ON THE EQUIVALENT CONDITION OF THE INVOLUTORY BCK-ALGEBRAS

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Abstract. We give a equivalent condition of the involutory BCK-algebras and use this condition to provide a negative answer to the open problem posed by Aslam and Thaheem in [1].

1. Introduction

In 1991, M. Aslam and A. B. Thaheem [1] introduced the concepts of annihilators and involutory ideals in commutative BCK-algebras, and studied their properties. They proved that (i) a commutative BCK-algebra satisfying D.C.C. is an involutory BCK-algebra, (ii) an implicative BCK-algebra is an involutory BCK-algebra, (iii) a finite commutative BCK-algebra is an involutory BCK-algebra. But they did not give an equivalent condition of the involutory BCK-algebras. In [1], they posed an open problem: Whether or not all ideals are involutory ideals in every commutative BCK-algebra. In other words, they asked that whether or not every commutative BCK-algebra is involutory. In this paper, we give an equivalent condition of the involutory BCK-algebras and use this equivalent condition to provide a negative answer to Aslam and Thaheem's open problem.

2. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is said to be a BCK-algebra if it satisfies: for all $x,y,z \in X$,

(I) $((x*y)*(x*z))*(z*y)=0$,

(II) $(x*(x*y))*y=0$,

(III) $x*x=0$,

(IV) $0*x=0$,

(V) $x*y=0$ and $y*x=0$ imply $x=y$.

for all $x,y \in X$ (see [15]). We can define a partial order "$\leq$" on X by $x \leq y$ if and only if $x*y=0$.

A BCK-algbera X has the following properties:

(1) $x*0=x$.

(2) $(x*y)*z=(x*z)*y$.

(3) $x \leq y$ implies that $x*z \leq y*z$ and $z*y \leq z*x$.

(4) $(x*z)*(y*z) \leq x*y$.

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(5) $x*(x*(x*y)) = x*y$.
(6) $0*(x*y) = (0*x)*(0*y)$.
(7) $x*0 = 0$ implies $x = 0$.

If $x \wedge y = y \wedge x$ where $x \wedge y = y*(y*x)$ for all x, y in a BCK-algebra X, we say that X is a commutative BCK-algebra.

A non-empty subset I of a BCK-algebra X is called an ideal of X if $0 \in I$, and $x*y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$. Let A be a subset of a BCK-algebra X. The set of all $x \in X$ satisfying

$$(\cdots((x*a_1)*a_2)*\cdots)*a_n = 0$$

for some $a_1, a_2, \cdots, a_n \in A$ is the minimal ideal of X containing A, which is called the ideal of X *generated* by A, and is denoted by $\langle A\rangle$. If $A = \{a\}$ then we denote $\langle\{a\}\rangle$ by $\langle a\rangle$.

For the convenience of notation, we denote

$$(\ldots((x*y_1)*y_2)*\ldots)*y_n = x*\prod_{i=1}^{n} y_i.$$

In case $y_1 = y_2 = \ldots = y_n = y$, we shall denote this by $x*y^n$. Obviously we have $\ldots \leq x*y^n \leq x*y^{n-1} \leq \ldots \leq x*y \leq x$. By Hoo [2], X is said to satisfy Descending Chain Condition, denoted by (D.C.C.), if any sequence of type $\{x*a^n\}$ $(x, a \in X)$ terminates in the sense that $x*a^{n+1} = x*a^n$ for some positive integer n. An implicative $BCK-$algebra satisfies D.C.C. and any finite commutative $BCK-$algebra dose, too [1].

Definition 2.1([1]). Let X be a commutative BCK-algebra and A a subset of X. We define the set

$$A^* = \{x \in X \mid x \wedge a = 0, \forall a \in A\}$$

as the annihilator of A.

We write A^{**} in place of $(A^*)^*$. Note that A^* is nonempty since $0 \in A^*$. Obviously we have $X^* = \{0\}$ and $\{0\}^* = X$. If A is an ideal it is easy to see that $A \cap A^* = \{0\}$. We observe that if $x \in A^*$ then $x \wedge a = 0$ for all $a \in A$. It follows that $a*(a*x) = 0$ and hence $a \leq a*x \leq a$, which implies that $a = a*x$. Thus $x \in A^*$ if and only if $a = a*x$ for all $a \in A$. Moreover if X is commutative, then $x \in A^*$ if and only if $x = x*a$ for all $a \in A$.

Lemma 2.2([1]). *If A is a subset of a commutative BCK-algebra X, A^* is an ideal of X.*

Definition 2.3([1]). An ideal A of a commutative BCK-algebra X is said to be *involutory* if $A = A^{**}$. Moreover a commutative BCK-algebra X is said to be *involutory* if every ideal of X is involutory.

