup of G. If N has a complement in G, then $N \cap \Phi(G) = \{1_G\}$.*

Proof. If N is complemented in G then $G = NH$ and $N \cap H = \{1_G\}$ for some proper subgroup H of G. Since $N \trianglelefteq G$ and $\Phi(G) \trianglelefteq G$, we have $N \cap \Phi(G) \trianglelefteq G$. Now $N \cap \Phi(G) \leq N$ and minimality of N implies that $N \cap \Phi(G) = \{1_G\}$ or $N \cap \Phi(G) = N$. If $N \cap \Phi(G) = N$ then $N \subseteq \Phi(G)$ and hence $NH \subseteq \Phi(G)H$, so that $G = \Phi(G)H$. But this implies that $H = G$ which is not possible, since $N \subseteq \Phi(G)$. Thus $N \cap \Phi(G) = \{1_G\}$ as required. $\square$

We now use Lemma 2.17 to give a more detailed proof for the following theorem which was stated as Lemma 4.6 by Eick in [2].

Theorem 2.18. *Let G be an extension of N by H, where N is abelian. Then G is a minimal Frattini extension of N, if and only if N is a minimal non-complemented normal subgroup of G.*

Proof. Let N be a minimal non-complemented normal subgroup of G. If $N \not\leq \Phi(G)$, then there exists a maximal subgroup M of G such that $N \not\leq M$. Now $N \trianglelefteq G$ and $M \leq G$, implies that $M < NM \leq G$. The maximality of M implies that $NM = G$. Since $M \cap N \trianglelefteq N$ and $N \cap M \trianglelefteq M$ we have that $N \cap M \trianglelefteq NM = G$. The minimality of N in G implies that $N \cap M = \{1_G\}$ and hence M is a complement of N, which is a contradiction. So $N \leq \Phi(G)$ and therefore G is a minimal Frattini extension. Conversely, if N is a complemented minimal normal subgroup of G, then by Lemma 2.17 we have that $N \cap \Phi(G) = \{1_G\}$ and hence N is not a subgroup of $\Phi(G)$. Therefore G is not a minimal Frattini extension. $\square$

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University of Natal, Private Bag X01, Scottsville, Pietermaritzburg,, 3209, South Africa
E-mail address: `moori@nu.ac.za, rodrigues@nu.ac.za`