

FUZZY MULTIPLY POSITIVE IMPLICATIVE IDEAL

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ABSTRACT. In this paper, we introduce the concept of fuzzy multiply positive implicative ideal of BCK-algebra and discuss some of its properties and relations among them.

1. Introduction

In 1965, L. A. Zadeh put forward the concept of fuzzy set, and in 1977, A. Rosenfeld applied it to the fundamental theory of groups. After that, many mathematical researchers further applied this concept to the other mathematical regions (cf. [3], [4], [5]). In this paper, we introduce the concept of fuzzy multiply positive implicative ideal of BCK-algebra and discuss its properties.

Definition 1([2]). Let X be a BCK-algebra and n a positive integer, then X is said to be n -positive implicative, if $x * y^{n+1} = x * y^n$ for any $x, y \in X$.

$$\text{where } x * y^n = (\cdots \overbrace{((x * y) * y * \cdots)}^n * y, x * y^0 = x$$

Definition 2([3]). A BCK-algebra X is said to be a multiply positive implicative, for any $x, y \in X$, there exists a positive integer $n = n(x, y)$, such that $x * y^{n+1} = x * y^n$

Definition 3([3]). A non-vacuous subset A of BCK-algebra X is said to be a multiply positive implicative ideal of X if it satisfies

(i) $0 \in A$;

(ii) If $(x * y) * z^n \in A, y * z^n \in A$, for any $x, y, z \in X$, there exists a positive integer $k = k(x, y, z)$, such that $x * z^k \in A$.

It is clear that every positive implicative BCK-algebra is a 1-positive implicative and every n -positive implicative BCK-algebra is multiply positive implicative. However, the reverse propositions are not true.

Example 1. Let $X = \{0, 1, 2, \cdots, n\}$ $n \geq 3$.

$$x * y = \begin{cases} 0 & x \leq y \\ x - y & x > y \end{cases}$$

We can easily check that $(X; *, 0)$ is a BCK-algebra.

Moreover, if $y = 0$, then $x * y^n = x * y^{n+1}$ and if $y \neq 0, x * y^n = 0 = x * y^{n+1}$, so $x * y^n = x * y^{n+1}$, hence X is a n -positive implicative. But X isn't a positive implicative BCK-algebra, because $3 * 1 \neq (3 * 1) * 1$

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Example 2. Let

$$X = N \cup \{0\}, x * y = \begin{cases} 0 & x \leq y \\ x - y & x > y \end{cases}$$

We can easily check that $(X; *, 0)$ is a BCK-algebra. If $y = 0$, then $x * y^n = x = x * y^{n+1}$ and if $y \neq 0$, let $n = \lceil \frac{x}{y} \rceil + 1$, then $x * y^n = 0 = x * y^{n+1}$, hence X is multiply positive implicative, but X isn't a n -positive implicative. In fact, for any number n , $(n+1) * 1^n = 1$, $(n+1) * 1^{n+1} = 0$, i.e., $(n+1) * 1^n \neq (n+1) * 1^{n+1}$.

Example 3. Let $X_i = N \cup \{0\}, i = 1, 2, \dots$

Let $Y = \prod_{i=1}^{\infty} X_i = X_1 \times X_2 \times \dots$. For $x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in Y$, define $x * y = \{z_i\}_{i=1}^{\infty}$, where

$$z_i = \begin{cases} 0 & x_i \leq y_i \\ x_i - y_i & x_i > y_i \end{cases} (i = 1, 2, \dots)$$

We can easily check that $(Y; *, 0)$ is a BCK-algebra, but Y isn't multiply positive implicative. In fact for any number n . Let $x = (1, 2, 3, \dots), y = (1, 1, 1, \dots)$, clearly, $x, y \in Y$ and $x * y^n = (\overbrace{0, 0, \dots, 0}^n, 1, 2, \dots), x * y^{n+1} = (\overbrace{0, 0, \dots, 0}^{n+1}, 1, \dots, 1, \dots)$ then $x * y^n \neq x * y^{n+1}$ for any number n .

