ON THE EXTENSIONS OF FK-TERNARY ALGEBRAS

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Abstract. In [3], we defined a $U(\varepsilon)$-algebra which is a generalization of Freudenthal-Kantor triple system defined by I. L. Kantor [8] and K. Yamaguti [13]. In this paper, we call it an FK-ternary algebra. We define a representation of the FK-ternary algebra. Moreover we define a cohomology space of order 3 of the FK-ternary algebra associated with the representation, and give an interpretation of this space in relation to extensions of the FK-ternary algebra.

Introduction

I. L. Kantor [8] constructed a graded Lie algebra from a ternary algebra satisfying two conditions. This ternary algebra is called a generalized Jordan triple system of the second order, which is a generalization of the Jordan triple system defined by K. Meyberg [9]. On the other hand, B. N. Allison [1] and W. Hein [6], [7] gave a notion of the $\mathfrak{J}$-ternary algebras, which is based on the results of H. Freudenthal [5] about the geometry of the exceptional Lie groups. K. Yamaguti [13] reformed the axioms of the $\mathfrak{J}$-ternary algebra, and he defined a Freudenthal triple system. Moreover, unifying the Freudenthal triple system and the generalized Jordan triple system of the second order, he defined a Freudenthal-Kantor triple system. In [3], we introduced an algebraic system which is a generalization of the Freudenthal-Kantor triple system, and called it a $U(\varepsilon)$-algebra. But, in this paper, we shall call it an FK-ternary algebra. The first purpose of this paper is to give the results on FK-ternary algebras corresponding to the results on Freudenthal-Kantor triple systems obtained by K. Yamaguti [14], [15]. In §2, we construct a graded Lie algebra of the second order using endomorphisms of the FK-ternary algebra satisfying certain conditions. These conditions were used in [14] in order to construct a graded Lie algebra of second order from the Frudenthal-Kantor triple system. In §3, we define a representation of FK-ternary algebra, and show that this representation induces that of the Lie triple system associated with the FK-ternary algebra. The main purpose of this paper is to define the cohomology space of order 3 of an FK-ternary algebra associated with the representation, and give an interpretation of the cohomolgy space in relation to extensions of the FK-ternary algebras. This carries out in §4.

Throughout this paper, it is assumed that any vector space is finite dimensional vector space over a field of characteristic different from two.

§1. Preliminaries

Let U be a vector space over a field F of characteristic different from two and let $B : U \times U \times U \longrightarrow U$ be a trilinear mapping. Then the pair (U, B) (or U) is called a *triple system* over F. We shall often write (abc) (or $[abc]$) in stead of $B(a, b, c)$. For subspaces V_i $(i = 1, 2, 3)$ of U, we denote by $(V_1V_2V_3)$ the subspace spanned by all elements of the form

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$(a_1a_2a_3)$ for $a_i \in V_i$. A subspace I of U is called an *ideal* if $(UUI)+(UIU)+(IUU) \subset I$ is valid. The whole space U and $\{0\}$ are called the trivial ideals. U is said to be *simple* if $(UUU) \neq \{0\}$ and U has no non-trivial ideal. An endomorphism D of U is called a *derivation* if $D(abc) = (Da\,b\,c) + (a\,Db\,c) + (a\,b\,Dc),\ a,b,c \in U$. We denote by $\mathfrak{D}(U)$ the set of all derivations of U. $\mathfrak{D}(U)$ is a Lie algebra under the usual Lie product. For $a, b \in U$, let us define the endomorphisms $L(a,b),\ M(a,b),\ R(a,b),\ K(a,b)$ on U by

$$L(a,b)x := (abx), \quad M(a,b)x := (axb), \quad R(a,b)x := (xab), \quad K(a,b)x := (axb) - (bxa).$$

A *Lie triple system* (or *LTS* simply) is a triple system U with a trilinear product $[abc]$ satisfying the following conditions for $a,b,c,d,e \in U$:

(LTS1) $[aab] = 0$,

(LTS2) $[abc] + [bca] + [cab] = 0$,

(LTS3) $[ab[cde]] = [[abc]de] + [c[abd]e] + [cd[abe]]$.

The condition (LTS3) shows that $L(a,b)$ is a derivation of the LTS T, which is called an *inner derivation*. We denote by $\mathfrak{D}_0(T)$ the set of all inner derivations of T. $\mathfrak{D}_0(T)$ is an ideal of $\mathfrak{D}(T)$. Let $\mathfrak{D}$ be a subalgebra of $\mathfrak{D}(T)$ including $\mathfrak{D}_0(T)$. It is known that the direct sum $\mathfrak{D} \oplus T$ as a vector space becomes a Lie algebra with respect to the product

$$[D+a, E+b] := L(a,b) + [D,E] + Db - Ea,$$

where $D, E \in \mathfrak{D},\ a, b \in T$. Especially, $\mathfrak{D}_0 \oplus T$ is called a *standard enveloping Lie algebra* of T. We defined the following triple system in [3] and called it a $U(\varepsilon)$-algebra. But we rename it in this paper.

Definition. A triple system (U,B) is called an *FK-ternary algebra* (or *FKTA* simply) if there exists an automorphism ε of (U,B) satisfying the following identities:

(U1) $[L(a,b), L(c,d)] = L(L(a,b)c,d) - L(c, L(b,\varepsilon a)d)$,

(U2) $K(K(a,b)c,d) = L(d,c)K(a,b) + K(a,b)L(c,\varepsilon d)$,

where $a,b,c,d \in U$. An FKTA (U,B) with an automorphism ε satisfying the above conditions is also denoted by (U,B,ε) (or (U,ε)).

The FKTA's $(U, \pm\mathrm{Id})$ are nothing but the *Freudenthal-Kantor triple systems* (or *FKTS* simply) $U(\varepsilon),\ \varepsilon = \pm 1$ (cf. [13]), particularly, the FKTA's (U,Id) are the generalized Jordan triple systems of second order and the FKTA's $(U,-\mathrm{Id})$ are the *Freudenthal triple systems* (or *FTS* simply) (cf. [5]).

