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ON THE CROSSING NUMBER OF THE SIMPLE CONNECTED GRAPHS

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Abstract. In [3] we give an algorithm for getting all non-isomorphic embeddings of the simple , connected, planar graphs. In this paper, we give an algorithm for getting the crossing number of the simple, connected graphs by using this algorithm. And we compute the number of the simple, connected graphs with order 10 or less that have crossing number 1 and the numbers of the simple, connected graphs with order 9 or less that have crossing number 2 and 3, respectively.

1 Introduction We can determine the crossing number $\nu(K_6)$ of the complete graph K_6 in the following manner. Since K_6 is non-planar, $\nu(K_6)$ is positive. An algorithm for planarity testing is given in [1] and another algorithm is given in [5]. Since $K_6 - (0,1)$, which is the only non-isomorphic subgraph with size 14 of K_6 , is non-planar, $\nu(K_6)$ is greater than 1. Since $K_6 - \{(0,1),(0,2)\}$ and $K_6 - \{(0,1),(2,3)\}$, which are all non-isomorphic subgraphs with size 13 of K_6, are non-planar, $\nu(K_6)$ is greater than 2. $K_6 - \{(0,1),(0,2),(1,3)\}$ is planar and it has unique embedding like the next figure. An algorithm for getting all non-isomorphic embeddings of simple , connected, planar graphs is given in [3].

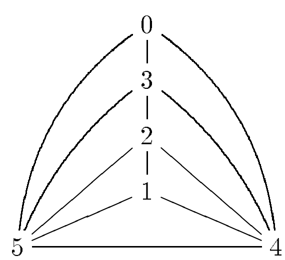

Figure 1

By adding edges (0,1), (0,2) and (1,3) to Figure 1, we have a drawing of K_6 like the next figure. Then we have $\nu(K_6) = 3$.

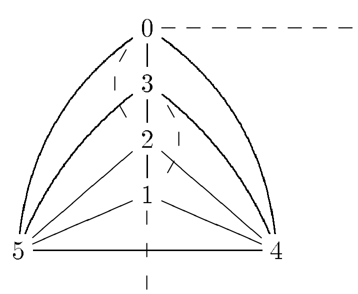

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Figure 2

Since $K_6 - \{(0,1),(0,2),(1,3)\}$ is 3-connected, it has only one non-isomorphic embedding. Next, let $G = (V, E)$, where $V = \{0,1,2,3,4,5,6,7\}$ and $E = \{(0,1),(0,2),(0,3),(0,6),(0,7),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3),(2,4),(2,5),(2,6),(2,7)\}$. G is non-planar and $G - (0,3)$ is the planar, 2-connected graph and has the following two non-isomorphic embeddings.

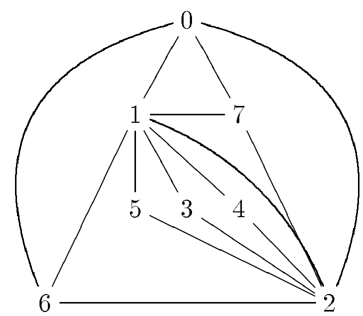

Figure 3

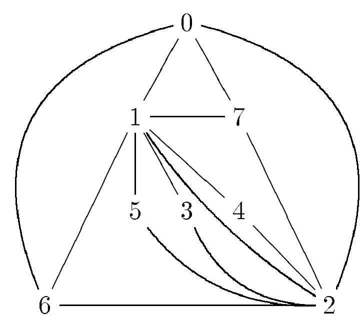

Figure 4

Since $G - (0,3)$ has an automorphism $(0)(1)(2)(3,4)(5)(6)(7)$, we can exchange 3 and 4 in Figure 4 and can draw (0,3) with one crossing. Then we have $\nu(G) = 1$.

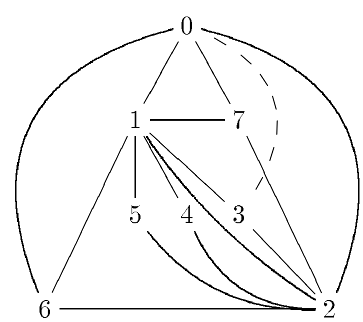

Figure 5

When the faces are adjoining with 2 or more sides, whether two edges cross or not depends on our choosing of the sides.