Xi Ougen introduced the concepts of fuzzy ideal and fuzzy implicative ideal of BCK-algebra ([4]). Moreover, we introduced the concept of fuzzy n -positive implicative ideal of BCK-algebra ([5]).

Definition 4 ([5]). Let X be a BCK-algebra, a fuzzy subset μ of X is said to be a fuzzy n -positive implicative ideal of X if:

- (i) $\mu(0) > \mu(x)$, for any $x \in X$
- (ii) $\mu(x * z^n) \geq \min\{\mu(x * y) * z^n, \mu(y * z^n)\}$

In particular, when $n = 1$, μ is a fuzzy implicative ideal of X . In this paper, we introduce the concepts of fuzzy multiply positive implicative ideal of BCK-algebra.

Definition 5. Let X be a BCK-algebra, a fuzzy subcept μ of X is said to be a fuzzy multiply positive implicative ideal of X if:

- (i) $\mu(0) \geq \mu(x)$, for any $x \in X$;
- (ii) For any $n, m \in N$, there exists a positive integer $k = k(x, y, z)$, such that $\mu(x * z^k) \geq \min\{\mu((x * y) * z^n), \mu(y * z^m)\}$ for any $x, y, z \in X$.

2. Main theorem.

Theorem 1. A fuzzy subset μ of BCK-algebra X is a fuzzy multiply implicative ideal of X if and only if, for every $\lambda \in [0, 1], \mu_\lambda = \{x | x \in X, \mu(x) \geq \lambda\}$ is a multiply implicative ideal of X , when $\mu_\lambda \neq \varphi$.

Proof. Suppose μ is a fuzzy multiply implicative ideal of X . According to definition 5, we have $\mu(0) \geq \mu(x)$ for any $x \in X$, therefore, $\mu(0) \geq \mu(x) \geq \lambda$. For $x \in \mu_\lambda$, so $0 \in \mu_\lambda$.

Moreover, suppose $(x * y) * z^n \in \mu_\lambda, y * z^m \in \mu_\lambda$, then $\mu((x * y) * z^n) \geq \lambda, \mu(y * z^m) \geq \lambda$. By definition 5 there exists $k \in N$, such that

$$\mu(x * z^k) \geq \min\{\mu((x * y) * z^n), \mu(y * z^m)\} \geq \lambda.$$

so $x * z^k \in \mu_\lambda$. Hence μ_λ is a multiply positive implicative ideal of X .

Conversely, we only need to prove (i) and (ii) of definition 5 are true. If (i) isn't true, then there exists $x_0 \in X$, such that $\mu(0) \leq \mu(x_0)$. Let $\lambda_0 = (\mu(x_0) + \mu(0))/2$, then $\mu(0) < \lambda_0$, and $0 \leq \lambda_0 < \mu(x_0) \leq 1$. So $x_0 \in \mu_{\lambda_0}$, and $\mu_{\lambda_0} \neq \phi$. But μ_{λ_0} is a multiply positive implicative of X , so $0 \in \mu_{\lambda_0}$ and $\mu(0) \geq \lambda_0$, it's contradictory.

Moreover if (ii) isn't true, then there exists $x_0, y_0, z_0 \in X$ such that

$\mu(x_0 * z_0^k) < \min\{\mu((x_0 * y_0) * z_0^n), \mu(y_0 * z_0^m)\}$. Let $\lambda_0 = \frac{1}{2}(\mu(x_0 * z_0^k) + \min\{\mu((x_0 * y_0) * z_0^n), \mu(y_0 * z_0^m)\})$ then $\lambda_0 > \mu(x_0 * z_0^k)$ and $0 \leq \lambda_0 < \min\{\mu((x_0 * y_0) * z_0^n), \mu(y_0 * z_0^m)\} \leq 1$, so we have $\mu((x_0 * y_0) * z_0^n) \geq \lambda_0$ and $\mu(y_0 * z_0^m) \geq \lambda_0$ then $(x_0 * y_0) * z_0^n \in \mu_{\lambda_0}$ and $y_0 * z_0^m \in \mu_{\lambda_0}$. Therefore $\mu_{\lambda_0} \neq \emptyset$. As μ_{λ_0} is a multiply positive implicative ideal, it implies $x_0 * z_0^k \in \mu_{\lambda_0}$, so $\mu(x_0 * z_0^k) \geq \lambda_0$, it's contradictory. Therefore, μ is a fuzzy multiply positive implicative ideal.

