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ON THE NON-ISOMORPHIC EMBEDDINGS OF THE SIMPLE, CONNECTED, PLANAR GRAPHS

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Abstract. In this paper, we find the necessary and sufficient condition for the simple, 2-connected, plane graphs to be isomorphic and the necessary and sufficient condition for the simple, connected, plane graphs to be isomorphic. And we find an algorithm to give all non-isomorphic embeddings of the simple, connected, planar graph by using this condition. Finally, we compute the numbers of the non-isomorphic embeddings of simple, connected, planar graphs and those of the simple, 2-connected, planar graphs with order p=8,9 and 10.

1 Theorems We generally follow the definitions and notation of [2].

Let G_1 and G_2 be two simple, connected, planar graphs. Let $\iota_1 : G_1 \to S^2$ and $\iota_2 : G_2 \to S^2$ be embedding of G_1 and G_2 into S^2, respectively. Two embeddings ι_1 and ι_2 are called to be isomorphic if there exists a homeomorphism $h : S^2 \to S^2$ such that h induces the isomorphism between $\iota_1(G_1)$ and $\iota_2(G_2)$. Two embeddings ι_1 and ι_2 are called to be orientation-preservingly isomorphic if there exists an orientation-preserving homeomorphism $h : S^2 \to S^2$ such that h induces the isomorphism between $\iota_1(G_1)$ and $\iota_2(G_2)$. Two embeddings ι_1 and ι_2 are called to be orientation-reversingly isomorphic if there exists an orientation-reversing homeomorphism $h : S^2 \to S^2$ such that h induces the isomorphism between $\iota_1(G_1)$ and $\iota_2(G_2)$. And therefore, two embeddings ι_1 and ι_2 are called to be isomorphic if and only if they are orientation-preservingly isomorphic or orientation-reversingly isomorphic.

A path addition to a graph G is the addition to G of a path between two existing vertices of G, such that the edges and internal vertices of the path are not in G.

A Whitney synthesis of a graph G from a graph H is a sequence of graph, $G_0, G_1, \cdots, G_l$, where $G_0 = H, G_l = G$, and G_i is the result of a path addition to G_{i-1}, for $i = 1, \cdots, l$.

Theorem 1. *(Whitney, 1932) A graph is 2-connected if and only if G is a cycle or a Whitney synthesis from a cycle.*

Lemma 1. *Let G_1 and G_2 be simple, 2-connected, planar graphs and $\iota_1 : G_1 \to S^2$ and $\iota_2 : G_2 \to S^2$ be embeddings such that the dual graph ι_1^* is isomorphic to the dual graph ι_2^* as abstract pseudo graphs. Then there exist a homeomorphism $h : S^2 \to S^2$ such that h induces isomorphism between $\iota_1(G_1)$ and $\iota_2(G_2)$.*

Proof. We use the Whitney's Synthesis of 2-connected graphs. We will prove with the induction on the number of edge of G_1. If G_1 is a cycle, then the result is trivial. Therefore, by the Whitney's synthesis of 2-connected graphs, let G_1 be a Whitney synthesis from a cycle $H_0, H_1, \cdots, H_l$. H_l is the result of a path addition to H_{l-1}. Let P be the path that is added to H_{l-1} and let Q be the path of G_2 that corresponds to P by the isomorphism $\iota_1^* \to \iota_2^*$. Let $G_2' = G_2 - Q$, $\iota_1' = \iota_1|H_{l-1}$, and $\iota_2' = \iota_2|G_2'$. Then we have an induced isomorphism

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$(\iota_1')^* \cong (\iota_2')^*$. By the assumption of the induction, we have a homeomorphism $g : S^2 \to S^2$ such that g induces the isomorphism $\iota_1'(H_{l-1}) \cong \iota_2'(G_2')$. By modifying the corespondance between the face including P and the face including Q of the homeomorphism, we can obtain the resulting homeomorphism. □

For the simple, 2-connected, planar graphs, we have the following theorem.