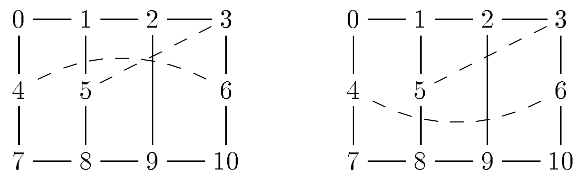

Figure 6

Furthermore, when the crossing number is calculated, the route of each edge must designate whether it passes what side of face in advance.

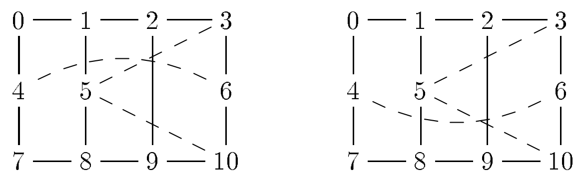

Figure 7

We can show such a route of edge with the following list.

$T, 47$ → $F, 51$ → $F, 92$ → $T, 63$

Here T shows that it is instructing the end vertex and F shows that it is instructing the intermediate edge. We designate the end vertex with the starting side of the edge when we revolve the face counterclockwise. This list shows that the route of the edge start at vertex 4 and pass the side (5,1) and (9,2) and end at vertex 6. These lists will be called E-pathes (pathes with designated edges).

Let ι be an imbedding of a simple, connected graph G and e be an edge which is not contained in G. When we draw e in ι, the length of e is the number of crossings of e and the edges of G and we call the path with minimal length shortest path.

Although we were drawing the edges in the shortest path in the examples, until now, we consider the following example.

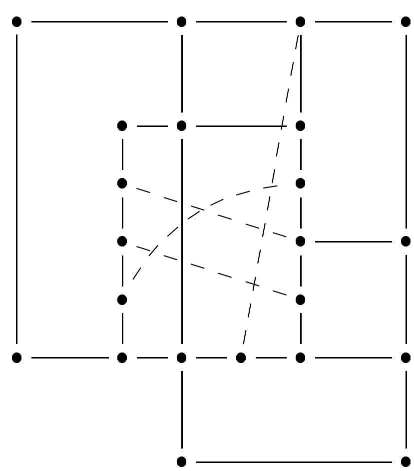

Figure 8

In Figure 8 the number of crossing is 9. However, we can draw it in the following manner.

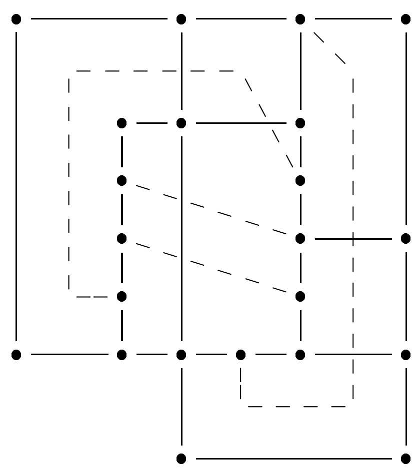

Figure 9

In Figure 9 the number of crossing is 6. ! ! Therefore, to consider only the shortest path is insufficient to obtain the crossing number. The next theorem is one answer to this problem.

Theorem 1. *Let ι be an embedding of a simple, connected graph G and $e_1, e_2, \cdots, e_n$ be edges which are not contained in G. When each edge e_i is taken in the shortest path in ι, let l_i be the length of the shortest path for e_i and k_i be the number of crossings with the pathes of other edges. We assume that the shortest path for e_m for some m must be replaced more longer path in order to get the crossing number. Then the length of the path for e_i is less than $l_i + (k_1 + k_2 + \cdots + k_n)/2$.*

Proof. Let m_i be the length of the path for e_i and g_i be the number of crossings with the path of other edges in the drawing which give the crossing number. By the assumption, we have

$$\sum_{i=1}^{n} m_i + (\sum_{i=1}^{n} g_i)/2 \ < \ \sum_{i=1}^{n} l_i + (\sum_{i=1}^{n} k_i)/2$$

Then we have

$$\sum_{i=1}^{n}(m_i - l_i) + (\sum_{i=1}^{n} g_i)/2 \ < \ (\sum_{i=1}^{n} k_i)/2$$

Since m_i is greater than or equal to l_i for each i and $(\sum_{i=1}^{n} g_i)/2$ is non-negative, we have

$$m_i \ < \ l_i + (\sum_{i=1}^{n} k_i)/2 \ for \ each \ i$$

□

By these studies we can give an algorithm that give the crossing number of the simple, connected graph.

2 Algorithm

Algorithm 1.
input *An embedding* ι *of a connected, planar graph* G *with* $\nu(G+e) \geq n$*, a set of edges* e*, and an integer* n
output *If the minimum number of crossings is* n*, when the edges* e *are added to the embedding* ι*, then return* n *else return* $n+1$.