Example 4. Let $X = \{0, a, b, c\}$ in which $*$ is defined by

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then X is a BCK-algebra. Let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 > t_1 > t_2$. Define $\mu : X \rightarrow [0, 1]$ by $\mu(0) = t_0, \mu(a) = \mu(b) = t_1$, and $\mu(c) = t_2$.

Moreover, $\mu_{t_0} = \{0\}, \mu_{t_1} = \{0, a, b\}, \mu_{t_2} = X$. Routine calculations give that μ is a fuzzy multiply positive implicative ideal of X if and only if, for every $\lambda \in [0, 1], \mu_\lambda = \{x | x \in X, \mu(x) \geq \lambda\}$ is a multiply positive implicative ideal.

Definition 6 ([7]). If $(X, *, 0), (X', *', 0')$ are BCK-algebras, then a map f of X onto X' is called a homomorphism if $f(x * y) = f(x) *' f(y), \forall x, y \in X$.

Definition 7 ([6]). If μ is a fuzzy subset of X , and f is a map of X onto X' , then the fuzzy subset ν in X' defined by

$$\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x), \quad \text{for } \forall y \in X'$$

is called the image of μ under f . Similarly, if ν is a fuzzy subset in X' , then the fuzzy subset $\mu = f \cdot \nu$ in X is called the preimage of ν under f .

Theorem 2. Let f be a homomorphism mapping from BCK-algebra $(X, *, 0)$ onto BCK-algebra $(X', *', 0')$, ν be a fuzzy multiply positive implicative ideal of X' . Then the homomorphism preimage μ of ν under f , is a multiply positive implicative ideal of X' .

Proof. By the definition 7, for any $x \in \nu$, we have $\nu(f(x)) = \mu(x)$, since $f(x) \in X'$ and ν is a fuzzy multiply positive implicative ideal of X' , then

$$\nu(0') \geq \nu(f(x)) = \mu(x), \text{ but}$$

$$\nu(0') = \nu(f(0)) = \mu(0), \text{ thus } \mu(0) \geq \mu(x), \text{ for any } x \in X$$

Furthermore, for any $n, m \in N$, there exists $k \in N$, such that

$$\begin{aligned} \mu(x * z^k) &= \nu(f(x * z^k)) = \nu(f(x) *' f^k(z)) \\ &\geq \min\{\nu((f(x) *' y') *' f^n(z)), \nu(f(x) *' f^m(z))\} \quad \text{for any } y' \in X'. \end{aligned}$$

Let y be an arbitrary preimage of y' under f then

$$\begin{aligned}\mu(x * z^k) &\geq \min(\nu(f(x) *' f(y)) *' f^n(z), \nu(f(y) *' f^n(z))) \\ &= \min(\nu((f(x * y) * z^n), \nu(f(y * z^n))) \\ &= \min(\mu(x * y) * z^n, \mu(y * z^n))\end{aligned}$$

Since y' is an arbitrary element of x' , the above result is true for any $y \in X$, i.e.,

$$\mu(x * z^k) \geq \min(\mu((x * y) * z^n), \mu(y * z^n)) \quad \text{for any } x, y, z \in X.$$

Hence μ is a fuzzy multiply positive implicative ideal of X .

Definition 8 ([6]). A fuzzy subset of μ in X has sup property if, for any subset $T \subseteq X$ there exists $x_0 \in T$ such that

$$\mu(x_0) = \sup_{t \in T} \mu(t).$$

Theorem 3. Suppose f is the same map as Theorem 2, μ is a fuzzy multiply positive implicative ideal of X with sup property, then the homomorphic image ν of μ under f is fuzzy multiply positive implicative ideal of X' .

Proof. Since μ is a fuzzy multiply positive implicative ideal of X , then for any $x \in X$, we have $\mu(0) \geq \mu(x)$. Moreover, the image of zero element 0 of X under f is the zero element $0'$ of X' , so $0 \in f^{-1}(0')$.