Theorem 2. *Let G be a simple, 2-connected, planar graph and $\iota_1 : G \to S^2$ and $\iota_2 : G \to S^2$ be two embeddings of G. Then ι_1 is isomorphic to ι_2 if and only if the dual graph of ι_1 is isomorphic, as abstract pseudo graphs, to the dual graph of ι_2.*

Proof. ($\Rightarrow$) The assertion is trivially true.
($\Leftarrow$) Let $G_1 = G$ and $G_2 = G$ in Lemma 1, then we have the result. □

For the simple, connected, plane graphs, we need more conditions. Let G be a simple, connected, planar graph and $\iota_1 : G \to S^2$ and $\iota_2 : G \to S^2$ be two embeddings of G. If the dual graph of ι_1 is isomorphic, as abstract pseudo graphs, to the dual graph of ι_2, then we have a one-to-one correspondence between the faces given by the embedding $\iota_1 : G \to S^2$ and the faces given by the embedding $\iota_2 : G \to S^2$. If each pair of the corresponding faces is the pair of the same polygons whose coresponding vertices have the same degrees, then we call the isomorphism DP (degree preserving). If each pair of the corresponding faces, where one face is reflected, is the pair of the same polygons whose coresponding vertices have the same degrees, then we call the isomorphism DR (degree reversing). If each pair of the corresponding faces is the pair of the same polygons whose boundaries are composed of the edges that are adjoining with the faces that corresond to the faces given by the isomorphism, then we call the isomorphism BP (boundary preserving). If each pair of the corresponding faces, where one face is reflected, is the pair of the same polygons whose boundaries are composed of the edges that are adjoining with the faces that corresond to the faces given by the isomorphism, then we call the isomorphism BR (boundary reversing).

Lemma 2. *Let G_1 and G_2 be simple, connected, planar graphs and $\iota_1 : G_1 \to S^2$ and $\iota_2 : G_2 \to S^2$ be the embeddings of G_1 and G_2, respectively. Suppose that there exists an isomorphism (as abstract pseudo graphs) φ from the dual graph of ι_1 to the dual graph of ι_2 such that (1) φ is DP and (2) φ is BP. Then, there exists an orientation-preserving homeomorphism $h : S^2 \to S^2$ such that h induces the isomorphism between $\iota_1(G_1)$ and $\iota_2(G_2)$.*

Proof. We prove with the induction on the number of edge of G_1. If G_1 consists of one point, then the assertion is trivially true. First of all we consider the case that G_1 has a vertex v_1 of degree 1. Let e_1 be the edge that is incident on v_1. Let e_2 be the edge of G_2 that corresponds to e_1 under the given isomorphism. Let $G_1' = G_1 - e_1, G_2' = G_2 - e_2, \iota_1' = \iota_1|G_1'$, and $\iota_2' = \iota_2|G_2'$. Then the given isomorphism induces the isomorphism between $(\iota_1')^*$ and $(\iota_2')^*$. It is trivially true that this isomorphism satisfies all conditions of Lemma 2. By the assumption of the induction, there exists an orientation-preserving homeomorphism $g : S^2 \to S^2$ that induces an isomorphism between $\iota_1'(G_1')$ and $\iota_2'(G_2')$. Let α_1 be the face including e_1 and let α_2 be the face including e_2. By the condition (1), we can modify the corespondance between the faces α_1 and α_2 of the homeomorphism, and we can obtain the resulting homeomorphism. If G_1 has no vertex of degree 1, then G_1 has a cycle. Let e_1 be an edge on a cycle of G_1 and let e_2 be the edge that corresponds to e_1 under the given isomorphism. Let $G_1' = G_1 - e_1, G_2' = G_2 - e_2, \iota_1' = \iota_1|G_1'$, and $\iota_2' = \iota_2|G_2'$. Then the given isomorphism induces the isomorphism between $(\iota_1')^*$ and $(\iota_2')^*$. It is trivially true that this isomorphism satisfies all conditions of Lemma 2. By the assumption of the induction, there exists an orientation-preserving homeomorphism $g : S^2 \to S^2$ that induces an isomorphism between $\iota_1'(G_1')$ and $\iota_2'(G_2')$. Let α_1 be the face including e_1 and let α_2 be the face including e_2. By the condition (2), we can modify the corespondance between the faces α_1 and α_2 of the homeomorphism, and we can obtain the resulting homeomorphism. □