1. *Let* $e = \{e_1, e_2, \cdots, e_m\}$
2. *Let* $e_k = (u_k, v_k)$ *for each* k
3. *Choose one path of the shortest distance from a face including* u_k *to a face including* v_k *in the embedding* ι *for each* e_k
4. *To the combination of the pathes that are chosen in Step 3*
   (a) *Choose one E-path for each path that is chosen in Step 3*
   (b) *To the combination of the E-pathes that are chosen in Step (a),*
   let nu be the total number of the crossings and extra be the number of the crossings among the E-pathes.
5. *if* $nu = n$ *then return* n
6. *if* $extra > 1$ *then*

   Get all path with the length of the shortest distance + extra - 1 or less from a face including u_k *to a face including* v_k *in the embedding* ι *for each* e_k

   else

   Get all path of the shortest distance from a face including u_k *to a face including* v_k *in the embedding* ι *for each* e_k
7. *Get all combinations of the pathes, that are given in Step 6, whose total distance is less than or equal to* $n+m$.
8. *For each combination, repeat the following:*
   (a) *Let* $\nu_0 = 0$
   (b) *for i=1 to m*
   $\nu_0 = \nu_0$ *+ length of the i-th path - 1*
   (c) *Get all E-pathes for each path that is contained in the combination*
   (d) *Get all combination of the E-pathes*
   (e) ν_1 *= large number*
   (f) *For each combination given in Step (d), repeat the following:*
      i. *cnt = the number of crossings among E-pathes*
      ii. *if cnt* $< \nu_1$ *then* ν_1 *= cnt*
   (g) *if* $\nu_0 + \nu_1 = n$ *then return* n
9. *return n+1*
10. *end*

Algorithm 2.
input *A connected, planar graph G with $\nu(G+e) \geq n$, a set of edges e, and an integer n*
output *If $\nu(G+e)=n$ then return n else return $n+1$*
remark *If G is 3-connected then G has a unique non-isomorphic embedding.*

1. *Get all non-isomorphic embedding of G*
2. *if G is not 3-connected then*
   (a) *Get the automorphism group of G*
   (b) *For each embedding ι, repeat the following:*
      i. *For each automorphism α of G, repeat the following:*
         A. *Exchange the vertices in the embedding ι by α*
         B. *By using algorithm 1, get number c of crossings when e is added to the embedding ι*
         C. *if $c = n$ then return n*
   (c) *return n+1*
3. *if G is 3-connected then*
   (a) *For each embedding ι, repeat the following:*
      i. *By using algorithm 1, get number of crossings c when e is added to the embedding ι*
      ii. *if $c = n$ then return n*
   (b) *return n+1*
4. *end*

We need three following function.
Function CrossGminusEs(G, n)
Check the crossing number of G by removing at most n edges from G
input A connected graph G and an integer n
output If $\nu(G) \leq n$ then return $\nu(G)$ else return $n+1$

1. If n=0 then
   if G is planar then return 0 else return 1
2. Let cnt = CrossGminusEs(G, n-1)
3. If $cnt \leq n-1$ then return cnt
4. For each edge e of G, repeat the following:
   (a) Let NewG be G-e
   (b) If NewG is not connected or is not new subgraph of G then choose next edge e of G and goto Step (a)
   (c) Let cnt = CrossGminusEs(NewG, n-1)
   (d) if cnt = 0 then
      i. Check $\nu(NewG+e)$ by using algorithm 2
      ii. if $\nu(NewG+e) = n$ then return n

        iii. Choose next edge e of G and goto Step (a)
   (e) if cnt = n then
       choose next edge e of G and goto Step (a)
   (f) Let cnt = subCrossGminusEs(NewG, $\{e\}$, n)
   (g) if cnt = n then return n

5. return n+1

Function subCrossGminusEs(G, E, n)
input A connected non-planar graph G , a set of edges E and an integer n
output If $\nu(G+E) = n$ then return n else return $n+1$

1. Let len be the number of edges in E
2. For each edge e of G, repeat the following:
   (a) Let NewG be G-e
   (b) If NewG is not connected or is not new subgraph of G then
       choose next edge e of G and goto Step (a)
   (c) Let cnt = CrossGminusEs(NewG, n-len-1)
   (d) If cnt = 0 then
       i. If NewG is not maximum planar subgraph of G+E then
          choose next edge e of G and goto Step (a)
       ii. Check $\nu(NewG + E \cup \{e\})$ by using algorithm 2
       iii. if $\nu(NewG + E \cup \{e\}) = n$ then return n
   (e) If $0 < cnt \leq n - len - 1$ then
       i. cnt = subCrossGminusEs(NewG, $E \cup \{e\}$, n)
       ii. If cnt = n then return n
3. return n+1
4. end