Clearly $\{0\}$ and X are involutory ideals.

Lemma 2.4([1]).

(i) *Let X be a commutative BCK-algebra satisfying D.C.C. Then every ideal of X is involutory, that is, X is an involutory BCK-algebra.*

(ii) *Any implicative BCK-algebra is an involutory BCK-algebra.*

(iii) *Any finite commutative BCK-algebra is an involutory BCK-algebra.*

Lemma 2.5([1]). *Let X be a commutative BCK-algebra and A, B be subsets of X. Then $A^* = A^{**}$ and $A \subseteq B$ implies $B^* \subseteq A^*$.*

Lemma 2.6([1]). *In an involutory BCK-algebra X, we have $(A \cap B)^* = \langle A^* \cup B^*\rangle$ for any ideals A and B of X.*

Lemma 2.7([1]). *Let X be an involutory BCK-algebra. Then for any subset A of X, $\langle A\rangle = A^{**}$.*

3. An equivalent condition of the involutory BCK-algebras

In this sction, we shall give an equivalent condition of the involutory BCK-algebras. For this we need following propositions.

Proposition 3.1. *Let X be an involutory BCK-algebra. Then $X = \langle A \cup A^*\rangle$ for any ideal A of X.*

Proof. Note that $A \cap A^* = \{0\}$. By Lemma 2.6 and note that X is involutory, we have

$$\langle A \cup A^*\rangle = \langle A^{**} \cup A^*\rangle = (A^* \cap A)^* = (0)^* = X$$

.

Proposition 3.2. *Let X be an involutory BCK-algebra. Then $X = \langle r \cup r^*\rangle$ for any $r \in X$, where r^* means $\{r\}^*$.*

Proof. By Lemma 2.7, $r^{**} = \langle r\rangle$. It follows from Lemma 2.5 that $\langle r\rangle^* = r^{***} = r^*$. By Proposition 3.1, we have

$$X = \langle\langle r\rangle \cup \langle r\rangle^*\rangle = \langle\langle r\rangle \cup r^*\rangle.$$

Therefore for any $x \in X$, there exist $a_1, a_2, ..., a_n \in \langle r\rangle$ and $b_1, b_2, ..., b_m \in r^*$ such that

$$(x * \prod_{i=1}^{n} a_i) * \prod_{j=1}^{m} b_j = 0.$$

In other word, $(x * \prod_{j=1}^{m} b_j) * \prod_{i=1}^{n} a_i = 0$. Note that $a_i \in \langle r\rangle$ and $\langle r\rangle$is an ideal of X, we have $x * \prod_{j=1}^{m} b_j \in \langle r\rangle$. This shows that there exists $l \in N$ such that $x * \prod_{j=1}^{m} b_j * r^l = 0$ and so $x \in \langle r \cup r^*\rangle$. Thus $X = \langle\langle r\rangle \cup r^*\rangle \subseteq \langle r \cup r^*\rangle$ and so $X = \langle\langle r\rangle \cup r^*\rangle = \langle r \cup r^*\rangle$, ending proof.

Theorem 3.3. *If X is an involutory BCK-algebra, then X satisfies D.C.C.*

Proof. Let X be an involutory BCK-algebra. Then every ideal of X is an involutory ideal. If X dosen't satisfies D.C.C., then there exist $x, r \in X$ such that $0 < ... < x*r^n < x*r^{n-1} < ... < x * r < x$ where $x * r^n \neq x * r^{n-1}$ for any $n \in N$. Now we claim that $x * r^n \notin r^*$ for any $n \in N$. Indeed, if $x * r^n \in r^*$, then $x * r^n * r = x * r^n$, or $x * r^{n+1} = x * r^n$, a contradiction. In other hand, $x \in X = \langle r \cup r^*\rangle$ by Proposition 3.2 and so there exists $m \in N$ and $a_1, a_2, ..., a_n \in r^*$ such that $(x * r^m) * \prod_{i=1}^{n} a_i = 0$. By Lemma 2.2, r^* is an ideal of X. Thus $(x * r^m) * \prod_{i=1}^{n} a_i = 0$ implies $x * r^m \in r^*$, this contradicts to the above claim. Therefore X must satisfy D.C.C.

Combining the Lemma 2.4 and Theorem 3.3 we get the following equivalent condition of an involutory BCK-algebra.

Theorem 3.4. *Let X be a commutative BCK-algebra. Then X is involutory if and only if X satisfies D.C.C.*

4. Apllication of the equivalent condition

In this section, we use the above equivalent condition to show that there exists a commutative BCK-algebra which is not involutory. Thus we give a negative answer to the open problem in [1].