Lemma 3. *Let G_1 and G_2 be simple, connected, planar graphs and $\iota_1 : G_1 \to S^2$ and $\iota_2 : G_2 \to S^2$ be the embeddings of G_1 and G_2, respectively. Suppose that there exists an isomorphism (as abstract pseudo graphs) φ from the dual graph of ι_1 to the dual graph of ι_2 such that (1) φ is DR and (2) φ is BR. Then, there exists an orientation-reversing homeomorphism $h : S^2 \to S^2$ such that h induces the isomorphism between $\iota_1(G_1)$ and $\iota_2(G_2)$.*

Proof. The proof is essentially same with Lemma 2. We omit it. □

Theorem 3. *Let G be a simple, connected, planar graph and $\iota_1 : G \to S^2$ and $\iota_2 : G \to S^2$ be two embeddings of G. Then the embedding $\iota_1 : G \to S^2$ is orientation-preservingly isomorphic to the embedding $\iota_2 : G \to S^2$ if and only if there exists an isomorphism (as abstract pseudo graphs) φ from the dual graph of ι_1 to the dual graph of ι_2 such that (1) φ is DP and (2) φ is BP. And the embedding $\iota_1 : G \to S^2$ is orientation-reversingly isomorphic to the embedding $\iota_2 : G \to S^2$ if and only if there exists an isomorphism (as abstract pseudo graphs) φ from the dual graph of ι_1 to the dual graph of ι_2 such that (1) φ is DR and (2) φ is BR.*

Proof. ($\Rightarrow$) The assertion is trivially true.
($\Leftarrow$) Let $G_1 = G$ and $G_2 = G$ in Lemma 2 and 3, then we have the results. □

2 Examples The next two embeddings are not isomorphic to each other but there is an isomorphism between their dual graphs that satisfies the conditions (2) of Theorem 3.

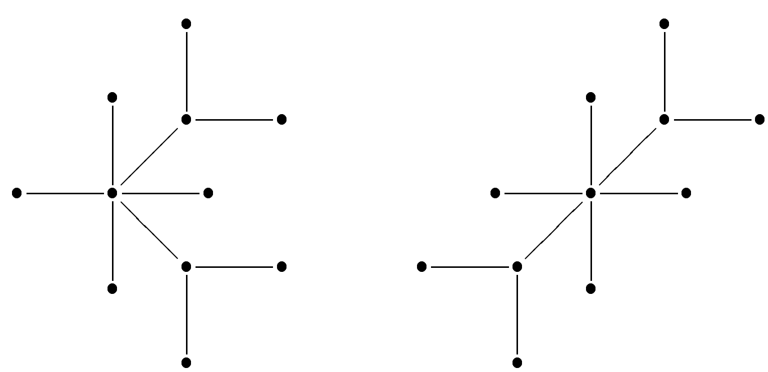


Figure 1

In this example the face is only an outside, respectively. We call this 0. The isomorphism h from the dual of the left graph to the dual of the right graph maps 0 to 0. Then the face sequences along the edges of both embeddings are $0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$. Then h is BP and BR. The degree sequence along the edges of the left graph is $6, 3, 1, 3, 1, 3, 6, 1, 6, 3, 1, 3, 1, 3, 6, 1, 6, 1, 6, 1$ and the degree sequence along the edges of the right graph is $6, 3, 1, 3, 1, 3, 6, 1, 6, 1, 6, 3, 1, 3, 1, 3, 6, 1, 6, 1$. Then h is not DP nor DR.

The next two embeddings are not isomorphic to each other but there is an isomorphism between their dual graphs that satisfies the conditions (1) of Theorem 3.

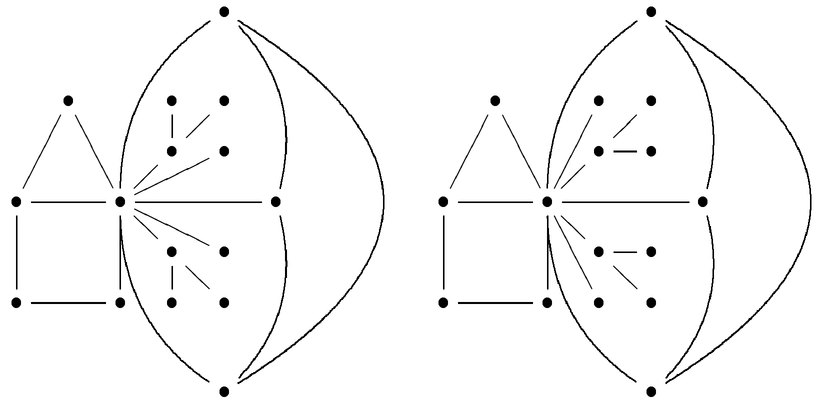


Figure 2

In this example we call the outside face 0, the left triangle 1, the left square 2, the middle upper face 3, the middle lower face 4, and the right triangle 5. Let $h(0) = 0, h(1) = 1, h(2) = 2, h(3) = 4, h(4) = 3$ *and* $h(5) = 5$. Then h is an isomorphism from the dual of the left graph to the dual of the right graph. Then the degree sequences along edges is given by the following list.