Thus $\nu(0') = \sup_{t \in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x)$ for any $x \in X$.

Furthermore, we have $\nu(0') \geq \sup_{t \in f^{-1}(0')} \mu(t)$ for any $x' \in X'$.

Thus $\nu(0') = \sup_{t \in f^{-1}(x')} \mu(t) = \nu(x')$ for any $x' \in X'$.

Moreover, for any $x', y', z' \in X'$ since $X' = f(X)$ then there exist $x, y, z \in X$ such that $x' = f(x), y' = f(y), z' = f(z)$.

For any $n, m \in N$, there exists $k \in N$ such that $\mu(x_0 * z^k) = \sup_{t \in f^{-1}(f(x * z^k))} \mu(t)$ for $x_0 * z^k \in f^{-1}(f(x * z^k))$ then

$$\begin{aligned}\nu(x' * z'^k) &= \nu(f(x) *' f^k(z)) = \nu(f(x * z^k)) \\ &= \sup_{t \in f^{-1}(f(x * z^k))} \mu(t) = \mu(x_0 * z^k) \\ &\geq \min(\mu(x_0 * y) * z^n, \mu(y * z^m)) \\ &= \min(\nu(f(x_0 * y) * z^n), \nu(f(y * z^m))) \\ &= \min(\nu((f(x_0) *' f^n(y)) *' f^n(z)), \nu(f(y) *' f^m(z))) \\ &= \min(\nu((f(x_0) *' f^n(z)) *' f'(y)), \nu(f(y) *' f^m(z))) \\ &= \min(\nu((f(x_0 * z^n) *' f(y)), \nu(f(y) *' f^m(z))) \\ &= \min(\nu(f(x * z^n) *' f(y)), \nu(f(y) *' f^m(z))) \\ &= \min(\nu((f(x) *' f^n(z)) *' f(y)), \nu(f(y) *' f^m(z))) \\ &= \min(\nu((f(x) *' f(y)) *' f^n(z)), \nu(f(y) *' f^m(z))) \\ &= \min(\nu((x' *' y') * z'^m), \nu(y' * z'^m))\end{aligned}$$

therefore, ν is a fuzzy multiply positive implicative ideal of X' .

For BCK-algebra, a multiply positive implicative ideal must be an ideal but an ideal may not be a multiply positive implicative ideal. However, when BCK-algebra is positive implicative, an ideal must be multiply positive implicative ideal. For fuzzy BCK-algebra, we get a similar conclusion.

Lemma 1 ([3]). *In a multiply positive implicative BCK-algebra X , the following conditions are equivalent.*

- (i) A is an ideal of X ;
- (ii) A is a multiply positive implicative ideal of X .

Lemma 2 ([4]). *Let X be a BCK-algebra, then is a fuzzy ideal of X if and only if, for every $\lambda \in [0, 1]$, $\mu_\lambda = \{x|x \in X, \mu(x) \geq \lambda\}$ is an ideal of X , when $\mu_\lambda \neq \phi$.*

Theorem 4. *In a multiply positive implicative BCK-algebra X . The following conditions are equivalent:*

- (i) μ is a fuzzy ideal of X ;
- (ii) μ is a fuzzy multiply positive implicative ideal of X .

Proof. (i) \Rightarrow (ii) suppose μ is a fuzzy ideal of X , by Lemma 2, for every $\lambda \in [0, 1]$, $\mu_\lambda = \{x|\mu(x) \geq \lambda\}$ is an ideal of X . Moreover, by Lemma 1, μ_λ is a multiply positive implicative ideal of X . Furthermore, by Theorem 1, μ is a fuzzy multiply positive implicative ideal of X .

(ii) \Rightarrow (i) Let $z = 0$ using $\mu(x * z^k) \geq \min(\mu((x * y) * z^n), \mu(y * z^m))$, we have $\mu(x) \geq \min(\mu(x * y), \mu(y))$, hence μ is a fuzzy ideal of X .

Corollary ([5]). *In a n-positive implicative BCK-algebra X , fuzzy n-positive implicative ideal and fuzzy ideal are equivalent.*

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