Function crosslessP(G, n)
input A connected graph G with $\nu(G) \geq n$ and an integer n
output If $\nu(G) = n$ then return n else return $n+1$

1. If n = 0 then
   if G is planar then return 0 else return 1
2. return CrossGminusEs(G, n)
3. end

Algorithm 3. *Calculating the crossing number of the connected graph*
input *A connected graph G*
output *Crossing number $\nu(G)$*

1. *Let i = 0*

2. *if crosslessP(i) = i then return i*
3. *Let i = i+1*
4. *goto Step 2*
5. *end*

Theorem 2. *The algorithm 3 calculates the crossing number of the simple, connected graph.*

Proof. Let G be the simple, connected graph and let H be a maximal planar subgraph of G. Adding the edges of G, which are not contained in H, to H increases at least one per one edge of the crossing number It is sufficient to check $\nu(G) = n$ that we remove n or less edges from G. Therefore, if $\nu(G) = n$,$n \geq 1$, then there is the edges $e_1, e_2, \cdots, e_m$, $m \leq n$ such that $G - \{e_1, e_2, \cdots, e_m\}$ is a maximum planar subgraph of G and the number of crossings is n when we draw the edges $e_1, e_2, \cdots, e_m$ in some embedding of $G - \{e_1, e_2, \cdots, e_m\}$. In this case, we have

$$\nu(G) > \nu(G - e_1) > \nu(G - \{e_1, e_2\}) > \cdots > \nu(G - \{e_1, e_2, \cdots, e_m\}) = 0$$

and

$$\nu(G) - k \geq \nu(G - \{e_1, e_2, \cdots, e_k\}) \ for \ each \ k$$

It is sufficient to calculate the minimum number of crossings, when we draw the edges $e_1, e_2, \cdots, e_m$ in some embedding of $G - \{e_1, e_2, \cdots, e_m\}$, that we pay attention to that expressed in the introduction. Obviously Algorithm 1 and 2 have realized these considerations. Next we consider the function CrossGminusEs(G, n). If $\nu(G) = 0$ then G is planar and CrossGminusEs(G,0) return 0. We assume $\nu(G) = n$. We consider CrossGminusEs(G,n). By induction hypothesis, we have CrossGminusEs(G,$n-1$)= n in Step 2. In Step 4 CrossGminusEs(G, n) searches all edge e_1 such that $\nu(G - e_1) \leq \nu(G) - 1$. If $\nu(G - e_1) = 0$ then $G - e_1$ is a maximal planar subgraph of G. We check the number of the crossings by using algorithm 2 in Step 4.(d). If e_1 is the desired edge then we get CrossGminusEs(G, n)= n. If $\nu(G - e_1) > 0$ then we call subCrossGminusEs(G, $\{e_1\}$, n) in Step 4.(f). Next we consider the function subCrossGminusEs(G, E, n). In Step 2 subCrossGminusEs(G, $\{e_1\}$, n) searches all edge e_2 such that $\nu(G - \{e_1, e_2\}) \leq \nu(G) - 2$. If $\nu(G - \{e_1, e_2\}) = 0$ then $G - \{e_1, e_2\}$ is a maximal planar subgraph of G. We check the number of the crossings by using algorithm 2 in Step 2.(d). If e_1, e_2 is the desired edges then we get subCrossGminusEs(G, $\{e_1\}$, n)= n and CrossGminusEs(G, n)= n. If $\nu(G - \{e_1, e_2\}) > 0$ then we call subCrossGminusEs(G, $\{e_1, e_2\}$, n) in Step 2.(e). Repeating this process subCrossGminusEs(G, E, n) finds edges $e_1, e_2, \cdots, e_m$ such that

$$\nu(G) - k \geq \nu(G - \{e_1, e_2, \cdots, e_k\}) \ for \ each \ k$$

and $G - \{e_1, e_2, \cdots, e_m\}$ is a maximal planar subgraph of G. Since $\nu(G - \{e_1, e_2, \cdots, e_m\}) = 0$, we check the number of crossings by using algorithm 2 in Step 2.(d). If $e_1, e_2, \cdots, e_m$ is the desired edge sequence then we get subCrossGminusEs(G, $\{e_1, e_2, \cdots, e_{m-1}\}$, n)= n and CrossGminusEs(G, n)= n. Since we consider all edge sequences $e_1, e_2, \cdots, e_m$ such that