Let (U,B) be a GJTS of the second order. A non-singular linear transformation φ is called a *weak automorphism* of (U,B) if there exists a non-singular linear transformation $\overline{\varphi}$ of U such that

$$\varphi B(a,b,c) = B(\varphi a, \overline{\varphi} b, \varphi c), \quad \overline{\varphi} B(a,b,c) = B(\overline{\varphi} a, \varphi b, \overline{\varphi} c).$$

It is clear that an automorphism of (U,B) is a weak automorphism of (U,B). We define the new triple product in U by $B_\varphi(a,b,c) := B(a,\varphi b,c)$. Then (U,B_φ) becomes an FKTA $(U,B_\varphi,\varepsilon)$ for $\varepsilon = (\overline{\varphi}\varphi)^{-1}$ and is called a *φ-modification* of (U,B) (cf. [3]). A notion of the φ-modifications was defined by H. Asano [2] for involutive automorphisms φ of (U,B). In this case, φ-modifications are also GJTS's of the second order.

Example. Let $\mathbb{H}$ be the set of all quaternion numbers and define a triple product in $\mathbb{H}$ by

$$B(x, y, z) := x\overline{y}z + z\overline{y}x - y\overline{x}z,$$

where $\overline{x}$ denotes the conjugate quaternion of x. Then it is easy to verify that the triple system $(\mathbb{H}, B)$ is a GJTS of the second order. Moreover, it is easily seen that the mapping $\varphi : x \mapsto ax$ is an automorphism of $(\mathbb{H}, B)$ for a fixed quaternion number a such that $|a| = 1$. Therefore $(\mathbb{H}, B_\varphi)$ becomes an FKTA for $\varepsilon = \varphi^{-2}$. Particulaly, if $a = \pm 1$, $(\mathbb{H}, B_\varphi)$ is a GJTS of the second order and if a is a pure quaternion number, $(\mathbb{H}, B_\varphi)$ is an FTS.

§2 Lie algebras constructed from FK-ternary algebras

K. Yamaguti constructed a graded Lie algebra of the second order from a Freudenthal-Kantor triple system U using endomorphisms of U satisfying certain conditions [14]. In this section, we also construct a graded Lie algebra of the second order from a given FK-ternary algebra (U, ε).

Let D, D^* be linear endomorphisms of an FKTA (U, ε), then the pair (D, D^*) is said to *satisfy the condition* (L) if

(L1) $[D, L(a, b)] = L(Da, b) - L(a, D^*b)$,

(L2) $[D^*, L(a, b)] = L(D^*a, b) - L(a, \varepsilon D\varepsilon^{-1}b)$

for all $a, b \in U$. (U1) implies that a pair $(L(a, b), L(b, \varepsilon a))$ satisfies the condition (L). If the trace form γ of (U, ε) is non-degenerate, $L(b, \varepsilon a)$ is the adjoint operator of $L(a, b)$ with respect to the trace form γ ([3] Lemma 2.4).

Let C be a linear endomorphism of U, then C is said to *satisfy the condition* (K) if

(K1) $K(Ca, b) = L(b, a)C + CL(a, \varepsilon b)$,

(K2) $K(a, b)C\varepsilon = L(b, C\varepsilon a) - L(a, C\varepsilon b)$

for all $a, b \in U$. By (U2) and [3] Lemma 2.1, we see that an endomorphism $K(a, b)$ satisfies the condition (K) (cf. [14]).

Lemma 2.1. Let (U, ε) be an FKTA. Let (D, D^*) be a pair of linear endomrphisms of U satisfying the condition (L) and C a linear endomorphism of U satisfying the condition (K). Then the following relations hold:

(2.1) $K(a, b)D^* + DK(a, b) = K(Da, b) + K(a, Db)$,

(2.2) $D^*K(a, b) + K(a, b)\varepsilon D\varepsilon^{-1} = K(D^*a, b) + K(a, D^*b)$,

(2.3) $CK(a, b)\varepsilon = L(Ca, b) - L(Cb, a)$

for all $a, b \in U$.

Proof. Let $c \in U$. By (L1), we have

$$\begin{aligned} K(a, b)D^*c + DK(a, b)c &= L(a, D^*c)b - L(b, D^*c)a + DL(a, c)b - DL(b, c)a \\ &= L(Da, c)b + L(a, c)Db - L(Db, c)a - L(b, c)Da \\ &= K(Da, b)c + K(a, Db)c. \end{aligned}$$

Hence (2.1) holds. (2.2) and (2.3) follow from (L2) and (K1) respectively. □

For linear endomorphisms A, B and C of an FKTA (U, ε), we define a triple product $< ABC >$ by

$$< ABC > := AB\varepsilon C + CB\varepsilon A.$$

Then the triple product $< ABC >$ is also a linear endomorphism of (U, ε).

Proposition 2.2. Let (D, D^*), (E, E^*) be pairs satisfying the condition (L) and A, B and C linear endomorphisms satisfying the condition (K). Then the following statements are valid:

(1) $([D, E], [E^*, D^*])$ satisfies the condition (L),

(2) $(BC\varepsilon, C\varepsilon B)$ satisfies the condition (L),

(3) $DB + BD^*$ satisfies the condition (K),

(4) $< A\,B\,C >$ satisfies the condition (K).

Proof. Let $a, b \in U$.

(1) Using Jacobi identity and (L1), we have

$$\begin{aligned}
[[D, E], L(a, b)] &= [D, [E, L(a, b)]] - [E, [D, L(a, b)]] \\
&= [D, L(Ea, b) - L(a, E^*b)] - [E, L(Da, b) - L(a, D^*b)] \\
&= L(DEa, b) + L(a, D^*E^*b) - L(EDa, b) - L(a, E^*D^*b) \\
&= L([D, E]a, b) - L(a, [E^*, D^*]b).
\end{aligned}$$

Hence $([D, E], [E^*, D^*])$ satisfies the condition (L1). Similarly it follows that this pair satisfies the condition (L2).