face											
face = 0	10	3	3	10	2	2	3	2			
1	10	2	3								
2	10	3	2	2							
3	10	3	1	3	1	3	10	1	10	3	3
4	10	1	10	3	1	3	1	3	10	3	3
5	3	3	3								

degree sequence along edges of the left graph

face											
face = 0	10	3	3	10	2	2	3	2			
1	10	2	3								
2	10	3	2	2							
3	10	1	10	3	1	3	1	3	10	3	3
4	10	3	1	3	1	3	10	1	10	3	3
5	3	3	3								

degree sequence along edges of the right graph

Then h is DP. But the face sequence along edges of face 3 of the left graph is $3, 3, 3, 3, 3, 3, 3, 3, 4, 5, 0$ and the face sequence along edges of face 4 of the right graph is $4, 4, 4, 4, 4, 4, 4, 4, 0, 5, 3$. Then h is not BP.

3 Algorithm We will review the definition of the appendage. Let H be a subgraph of a connected graph G. Two edges e_1 and e_2 of $E_G - E_H$ are unseparated by subgraph H if there exists a walk in G that contains both e_1 and e_2, but whose internal vertices are not in H. An appendage to subgraph H is the induced subgraph on an equivalence class of edges of $E_G - E_H$ under the relation unseparated by H. An appendage to H is called a chord if it contains only one edge. Let B be an appendage to H. Then a contact point of B is a vertex of $B \cap H$. Let C be a cycle of a graph G and let B be an appendage to C. Let $\{v_1, v_2, \cdots, v_n\}$ be the contact points. If n is equal to 2, let K be the edge that joins v_1 and v_2, and if n is greater than or equal to 3, let K be the cycle $< v_1, v_2, v_3, \cdots, v_n, v_1 >$. We call the graph $B + K$ modified appendage. If the appendage has only one contact point, we call the appendage modified appendage. Let C be a cycle in a graph. The appendages B_1 and B_2 of C are said to overlap if either of the following conditions holds:

1. Two contact points of B_1 alternate with two contact points of B_2 on cycle C.
2. B_1 and B_2 have three contact points in common.

Let C be a cycle in a planar drawing of a graph, and let B_1 and B_2 be overlapping appendages of C. Then the appendages do not both lie on the same side of that cycle in the plane. Let C be a cycle of a connected graph G, and suppose that C has been drawn in the plane. Relative to that drawing, an appendage of C is said to be inner or outer, according to whether that appendage is drawn inside or outside of C.

We use the planarity algorithm given in [4]. By modifying the planarity algorithm, we construct the list of the possible arrangement of each appendage. For example we make the list given in Figure 4 to the graph given in Figure 3.