Suppose $N = \{0, 1, 2, ...\}, A = \{a_n | n \in N\}$ and $X = N \cup A$. Define the operation $*$ as follows:

$$n * m = \begin{cases} 0 & \text{if } n < m, \\ n - m & \text{if } n \geq m, \end{cases}$$

$$a_n * a_m = \begin{cases} 0 & \text{if } m < n, \\ m - n & \text{if } m \geq n, \end{cases}$$

$$n * a_m = 0, a_m * n = a_{m+n}$$

where $m, n \in N$ and $a_n, a_m \in A$. Then we have the following facts.

Proposition 4.1 ([5,§6.1,Example]). *$(X, *, 0)$ is a BCK-algebra.*

Proposition 4.2. *$(X, *, 0)$ is a commutative BCK-algebra.*

Proof. We consider the following three cases.

(i) $x = a_n, y = a_m$.

$$\begin{aligned} x * (x * y) &= a_n * (a_n * a_m) \\ &= \begin{cases} a_n & \text{if } m < n, \\ a_n * (m - n) & \text{if } m \geq n, \end{cases} \\ &= \begin{cases} a_n & \text{if } m < n, \\ a_{n+(m-n)} & \text{if } m \geq n, \end{cases} \\ &= \begin{cases} a_n & \text{if } m < n, \\ a_m & \text{if } m \geq n, \end{cases} \end{aligned}$$

$$\begin{aligned} y * (y * x) &= a_m * (a_m * a_n) \\ &= \begin{cases} a_m * (n - m) & \text{if } m < n, \\ a_m * 0 & \text{if } m \geq n, \end{cases} \\ &= \begin{cases} a_{m+(n-m)} & \text{if } m < n, \\ a_m & \text{if } m \geq n, \end{cases} \\ &= \begin{cases} a_n & \text{if } m < n, \\ a_m & \text{if } m \geq n, \end{cases} \end{aligned}$$

Thus $x * (x * y) = y * (y * x)$ in case (i).

(ii) $x = n, y = m$.

$$\begin{aligned} x * (x * y) &= n * (n * m) \\ &= \begin{cases} n & \text{if } n < m, \\ n * (n - m) & \text{if } n \geq m, \end{cases} \\ &= \begin{cases} n & \text{if } n < m, \\ n - (n - m) & \text{if } n \geq m, \end{cases} \\ &= \begin{cases} n & \text{if } n < m, \\ m & \text{if } n \geq m, \end{cases} \end{aligned}$$

$$\begin{aligned} y*(y*x) &= m*(m*n) \\ &= \begin{cases} m*(m-n) & \text{if } n<m, \\ m*0 & \text{if } n \geq m, \end{cases} \\ &= \begin{cases} m-(m-n) & \text{if } n<m, \\ m & \text{if } n \geq m, \end{cases} \\ &= \begin{cases} n & \text{if } n<m, \\ m & \text{if } n \geq m, \end{cases} \end{aligned}$$

Hence $x*(x*y) = y*(y*x)$.

(iii) $x = a_n, y = m$.

$$\begin{aligned} x*(x*y) &= a_n*(a_n*m) \\ &= a_n*a_{n+m} \\ &= (n+m)-n = m \end{aligned}$$

$$\begin{aligned} y*(y*x) &= m*(m*a_n) \\ &= m*0 = m \end{aligned}$$

It follows that $x*(x*y) = y*(y*x)$ in case (iii).

Combining the above arguments we get that X is a commutative BCK-algebra.

Proposition 4.3. *X doesn't satisfy D.C.C.*

Proof. Consider a_0 and 1 in X. We have $a_0*1 = a_{0+1} = a_1$ and $a_0*1^2 = (a_0*1)*1 = a_1*1 = a_{1+1} = a_2$. In general, we assume $a_0*1^{n-1} = a_{n-1}$. Then $a_0*1^n = (a_o*1^{n-1})*1 = a_{n-1}*1 = a_{n-1+1} = a_n$. By the induction we get $a_0*1^n = a_n$ for all $n \in N$. Therefore the sequence of type $\{a_0*1^n\}$ doesn't terminate since $a_0*1^{n+1} \neq a_0*1^n$ for any $n \in N$. Hence X doesn't satisfy D.C.C.

By the Theorem 3.4 and Proposition 4.3 we have the following.

Proposition 4.4. *X is not an involutory BCK-algebra, that is, there exists at least one ideal of X such that it is not an involutory ideal of X.*

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