(2) From (K1), (2.3) and (K2),we obtain

$$\begin{aligned}
[BC\varepsilon, L(a, b)] &= BCL(\varepsilon a, \varepsilon b)\varepsilon - L(a, b)BC\varepsilon = BK(C\varepsilon a, b)\varepsilon - K(Bb, a)C\varepsilon \\
&= L(BC\varepsilon a, b) - L(a, C\varepsilon Bb).
\end{aligned}$$

Hence $(BC\varepsilon, C\varepsilon B)$ satisfies (L1). Similarly we can verify that this pair satisfies the condition (L2).

(3) Using (K1), (L1), (L2) and (2.1), we obtain

$$\begin{aligned}
&K((DB + BD^*)a, b) - L(b, a)(DB + BD^*) - (DB + BD^*)L(a, \varepsilon b) \\
&= K(DBa, b) - K(Ba, b)D^* - DK(Ba, b) \\
&+B(L(D^*a, \varepsilon b)L(a, \varepsilon b)D^* - D^*L(a, \varepsilon b) + (DL(b, a) - L(b, a)D + L(b, D^*a))B \\
&= K(DBa, b) - K(Ba, b)D^* - DK(Ba, b) + BL(a, \varepsilon Db) + L(Db, a)B \\
&= K(DBa, b) - K(Ba, b)D^* - DK(Ba, b) + K(Ba, Db) = 0.
\end{aligned}$$

Hence $DB + BD^*$ satisfies (K1). Using (2.2) and (K2), we have

$$\begin{aligned}
K(a, b)DB\varepsilon &= \varepsilon^{-1}K(\varepsilon a, \varepsilon b)\varepsilon D\varepsilon^{-1}\varepsilon B\varepsilon \\
&= \varepsilon^{-1}(K(D^*\varepsilon a, \varepsilon b) + K(\varepsilon a, D^*\varepsilon b) - D^*K(\varepsilon a, \varepsilon b))\varepsilon B\varepsilon \\
&= K(\varepsilon^{-1}D^*\varepsilon a, b)B\varepsilon + K(a, \varepsilon^{-1}D^*\varepsilon b)B\varepsilon - \varepsilon^{-1}D^*\varepsilon K(a, b)B\varepsilon \\
&= L(b, BD^*\varepsilon a) - L(a, BD^*\varepsilon b) - L(\varepsilon^{-1}D^*\varepsilon a, B\varepsilon b) + L(\varepsilon^{-1}D^*\varepsilon b, B\varepsilon a) - \varepsilon^{-1}D^*\varepsilon K(a, b)B\varepsilon.
\end{aligned}$$

From this we get

$$\begin{aligned}
&K(a, b)DB\varepsilon + L(a, BD^*\varepsilon b) - L(b, BD^*\varepsilon a) \\
&= L(\varepsilon^{-1}D^*\varepsilon b, B\varepsilon a) - L(\varepsilon^{-1}D^*\varepsilon a, B\varepsilon b) - \varepsilon^{-1}D^*\varepsilon K(a, b)B\varepsilon \cdots \text{ (i)}.
\end{aligned}$$

Using (K2), (L2) and (2.1), we have

$K(a,b)BD^*\varepsilon = K(a,b)B\varepsilon\varepsilon^{-1}D^*\varepsilon = (L(b,B\varepsilon a) - L(a,B\varepsilon b))\varepsilon^{-1}D^*\varepsilon$
$= \varepsilon^{-1}(L(\varepsilon b,\varepsilon B\varepsilon a)D^* - L(\varepsilon a,\varepsilon B\varepsilon b)D^*)\varepsilon$
$= \varepsilon^{-1}(D^*L(\varepsilon b,\varepsilon B\varepsilon a) - L(D^*\varepsilon b,\varepsilon B\varepsilon a) + L(\varepsilon b,\varepsilon DB\varepsilon a)$
$-D^*L(\varepsilon a,\varepsilon B\varepsilon b) + L(D^*\varepsilon a,\varepsilon B\varepsilon b) - L(\varepsilon a,\omega DB\varepsilon b))\varepsilon$
$= \varepsilon^{-1}D^*\varepsilon K(a,b)B\varepsilon - L(\varepsilon^{-1}D^*\varepsilon b,B\varepsilon a) + L(b,DB\varepsilon a) + L(\varepsilon^{-1}D^*\varepsilon a,B\varepsilon b) - L(a,DB\varepsilon b).$

From this we obtain

$K(a,b)BD^*\varepsilon + L(a,DB\varepsilon b) - L(b,DB\varepsilon a)$
$= L(\varepsilon^{-1}D^*\varepsilon a,B\varepsilon b) - L(\varepsilon^{-1}D^*\varepsilon b,B\varepsilon a) + \varepsilon^{-1}D^*\varepsilon K(a,b)B\varepsilon \cdots$ (ii).

Adding (i) and (ii), it follows that $K(a,b)(DB+BD^*)\varepsilon - L(a,(DB+BD^*)\varepsilon b) + L(b,(DB+BD^*)\varepsilon a) = 0$, therefore (K2) is satisfied.

(4) follows from (2) and (3) immediately. □

Let (U,ε) be an FKTA, and let us consider a vector space direct sum $T = U \oplus U$. An element $a \oplus b$ of T is also denoted as $\begin{pmatrix} a \\ b \end{pmatrix}$ in column vector form. In this case, endomorphisms of T are denoted in the form of 2×2 matrices. Let $\mathfrak{D}$ be a vector space spanned by endomorphisms of T of the form $\begin{pmatrix} D & B \\ C\varepsilon & -D^* \end{pmatrix}$, where (D,D^*) is a pair of linear endomorphisms of U satisfying the condition (L) and B, C are linear endomorphisms of U satisfying the condition (K). Let $\mathfrak{D}_0$ be a subspace of $\mathfrak{D}$ which is spanned by endomorphisms of the form

$$\begin{pmatrix} L(a,b) & K(c,d) \\ K(e,f)\varepsilon & -L(b,\varepsilon a) \end{pmatrix},$$

where $a,b,c,d,e,f \in U$.