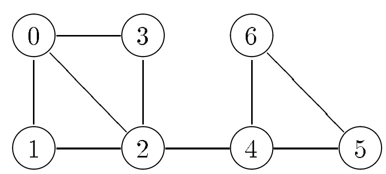


Figure 3

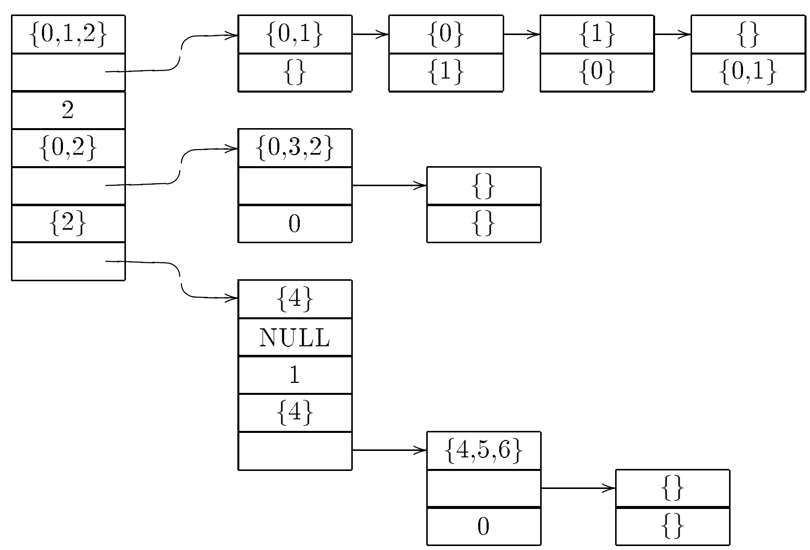


Figure 4

This list is composed in the following manner. First we find a cycle $< 0, 1, 2, 0 >$. We express this cycle as $\{0, 1, 2\}$. We set this cycle $\{0, 1, 2\}$ to the first slot of the list. There exists two appendages to the cycle.

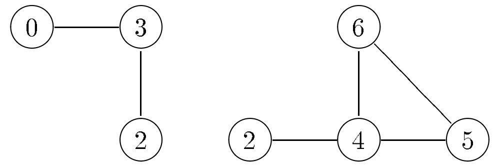


Figure 5

These appendages do not overlap. There exists four distributions to the inside or the outside of the cycle of these appendages. We set the pointer to this combination to the second slot of the list and set the number of the appendages to the third slot of the list. The modified appendage of the left appendage is the cycle $\{0, 3, 2\}$. Then we set the contact points $\{0, 2\}$ of the left appendage to the 4th slot of the list and set the pointer to the list of the modified appendage of the left appendage to the 5th slot of the list. The right appendage has only one contact point and has not any cycle that contains the contact point. Then we make a list that consists of the appendages to the vertex 4 to show that the vertices 2 and 4 are adjacent. We set the contact point $\{2\}$ to the 6th slot of the list and set the pointer of the above list to the 7th slot of the list. We call this list appendage tree.

Generally we compose the node of the appendage tree in the following manner:
We set a cycle or vertex to which we take appendages to the 1st slot of the node. We set the pointer to the list that consists of all possible distributions to the outside or inside of the appendages to the second slot of the node. We set the number of the appendages to the third slot of the node. We arrange the contact points and the pointer to the appendage tree that is made from the modified appendage of each appendage to the slots that continue.

The algorithm that gives all non-isomorphic embeddings of a simple, connected, planar graph is as follows.

Algorithm 1.
input *a simple, connected, non-trivial, planar graph G*
output *all non-isomorphic embeddings in S^2 of G*

1. *Construct the appendage tree.*
2. *Compose recursively all possible embeddings of G, by using the above tree.*
3. *By using Theorem 3, select the non-isomorphic embeddings.*

Details of Step 1

1. Find a cycle in G
2. If the cycle does not exist, then
   (a) Let x be a vertex of G whose degree is equal to 1
   (b) Let w be the adjent vertex to x
   (c) Let G' be $G - x$
   (d) Set $\{x\}$ to the first slot of the node
   Set NULL to the second slot of the node
   Set 1 to the third slot of the node
   Set $\{x\}$ to the 4th slot of the node
   Set the value, that planegivenvertex with the arguments of G' and w returns, to the 5th slot of the node

(e) The tree that was made is the resulting tree.

3. If the cycle exists, then

   The value that planegivencycle with the arguments of G and the cycle returns is the resulting tree.

tree *planegivenvertex(graph G, int v)

1. If G is a single vertex, then

   return the node that consists of $\{v\}$, NULL, and 0.

2. Let H be $\{v\}$ and get the appendages of G to H.

3. Let n be the number of the appendages.

4. Set $\{v\}$, NULL, and n to the node.

5. for i=1 to n

   (a) Implement the following thing toward the ith appendage B_i.
   (b) Set $\{v\}$ to the $2i+2$th slot of the node as the contact points.
   (c) Find a cycle in B_i that contains v.
   (d) If the cycle does not exist, then
   Let w be the adjacent vertex of v in B_i and let H be B_i-v. Set the value, that planegivenvertex with the arguments of H and w returns, to the $2i+3$th slot of the node.
   (e) If the cycle exist, then
   Set the value, that planegivencycle with the arguments of the appendage B_i and the cycle returns, to the $2i+3$th slot of the node.