$$\nu(G) - k \geq \nu(G - \{e_1, e_2, \cdots, e_k\}) \ for \ each \ k$$

and $G - \{e_1, e_2, \cdots, e_m\}$ is a maximal planar subgraph of G, we finally find the desired edge sequence. Therefore, our function CrossGminusEs(G, n) return n if $\nu(G) = n$. If $\nu(G) > n$ then clearly CrossGminusEs(G, n) return $n + 1$. We are repeating this step in small order. Then we can obtain the crossing number. □

3 Some Computations We can obtain the next theorems with a personal computer by using above algorithm. Our program is written by C++ and has about 10000 lines.

Theorem 3. *We obtain the result like the next table about the numbers of the simple, connected graphs with crossing number one and those of the simple, 2-connected graphs with crossing number one and those of the simple, 3-connected graphs with crossing number one.*

the numbers of the simple, connected graphs with crossing number one

order	5	6	7	8	9	10
size = 9		1				
10	1	1	2			
11		4	8	10		
12		3	29	57	41	
13		2	42	239	351	182
14			43	533	1842	2047
15			19	809	5740	13277
16			6	750	12188	53556
17				445	17464	149466
18				140	17056	293764
19				25	10931	411340
20					4520	408708
21					1071	287365
22					131	139682
23						45132
24						8690
25						812

the numbers of the simple, 2-connected graphs with crossing number one

order	5	6	7	8	9	10
size = 9		1				
10	1	1	1			
11		3	5	3		
12		3	18	23	7	
13		2	32	116	84	16
14			38	325	612	281
15			19	597	2581	2825
16			6	648	7031	16567
17				422	12316	63015
18				140	13992	159319
19				25	9965	272730
20					4382	316449
21					1071	249059
22					131	130671
23						44164
24						8690
25						812

the numbers of the simple, 3-connected graphs with crossing number one

order	5	6	7	8	9	10
size = 9		1				
10	1	1				
11		2	1			
12		2	5	2		
13		2	12	12		
14			18	52	10	
15			12	146	112	7
16			6	225	614	138
17				206	1841	1495
18				95	3279	8129
19				25	3447	25477
20					2178	48728
21					747	59288
22					131	46017
23						22363
24						6180
25						812

Theorem 4. *We obtain the result like the next table about the numbers of the simple, connected graphs with crossing number two and those of the simple, 2-connected graphs with crossing number two and those of the simple, 3-connected graphs with crossing number two.*

the numbers of the simple, connected graphs with crossing number two

order	6	7	8	9
size = 12		1		
13		2	4	
14	1	5	20	23
15		14	78	184
16		11	249	1052
17		5	386	4307
18			348	10357
19			143	15053
20			29	12727
21				6216
22				1603
23				195

the numbers of the simple, 2-connected graphs with crossing number two

order	6	7	8	9
size = 12		1		
13		2	2	
14	1	5	14	8
15		12	60	93
16		11	195	633
17		5	338	2885
18			330	7879
19			143	12787
20			29	11779
21				6061
22				1603
23				195

the numbers of the simple, 3-connected graphs with crossing number two

order	6	7	8	9
size = 12		1		
13		2	1	
14	1	5	7	1
15		9	32	18
16		9	104	154
17		5	197	828
18			218	2733
19			112	5413
20			29	6114
21				3875
22				1268
23				195

Theorem 5. *We obtain the result like the next table about the numbers of the simple, connected graphs with crossing number three and those of the simple, 2-connected graphs with crossing number three and those of the simple, 3-connected graphs with crossing number three.*

the numbers of the simple, connected graphs with crossing number three

order	6	7	8	9
size = 15	1	2	2	
16		4	15	11
17		5	65	162
18		4	145	1089
19			193	4108
20			104	9008
21			22	10293
22				5966
23				1604
24				184

the numbers of the simple, 2-connected graphs with crossing number three

order	6	7	8	9
size = 15	1	2	2	
16		3	11	7
17		5	51	100
18		4	127	776
19			181	3281
20			104	7855
21			22	9638
22				5851
23				1604
24				184

the numbers of the simple, 3-connected graphs with crossing number three

order	6	7	8	9
size = 15	1	2	2	
16		2	7	4
17		4	32	45
18		4	88	361
19			135	1694
20			87	4526
21			22	6258
22				4331
23				1375
24				184

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