Proposition 2.3. $\mathfrak{D}$ becomes a Lie algebra with repect to the commutator product [,], and the subspace $\mathfrak{D}_0$ is an ideal of $\mathfrak{D}$.

Proof. From Proposition 2.2, it follows that $\mathfrak{D}$ is closed with respect to the commutator product [,]. Using Lemma 2.1, Proposition 2.2 and the conditions (L), (K), it is easy to show that $[\mathfrak{D}_0,\mathfrak{D}] \subset \mathfrak{D}_0$. □

Put $\mathfrak{L} = \mathfrak{D} \oplus T$ and define an anti-commutative product [,] in $\mathfrak{L}$ as follows:
For $P,Q \in \mathfrak{D}$, $X,Y \in T$,

$$\begin{aligned} &[P,Q] := PQ - QP, \\ &[P,X] := -[X,P] := PX, \\ &[X,Y] := \begin{pmatrix} L(a,y) - L(b,x) & K(a,b) \\ K(x,y)\varepsilon & L(x,\varepsilon b) - L(y,\varepsilon a) \end{pmatrix}, \end{aligned} \tag{2.4}$$

where $X = a \oplus x$, $Y = b \oplus y$. Then, we can verify that the Jacobi identity holds by using Lemma 2.1, Proposition 2.2 and the conditions (L), (K), therefore we have

Theorem 2.4. The vector space $\mathfrak{L}$ becomes a Lie algebra with respect to the product defined by (2.4).

Now let L_i $(i = 0,\pm 1,\pm 2)$ be subspaces of $\mathfrak{L}$ as follows:

L_{-2} = the subspace spanned by all operatos $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{D}$,
$L_{-1} = U \oplus \{0\}$,
L_0 = the subspace spanned by all operatos $\begin{pmatrix} D & 0 \\ 0 & -D^* \end{pmatrix} \in \mathfrak{D}$,
$L_1 = \{0\} \oplus U$,
L_2 = the subspace spanned by all operators $\begin{pmatrix} 0 & 0 \\ C\varepsilon & 0 \end{pmatrix} \in \mathfrak{D}$.

Then it is easily shown that

$$\mathfrak{L} = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \quad [L_i, L_j] \subset L_{i+j},$$

that is, $\mathfrak{L}$ is a graded Lie algebra (or GLA simply) of the second order.

The subspace $T = L_{-1} \oplus L_1$ of $\mathfrak{L}$ becomes an LTS with respect to the trilinear product defined by

$$(2.5)\quad \left[\begin{pmatrix} a \\ x \end{pmatrix}\begin{pmatrix} b \\ y \end{pmatrix}\begin{pmatrix} c \\ z \end{pmatrix}\right] := \left[\left[\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix}\right], \begin{pmatrix} c \\ z \end{pmatrix}\right] = \begin{pmatrix} L(a,y) - L(b,x) & K(a,b) \\ K(x,y)\varepsilon & L(x,\varepsilon b) - L(y,\varepsilon a) \end{pmatrix}\begin{pmatrix} c \\ z \end{pmatrix},$$

where $a, b, c, x, y, z \in U$ (cf. [3]). T is called the *LTS associated with an FKTA* (U, ε). The standard enveloping Lie algebra $\mathfrak{L}_0 = \mathfrak{D}(T) \oplus T$ of T is an ideal of $\mathfrak{L}$, where $\mathfrak{D}(T)$ is the Lie algebra of inner derivations of T which coincides with $\mathfrak{D}_0$. $\mathfrak{L}_0$ is called the *GLA associated with* (U, ε).

§3. Representations of FK-ternary algebras

K. Yamaguti defined a representation of Freudenthal-Kantor triple system and constructed a representation of the Lie triple system associated with the Freudenthal-Kantor triple system [15]. In this section, we consider representations of FK-ternary algebras. We first recall the definition of a representation of Lie triple system.

A representation of a Lie triple system T into a vector space V is a pair (λ, ρ) of bilinear mappings of T into $\mathrm{End}(V)$ satisfying the following identities:

(3.1) $\lambda(X, Y) = \rho(Y, X) - \rho(X, Y)$,

(3.2) $[\lambda(X, Y), \rho(Z, W)] = \rho([XYZ], W) + \rho(Z, [XYW])$,

(3.3) $\rho(X, [YZW]) = \rho(Z, W)\rho(X, Y) - \rho(Y, W)\rho(X, Z) + \lambda(Y, Z)\rho(X, W)$

for all $X, Y, Z, W \in T$. For details of definitions and properties of representations of LTS's we refer to [11], [12]. The pair (L, R) of the left and right multiplications $L(X, Y)$, $R(X, Y)$ is a representation of T into itself, which is called a *regular representation.*

Definition. Let V be a vector space and E a non-singular linear endomorphism of V. A representation of an FKTA (U, ε) into (V, E) is a triple (λ, μ, ρ) of bilinear mappings of U into $\mathrm{End}(V)$ satisfying the following identities:

(3.4) $E\lambda(a, b) = \lambda(\varepsilon a, \varepsilon b)E$, $E\mu(a, b) = \mu(\varepsilon a, \varepsilon b)E$, $E\rho(a, b) = \rho(\varepsilon a, \varepsilon b)E$,

(3.5) $[\lambda(a, b), \lambda(c, d)] = \lambda((abc), d) - \lambda(c, (b\,\varepsilon a\,d))$,

(3.6) $[\lambda(a, b), \rho(c, d)] = -\rho((b\,\varepsilon a\,c), d) + \rho(c, (abd))$,

(3.7) $\lambda(a, b)\mu(c, d) + \mu(c, d)\lambda(b, \varepsilon a) = \mu((abc), d) + \mu(c, (abd))$, (3.8) $\mu(a,(bcd)) = \lambda(b,c)\mu(a,d) + \rho(c,d)\mu(a,b) - \mu(b,d)\rho(\varepsilon a,c)$,

(3.9) $\rho(a,(bcd)) = \lambda(b,c)\rho(a,d) + \rho(c,d)\rho(a,b) - \mu(b,d)\mu(a,c)E$,

(3.10) $\kappa(a,b)\mu(c,d)E + \phi(d,K(a,b)c) + \rho(c,K(a,b)d) = 0$,

(3.11) $\kappa(a,b)\rho(\varepsilon c,d) - \phi(d,c)\kappa(a,b) + \mu(c,K(a,b)d) = 0$,

(3.12) $\kappa(a,b)\phi(\varepsilon c,d) - \rho(d,c)\kappa(a,b) + \mu(K(a,b)d,c) = 0$,

(3.13) $\phi(a,b)\phi(c,d) - \lambda(b,c)\phi(a,d) - \phi((c\,\varepsilon b\,a),d) = 0$,

where $\kappa(a,b) := \mu(a,b) - \mu(b,a)$, $\phi(a,b) := \rho(a,b) - \lambda(b,a)$, $a,b,c,d \in U$.