6. Return the tree that was made.

tree *planegivencycle(graph G, path cycle)

1. Set the cycle to the first slot of the node.

2. Get the appendages of G to the cycle.

3. Rearrange them so that chords come first if they exist.

4. Set the pointer to the list that consists of all possible distributions to the outside or inside of the appendages to the second slot of the node.

5. Let n be the number of the appendages.

6. Set n to the third slot of the node.

7. for i=1 to n

   (a) Implement the following thing toward the ith appendage B_i.
   (b) Set the contact points of B_i to the $2i+2$th slot of the node.
   (c) If B_i is a chord, then
   Set NULL to the $2i+3$th slot of the node.

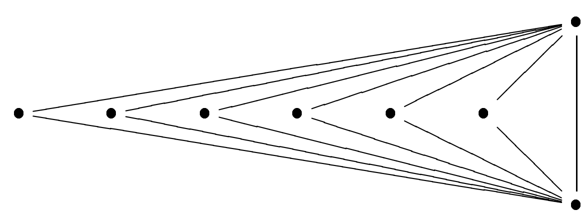
(d) If the contact points of B_i is a single vertex v, then
    i. Find a cycle C in B_i that contains v.
    ii. If C does not exist, then
        A. Let w be the adjacent vertex of v in the appendage.
        B. Let H be $B_i - v$.
        C. Set the value, that planegivenvertex with the argument of H and w returns, to the $2i + 3$th slot to the node.
    iii. If C exists, then
    Set the value, that planegivencycle with the argument of B_i and C returns, to the $2i + 3$th slot of the node.

(e) If the contact points of B_i consists two or more vertices and B_i is not a chord, then
    i. Let $\{v_1, v_2, v_3, \cdots, v_n\}$ be the contact points.
    ii. Find a shortest $v_1 - v_2$ path P in B_i that contains no other contact points.
    iii. Let C be the cycle $< P, v_3, \cdots, v_n, v_1 >$.
    iv. Set the value, that planegivencycle with the argument of the modified appendage and C returns, to the $2i + 3$th slot of the node.

8. Return the tree that was made.

Details of Step 2

For example, the useless embeddings increase very much when we consider all possible combinations in the case of the following graph.



Therefore, when we make the intermidiate embeddings, we do cropping as follows.

1. Make the set of all possible embeddings that are attached to the modified appendage to the cycle which is in the first slot.
2. Restrict the isomorphisms of the dual graphs only to the one that are correspondences between the faces that contain the same vertices of the cycle.
3. Leave only orientation-preserving non-isomorphic embeddings under the above isomorphism.

Now we explain the details of the construction of the embeddings. We construct recursively the embedding as follows.

1. In the case that the first slot of the tree is a vertex v
    (a) In the case that the third slot of the tree is 0
    Make the embedding that consists only one vertex v.
    (b) In the case that the third slot of the tree is 1

i. In the case that the first slot of the modified appendage is vertex w and the third slot of the modified appendage is 0
Make the embedding that consists only one edge $\{v, w\}$.

ii. In the case that the first slot of the modified appendage is vertex w and the third slot of the modified appendage is positive integer
Make all posible embeddings that are attached the edge $\{v, w\}$ to the embedding of the modified appendage to the outside.

iii. In the other case
Use the embedding of the appendage as it is.

(c) In the case that the third slot of the tree is two or more
If a modified appendage is attached to the vertex for the first time, then

i. In the case that the first slot of the modified appendage is a vertex w and the third slot of the modified appendage is 0
Make the embedding that consists only one edge $\{v, w\}$.

ii. In the case that the first slot of the modified appendage is a vertex w and the third slot of the modified appendage is positive integer
Make all posible embeddings that are attached the edge $\{v, w\}$ to the embedding of the modified appendage to the outside.

iii. In the other case
Use the embedding of the appendage as it is.

else

i. In the case that the first slot of the modified appendage is a vertex w and the third slot of the modified appendage is 0
Make all possible embeddings that are attached the edge $\{v, w\}$ to the embedding that has already been made. Classify the faces into two types by whether v is contained or not. Leave only orientation-preserving non-isomorphic embeddings, when the isomorphisms of the dual graphs are restricted to the one that are correspondeces between the faces of the same type.