Using the condition (U1), (U2) and [3] Lemma 2.1, we see that the triple (L,M,R) is a representation of (U,ε) into itself, which is called a *regular representation*. From (3.5), (3.6), (3.7), (3.8), (3.11) and (3.12), it follows that

(3.14) $[\lambda(a,b),\phi(c,d)] = \phi(c,(abd)) - \phi((b\,\varepsilon a\,c),d)$,

(3.15) $\kappa(K(a,b)c,d) = \lambda(d,c)\kappa(a,b) + \kappa(a,b)\lambda(c,\varepsilon d)$,

(3.16) $\mu(a,K(b,c)d) + \phi(d,b)\mu(a,c) = \phi(d,c)\mu(a,b) - \kappa(b,c)\rho(\varepsilon a,d)$,

(3.17) $\kappa(a,b)\phi(\varepsilon d,c) + \mu(a,d)\phi(\varepsilon b,c) - \mu(b,d)\phi(\varepsilon a,c) = 0$.

Remark. If (U,ε) is a Freudenthal-Kantor triple system, we can consider that $E = \pm\mathrm{Id}$. Consequently, we need not consider the pair (V,E).

Let (λ,μ,ρ) be a representation of an FKTA (U,ε) into (V,E). Then let us consider a direct product $V\times U$. An element (x,a) of $V\times U$ is also denoted as $\begin{pmatrix} x \\ a \end{pmatrix}$ in column vector form. Define a triple product $\{\ ,\ ,\}$ in $V\times U$ by

$$\left\{\begin{pmatrix} x \\ a \end{pmatrix}\begin{pmatrix} y \\ b \end{pmatrix}\begin{pmatrix} z \\ c \end{pmatrix}\right\} := \begin{pmatrix} \rho(b,c)x + \mu(a,c)y + \lambda(a,b)z \\ (abc) \end{pmatrix}. \tag{3.18}$$

It is easily seen that the endomorphism $E\times\varepsilon : (x,a) \mapsto (Ex,\varepsilon a)$ is an automorphism of the triple system $V\times U$.

Proposition 3.1. $(V\times U, \{\ ,\ ,\}, E\times\varepsilon)$ is an FK-ternary algebra.

Proof. For $X_i \in V\times U$ $(i=1,2,3,4,5)$, put

$$(x,a) = ([L(X_1,X_2),L(X_3,X_4)] - L(L(X_1,X_2)X_3,X_4) + L(X_3,L(X_2,(E\times\varepsilon)X_1)X_4))X_5.$$

By the bilinearity, in order to prove the condition (U1) it is sufficient to verify that $x = a = 0$ in the following cases:

Case (1) $X_1 = (x_1,0)$, $X_2 = (x_2,0)$, $X_i = (x_i,a_i)$ $(i=3,4,5)$,
Case (2) $X_1 = (x_1,0)$, $X_2 = (0,a_2)$, $X_i = (x_i,a_i)$ $(i=3,4,5)$,
Case (3) $X_1 = (0,a_1)$, $X_2 = (x_2,0)$, $X_i = (x_i,a_i)$ $(i=3,4,5)$,
Case (4) $X_1 = (0,a_1)$, $X_2 = (0,a_2)$, $X_i = (x_i,a_i)$ $(i=3,4,5)$.

Case (1) is clear.

Case (2) $a = 0$.

$$\begin{aligned} x &= [\rho(a_2,(a_3a_4a_5)) - \rho(a_4,a_5)\rho(a_2,a_3) + \mu(a_3,a_5)\mu(a_2,a_4)E - \lambda(a_3,a_4)\rho(a_2,a_5)]x_1 \\ &= 0 \text{ from (3.9).} \end{aligned}$$

Case (3) $a = 0$.

$$x = [\mu(a_1, (a_3 a_4 a_5)) - \rho(a_4, a_5)\mu(a_1, a_3) + \mu(a_3, a_5)\rho(\varepsilon a_1, a_4) - \lambda(a_3, a_4)\mu(a_1, a_5)]x_2$$
$$= 0 \text{ from (3.8)}.$$

Case (4) $a = (a_1 a_2 (a_3 a_4 a_5)) - ((a_1 a_2 a_3) a_4 a_5) + (a_3 (a_2\, \varepsilon a_1 a_4) a_5) - (a_3 a_4 (a_1 a_2 a_5))$
$= 0$ from (U1).

$$\begin{aligned} x = {} & [\lambda(a_1, a_2)\lambda(a_3, a_4) - \lambda((a_1 a_2 a_3), a_4) + \lambda(a_3, (a_2\, \varepsilon a_1\, a_4)) - \lambda(a_3, a_4)\lambda(a_1, a_2)]x_5 \\ & + [\lambda(a_1, a_2)\mu(a_3, a_5) - \mu((a_1 a_2 a_3), a_5) + \mu(a_3, a_5)\lambda(a_2, \varepsilon a_1) - \mu(a_3, (a_1 a_2 a_5))]x_4 \\ & + [\lambda(a_1, a_2)\rho(a_4, a_5) - \rho(a_4, a_5)\lambda(a_1, a_2) + \rho((a_2\, \varepsilon a_1\, a_4), a_5) - \rho(a_4, (a_1 a_2 a_5))]x_3 \\ = {} & 0 \text{ from (3.5), (3.7) and (3.6)}. \end{aligned}$$

Hence the condition (U1) is proved. Next in order to prove the condition (U2) we put

$$(y, b) = [K(K(X_1, X_2)X_3, X_4) - L(X_4, X_3)K(X_1, X_2) - K(X_1, X_2)L(X_3, (E \times \varepsilon)X_4)]X_5$$

for $X_i \in V \times U$ $(i = 1, 2, 3, 4, 5)$. By the bilinearity and the anti-commutativity of $K(X_1, X_2)$, it is sufficient to verify that $y = b = 0$ in the above three cases (1), (2) and (4):

Case (1) is clear.

Case (2) $b = 0$

$$y = [\phi(a_5, a_4)\phi(a_3, a_2) - \lambda(a_4, a_3)\phi(a_5, a_2) - \phi((a_3\, \varepsilon a_4\, a_5), a_2)]x_1 = 0 \text{ from (3.13)}.$$

Case (4) From the condition (U2) of (U, ε),

$$b = [K(K(a_1, a_2)a_3, a_4) - L(a_4, a_3)K(a_1 a_2) - K(a_1, a_2)L(a_3, \varepsilon a_4)]a_5 = 0.$$

$$\begin{aligned} y = {} & [\phi(a_5, a_4)\kappa(a_1, a_2) - \mu(a_4, K(a_1, a_2)a_5) - \kappa(a_1, a_2)\rho(\varepsilon a_4, a_5)]x_3 \\ & - [\phi(a_5, K(a_1, a_2)a_3) + \rho(a_3, K(a_1, a_2)a_5) + k(a_1, a_2)\mu(a_3, a_5)E]x_4 \\ & + [\kappa(K(a_1, a_2)a_3, a_4) - \lambda(a_4, a_3)\kappa(a_1, a_2) - \kappa(a_1, a_2)L(a_3, \varepsilon a_4)]x_5 \\ = {} & 0 \text{ from (3.11), (3.10) and (3.15)}. \end{aligned}$$

Hence the condition (U2) is proved. □

Let (U, ε) be an FKTA and T the LTS associated with (U, ε). Let V be a vector space with a non-singular endomorphism E and (λ, μ, ρ) a representation of (U, ε) into (V, E). Define bilinear mappings ρ^* and λ^* of T into $V \oplus V$ by

$$(3.19)\quad \rho^*\left(\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix}\right)\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \phi(y, a) + \rho(x, b) & -\mu(a, b) \\ -\mu(x, y)E & \phi(\varepsilon b, x) + \rho(\varepsilon a, y) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

$$\begin{aligned} (3.20)\quad \lambda^*\left(\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix}\right) & := \rho^*\left(\begin{pmatrix} b \\ y \end{pmatrix}, \begin{pmatrix} a \\ x \end{pmatrix}\right) - \rho^*\left(\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix}\right) \\ & = \begin{pmatrix} \lambda(a, y) - \lambda(b, x) & \kappa(a, b) \\ \kappa(x, y)E & \lambda(x, \varepsilon b) - \lambda(y, \varepsilon a) \end{pmatrix}, \end{aligned}$$

where $a, b, x, y \in U$, $u, v \in V$. The following result is a generalization of [15] Theorem 4.1 to the case U is an FKTA.

Proposition 3.2. Let (V, E) be a pair of a vector space and its non-singular endomorphism. Let (λ, μ, ρ) be a representation of an FKTA (U, ε) into (V, E) satisfying

$$(3.21)\quad \mu(c, d)\kappa(a, b)E - \phi(K(a, b)\varepsilon d, c) - \rho(K(a, b)\varepsilon c, d) = 0,$$

$$(3.22)\quad \rho(a, b)\phi(d, c) - \rho(d, b)\phi(a, c) - \phi(K(d, a)\varepsilon b, c) = 0,$$

$$(3.23)\quad \phi(d, c)\rho(a, b) - \phi(d, b)\rho(a, c) + \phi(d, K(b, c)a) = 0$$

for all $a, b, c, d \in U$. Then the pair (λ^*, ρ^*) defined by (3.19) and (3.20) is a representation of the LTS T associated with (U, ε) into a vector space $V \oplus V$.

Proof. For $X, Y, Z, W \in T$, $u \oplus v \in V \oplus V$, put

$$p \oplus q = ([\lambda^*(X,Y), \rho^*(Z,W)] - \rho^*([XYZ],W) - \rho^*(Z,[XYW]))(u \oplus v).$$

Since $\lambda^*(X,Y)$ and $[XYZ]$ are anti-commutative on arguments X, Y, in order to prove the condition (3.2) it suffices to verify that $p = q = 0$ in the following cases:

Case (1) $X = a \oplus 0,\quad Y = b \oplus 0,\quad Z = c \oplus z,\quad W = d \oplus w,$
Case (2) $X = a \oplus 0,\quad Y = 0 \oplus y,\quad Z = c \oplus z,\quad W = d \oplus w,$
Case (3) $X = 0 \oplus x,\quad Y = 0 \oplus y,\quad Z = c \oplus z,\quad W = d \oplus w.$

Case (1) $p = (-\kappa(a,b)\mu(z,w)E - \phi(w, K(a,b)z) - \rho(z, K(a,b)w))u$
$+(\kappa(a,b)\phi(\varepsilon d, z) - \rho(z,d)\kappa(a,b) + \mu(K(a,b)z, d) + \kappa(a,b)\rho(\varepsilon c, w) - \phi(w,c)\kappa(a,b)$
$+\mu(c, K(a,b)w))y = 0$ from (3.10), (3.12) and (3.11).
$q = (\mu(z,w)E\kappa(a,b) - \rho(\varepsilon K(a,b)z, w) - \phi(\varepsilon K(a,b)w, z))v = 0$ from (3.4) and (3.21).