ii. In the case that the first slot of the modified appendage is a vertex w and the third slot of the modified appendage is positive integer
Make all possible embeddings that are attached the edge $\{v, w\}$ to the embedding of the modified appendage to the outside. Incorporate them as much as possible into the embeddings that have already been made. Classify the faces into two types by whether v is contained or not. Leave only orientation-preserving non-isomorphic embeddings, when the isomorphisms of the dual graphs are restricted to the one that are correspondeces between the faces of the same type.

iii. In the other case
Assign the appendages of the embedding that has already made to the faces of the modified appendage that contain the contact point. The assignment renders all the combination including the permutations of the appendages. Classify the faces into two types by whether v is contained or not. Leave only orientation-preserving non-isomorphic embeddings, when the isomorphisms of the dual graphs are restricted to the one that are correspondeces between the faces of the same type.

2. In the case that the first slot of the tree is a cycle C

Make the embedding that consists only cycle C and attach the modified appendages of the node of the tree to it.

- If this is the embedding of the intermediate modified appendage, it is sufficient to make the embedding which has the outside face that contains all contact points. Therefore, it does the following manner, in the case that the number of contact points of the modified appendage is 3 or more.

  If there exists the chord that ties v_0 and v_1 of the contact points of the modified appendage and is attached in the outside, then

  Attach it first

  else

  The distribution does not compose.
- If the embedding is the final result, we make all combination.
- If the modified appendage has only one contact point of degree one, we make all combinations of the way of attaching of a possible branch and make the embedding of the modified appendage.
- If the modified appendage has only one contact point of degree 2 or more, we assign the appendages of the embedding that has already made to the faces of modified appendage that contain the contact point. The assignment renders all the combination including the permutations of the appendages.

Theorem 4. *The cropping does not change the final results.*

Proof. Let E_1 and E_2 be two intermidiate embeddings that are restricted orientation-preserving isomorphic to each other. Let $f : S^2 \to S^2$ be the orientation-preserving homeomorphism to which the restricted orientation-preserving isomorphism corresponds. We are sufficient to consider two cases.

In the case that other modified appendage A is attached to these embeddings:

Let F_1 be the face of E_1 to which A is attached. Let f map F_1 to the face F_2 of E_2. By the assumption, F_2 contains all contact points of A. Then f can be extend to the orientation-preserving isomorphism.

In the case that these embeddings are attached to some embedding B:

If the number of the contact points is greater than or equal to 3, then E_1 and E_2 contain the unique face that consists only contact points. Then we can modify the identity homeomorphism between S^2 and S^2 to the resulting orientation-preserving homeomorphism by using f. If the number of the contact points is equal to 2, then E_1 and E_2 contain two faces that contain all contact points. The orientation-preserving homeomorphism between E_1 and E_2 distinguishes these faces with other faces. Then we can modify the identity homeomorphism between S^2 and S^2 to the resulting orientation-preserving homeomorphism by using f. If the number of the contact points is equal to one and the contact point has degree one, then by attaching an edge to the faces, which correspond under f , respectively, we can modify the identity homeomorphism between S^2 and S^2 to the resulting orientation-preserving homeomorphism. And if the number of the contact points is equal to one and the contact point has degree greater than one, then by distributing the faces of B to the faces of E_1 and E_2 that correspond under f, we can modify f to the resulting orientation-preserving homeomorphism.

□

Theorem 5. *Algorithm 1 give all non-isomorphic embedding of the simple, connected, non-trivial, planar graph.*

Proof. Let G be a simple, connected, non-trivial, planar graph. If G has a cycle C then let $B_1, B_2, \cdots, B_m$ be the appendages to C. If all possible distributions to the inside or the outside of the cycle C of $B_1, B_2, \cdots, B_m$ are considered and the appendages are added with all the possible method to the cycle C then all the possible emmbedings are obtained. When we add the chord B_i to the intermediate embedding, we add it to the faces which contains two constant points of B_i. When we add the appendage B_i with three or more contact points to the intermediate embedding, we add it to the faces which contains all constant points of B_i. When we add the appendage B_i with only one contact point to the intermediate embedding, we divide it into two case. If the contact point of B_i has degree one then we add it to the faces which contains the constant point of B_i. If the contact point of B_i has degree two or more then we add it in the following manner. Let v be the contact point and let H be the intermediate embedding. Let $D_1, D_2, \cdots, D_n$ be the appendagea to $\{v\}$. Then we add $D_1, D_2, \cdots, D_n$ to the faces of B_i wchich contains v. The assignment renders all the combination including the permutations of $D_1, D_2, \cdots, D_n$. If G has not a cycle then let v be the vertex of G whose degree is equal to 1 and let w be the adjent vertex to v. To complete the embedding, we add an edge vw to the intermidiate embedding in all the possible manner. Even when the appendages are taken to vertex $\{v\}$, we do similarly. It is clear that all embeddings are obtained with this method. Finally, we select the non-isomorphic embedding by using Theorem 3. Therefore, we can obtained all non-isomorphic embedding of the simple, connected, non-trivial, planar graph G. Our Details of Step 1 and Step 2 only are expressing the technical skill for the implementation of this algorithm. □