Case (2) $p = ([\lambda(a,y), \phi(w,c)] - \phi(w,(ayc)) + \phi((y\,\varepsilon a\,w), c) + [\lambda(a,y), \rho(z,d)]$
$+\rho((y\,\varepsilon a\,z), d) - \rho(z, (ayd)))u$
$+(-\lambda(a,y)\mu(c,d) - \mu(c,d)\lambda(y,\varepsilon a) + \mu((ayc), d) + \mu(c,(ayd)))v$
$= 0$ from (3.14), (3.6) and (3.7).
$q = (\lambda(y,\varepsilon a)\mu(z,w)E + \mu(z,w)E\lambda(a,y) - \mu((y\,\varepsilon a\,z), w)E - \mu(z,(y\,\varepsilon a\,w))E)u$
$+(-[\lambda(y,\varepsilon a), \phi(\varepsilon d, z)] + \phi(\varepsilon d, (y\,\varepsilon a\,z)) - \phi(\varepsilon(ayd), z) - [\lambda(y,\varepsilon a), \rho(\varepsilon c, w)]$
$-\rho(\varepsilon(ayc), w) + \rho(\varepsilon c, (y\,\varepsilon a\,w)))v = 0$ from (3.4), (3.7), (3.14) and (3.6).

Case (3) $p = (\mu(c,d)\kappa(x,y)E - \rho(K(x,y)\varepsilon c, d) - \phi(K(x,y)\varepsilon d, c))u = 0$ from (3.21).
$q = (\kappa(x,y)E\phi(w,c) - \rho(\varepsilon c, w)\kappa(x,y)E + \mu(K(x,y)\varepsilon c, w)E + \kappa(x,y)E\rho(z,d)$
$-\phi(\varepsilon d, z)\kappa(x,y)E + \mu(z, K(x,y)\varepsilon d)E)u$
$+(-\kappa(x,y)E\mu(c,d) - \phi(\varepsilon d, K(x,y)\varepsilon c) - \rho(\varepsilon c, K(x,y)\varepsilon d))v$
$= 0$ from (3.12), (3.11), (3.4) and (3.10).

Hence the condition (3.2) is proved. Next we put

$$r \oplus s = (\rho^*(X,[YZW]) - \rho^*(Z,W)\rho^*(X,Y) + \rho^*(Y,W)\rho^*(X,Z) - \lambda^*(Y,Z)\rho^*(X,W))(u \oplus v).$$

In order to prove the condition (3.3) it is sufficient to verify that $r = s = 0$ in the following three cases:

Case (1) $X = a \oplus x,\quad Y = b \oplus 0,\quad Z = c \oplus 0,\quad W = d \oplus w,$
Case (2) $X = a \oplus x,\quad Y = b \oplus 0,\quad Z = 0 \oplus z,\quad W = d \oplus w,$
Case (3) $X = a \oplus x,\quad Y = 0 \oplus y,\quad Z = 0 \oplus z,\quad W = d \oplus w.$

Case (1) $r = (\rho(x, K(b,c)w) - \phi(w,c)\rho(x,b) + \phi(w,b)\rho(x,c) + \kappa(b,c)\mu(x,w)E)u$
$+(-\mu(a, K(b,c)w) + \phi(w,c)\mu(a,b) - \phi(w,b)\mu(a,c) - \kappa(b,c)\rho(\varepsilon a, w) - \mu(c,d)\phi(\varepsilon b, x)$
$+\mu(b,d)\phi(\varepsilon c, x) + \kappa(b,c)\phi(\varepsilon d, x))v = 0$ from (3.23), (3.10), (3.16) and (3.17).
$s = (\phi(\varepsilon K(b,c)w, x) - \rho(\varepsilon c, w)\phi(\varepsilon b, x) + \rho(\varepsilon b, w)\phi(\varepsilon c, x))v = 0$ from (3.22).

Case (2) $r = (-\phi((z\,\varepsilon b\,w), a) + \phi(w,b)\phi(z,a) - \lambda(b,z)\phi(w,a) + \rho(x,(bzd))$
$-\lambda(b,z)\rho(x,d) - \rho(z,d)\rho(x,b) - \mu(b,d)\mu(x,z)E)u$
$+(-\mu(a,(bzd)) + \rho(z,d)\mu(a,b) - \mu(b,d)\rho(\varepsilon a, z) - \lambda(b,z)\mu(a,d))v$
$= 0$ from (3.13), (3.9) and (3.8).
$s = (\mu(x,(z\,\varepsilon b\,w))E + \mu(z,w)E\rho(x,b) + \rho(\varepsilon b, w)\mu(x,z)E - \lambda(z,\varepsilon b)\mu(x,w)E)u$
$+(\phi(\varepsilon(bdz), x) - \phi(\varepsilon d, z)\phi(\varepsilon b, x) + \lambda(z,\varepsilon b)\phi(\varepsilon d, x) - \rho(\varepsilon a, (z\,\varepsilon b\,w)) - \mu(z,w)E\mu(a,b)$

$+\rho(\varepsilon b, w)\rho(\varepsilon a, z) + \lambda(z, \varepsilon b)\rho(\varepsilon a, w))v = 0$ from (3.4), (3.8), (3.13) and (3.9).

Case (3) $r = (\phi(K(y,z)\varepsilon d, a) - \rho(z,d)\phi(y,a) + \rho(y,d)\phi(z,a))u = 0$ from (3.22).

$$\begin{aligned}s = (&-\mu(x, K(y,z)\varepsilon d)E + \mu(z,w)E\phi(y,a) - \mu(y,w)E\phi(z,a) - \kappa(y,z)E\phi(w,a)\\ &+\phi(\varepsilon d, z)\mu(x,y)E + \phi(\varepsilon d, y)\mu(x,z)E - \kappa(y,z)E\rho(x,d))u\\ &+(r(\varepsilon a, K(y,z)\varepsilon d) - \phi(\varepsilon d, z)\rho(\varepsilon a, y) + \phi(\varepsilon d, y)\rho(\varepsilon a, z) + \kappa(y,z)E\mu(a,d))v\\ &= 0 \text{ from (3.4), (3.17), (3.16), (3.10) and (3.23).}\end{aligned}$$

Hence the condition (3.3) is proved. □

From [3] Lemma 2.1, we see that the regular representation (L, M, R) of (U, ε) satisfies the conditions of Propositionm 3.2.