4 Computations In [3], the non-isomorphic embeddings of the simple, 2-connected, planar graphs with order p=7 or less are given and the numbers of the non-isomorphic, simple, connected, planar graphs with order p=8 or less are listed. We will expand this list.

We can obtain the next theorem with a personal computer by using this algorithm. The program that we made for this paper consists of about 7000 lines with $C++$.

Theorem 6. *We obtain the result like the next table about the numbers of the non-isomorphic embedding of the simple, 2-connected, planar graphs and those of the simple, connected, planar graphs.*

The numbers of the non-isomorphic embeddings of the simple, 2-connected, planar graphs

order	8	9	10
size = 8	1		
9	6	1	
10	59	7	1
11	328	104	9
12	1146	915	181
13	2114	5046	2239
14	2144	16009	17876
15	1246	30183	85550
16	447	33719	254831
17	88	23749	478913
18	14	10585	581324
19		3017	468388
20		489	255156
21		50	93028
22			22077
23			3071
24			233

Remark Our result of order 7 or less agrees with the one that is listed in [3].

The numbers of the non-isomorphic embeddings of the simple, connected, planar graphs

order	3	4	5	6	7	8	9	10
size = 2	1							
3	1	2						
4		2	3					
5		1	7	6				
6		1	7	22	12			
7			5	42	76	27		
8			2	49	237	271	65	
9			1	35	442	1293	1000	175
10				18	510	3539	6743	3752
11				5	412	6205	25811	34035
12				2	218	7482	63233	173058
13					84	6318	106974	562486
14					18	3833	129780	1264809
15					5	1623	115988	2064232
16						485	76582	2520468
17						88	37421	2340393
18						14	13111	1665251
19							3228	904432
20							489	370667
21							50	111177
22								23376
23								3071
24								233

We give even the result of the planar graphs for the reference, because our program generates even all non-isomorphic planar graphs of order ten or less.

number of non-isomorphic, simple, 2-connected, planar graph

order	8	9	10
size = 8	1		
9	6	1	
10	40	7	1
11	158	70	9
12	406	426	121
13	662	1645	1018
14	737	4176	5617
15	538	7307	20515
16	259	8871	52068
17	72	7541	94166
18	14	4353	123357
19		1671	116879
20		378	79593
21		50	37859
22			12066
23			2306
24			233

number of non-isomorphic, simple, connected, planar graph

order	8	9	10
size = 7	23		
8	89	47	
9	236	240	106
10	486	797	657
11	804	2075	2678
12	1112	4454	8548
13	1211	8053	22768
14	1026	11990	51816
15	626	14379	99212
16	275	13380	156780
17	72	9464	199758
18	14	4844	201912
19		1734	158312
20		378	94321
21		50	41004
22			12394
23			2306
24			233

Remark We calculate also these values by the algorithm given in [2] and get the same results. Our result of order 7 or less agrees with the one that is listed in [3].

References

[1] Gary Chartrand and Ortrud R. Oellermann, Applied and Algorithmic Graph Theory, McGraw-Hill, New York, 1993

[2] Jonathan Gross and Jay Yellen, Graph Theory and Its Applications, CRC Press, Boca Raton, 1999

[3] Ronald C. Read and Robin J. Wilson, An Atlas of Graphs, Clarendon press, Oxford, 1998

[4] Steven S. Skiena, Implementing Discrete Mathematics, Addison-Wesley, 1990
Japanese translation: Addison-Wesley Toppan, Tokyo, 1992.

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