§4. Extensions of FK-ternary algebras

In this section, we define the cohomology space of order 3 of an FK-ternary algebra (U, ε) associated with a representation (λ, μ, ρ), and give an interpretation of it in relation to extensions of (U, ε) following the method of [4].

Let (U, ε) be an FKTA and (V, E) a pair of a vector space and a non-singular endomorphism of V. Let (λ, μ, ρ) be a representation of (U, ε) into (V, E). We denote by $C^1(U, V)$ the vector space spanned by linear mappings f of U into V such that

$$f(\varepsilon a) = Ef(a) \tag{4.1}$$

for all $a \in U$, and denote by $C^3(U, V)$ the vector space spanned by trilinear mappings of $U \times U \times U$ into V satisfying

$$f(\varepsilon a_1, \varepsilon a_2, \varepsilon a_3) = Ef(a_1, a_2, a_3) \tag{4.2}$$

and

$$\begin{aligned}&f(K(a_1,a_2)a_3, a_5, a_4) - f(a_4, a_5, K(a_1,a_2)a_3) - f(a_4, a_3, K(a_1,a_2)a_5)\\ &-f(a_1, (a_3\,\varepsilon a_4\, a_5), a_2) + f(a_2, (a_3\,\varepsilon a_4\, a_5), a_1) - \kappa(a_1,a_2)f(a_3, \varepsilon a_4, a_5)\\ &+\phi(a_5,a_4)(f(a_1,a_3,a_2) - f(a_2,a_3,a_1)) - \lambda(a_4,a_3)(f(a_1,a_5,a_2) - f(a_2,a_5,a_1)) = 0,\end{aligned} \tag{4.3}$$

where $a_i \in U$ $(i = 1,2,3,4,5)$. Moreover we denote the vector space spanned by 5-linear mappings f of $U \times U \times U \times U \times U$ into V by $C^5(U, V)$.

For $f \in C^1(U, V)$, we define a trilinear mapping $\delta^1 f : U \times U \times U \longrightarrow V$ as follows:

$$\delta^1 f(a_1,a_2,a_3) := -\rho(a_2,a_3)f(a_1) - \mu(a_1,a_3)f(a_2) - \lambda(a_1,a_2)f(a_3) + f((a_1 a_2 a_3)), \tag{4.4}$$

where $a_i \in U$ $(i = 1,2,3)$. We shall show that $\delta^1 f \in C^3(U, V)$. It is easy to check that $\delta^1 f$ satisfies the condition (4.2). For $a_i \in U$ $(i = 1,2,3,4,5)$,

$$\begin{aligned}&\delta^1 f(K(a_1,a_2)a_3, a_5, a_4) - \delta^1 f(a_4, a_5, K(a_1,a_2)a_3)) - \delta^1 f(a_4, a_3, K(a_1,a_2)a_5)\\ &-\delta^1 f(a_1, (a_3\,\varepsilon a_4\, a_5), a_2) + \delta^1 f(a_2, (a_3\,\varepsilon a_4\, a_5), a_1) - \kappa(a_1,a_2)\delta^1 f(a_3, \varepsilon a_4, a_5)\\ &+\phi(a_5,a_4)(\delta^1 f(a_1,a_3,a_2) - \delta^1 f(a_2,a_3,a_1)) - \lambda(a_4,a_3)(\delta^1 f(a_1,a_5,a_2) - \delta^1 f(a_2,a_5,a_1))\\ &= (-\kappa(K(a_1,a_2)a_3, a_4) + \lambda(a_4,a_3)\kappa(a_1,a_2) + \kappa(a_1,a_2)\lambda(a_3, \varepsilon a_4))f(a_5)\\ &+(\phi(a_5, K(a_1,a_2)a_3) + \rho(a_3, K(a_1,a_2)a_5) + \kappa(a_1,a_2)\mu(a_3,a_5)E)f(a_4)\\ &+(\mu(a_4, K(a_1,a_2)a_5) - \phi(a_5,a_4)\kappa(a_1,a_2) + \kappa(a_1,a_2)\rho(\varepsilon a_4, a_5))f(a_3)\end{aligned}$$

$+(\phi(a_5, a_4)\phi(a_3, a_1) - \lambda(a_4, a_3)\phi(a_5, a_1) - \phi((a_3\, \varepsilon a_4\, a_5), a_1))f(a_2)$
$+(\phi((a_3\, \varepsilon a_4\, a_5), a_2) + \lambda(a_4, a_3)\phi(a_5, a_2) - \phi(a_5, a_4)\phi(a_3, a_2))f(a_1)$
$+f(K(K(a_1, a_2)a_3, a_4)a_5) - f(L(a_4, a_3)K(a_1, a_2)a_5) - f(K(a_1, a_2)L(a_3, \varepsilon a_4)a_5)$
$= 0$ from (3.15), (3.4), (3.10), (3.11), (3.13), and (U2). Hence $\delta^1 f \in C^3(U, V)$.

Next we define a linear mapping δ^3 of $C^3(U, V)$ into $C^5(U, V)$ by the following formula:

(4.5) $\delta^3 f(a_1, a_2, a_3, a_4, a_5)$

$$:= -\rho(a_4, a_5)f(a_1, a_2, a_3) + \mu(a_3, a_5)f(a_2, \varepsilon a_1, a_4) + \lambda(a_1, a_2)f(a_3, a_4, a_5)$$
$$-\lambda(a_3, a_4)f(a_1, a_2, a_5) - f((a_1 a_2 a_3), a_4, a_5) + f(a_3, (a_2\, \varepsilon a_1\, a_4), a_5)$$
$$-f(a_3, a_4, (a_1 a_2 a_5)) + f(a_1, a_2, (a_3 a_4 a_5)),$$

where $f \in C^3(U, V)$, $a_i \in U$ $(i = 1, 2, 3, 4, 5)$.

Proposition 4.1. $\delta^3 \delta^1 f = 0$ for any $f \in C^1(U, V)$.

Proof. For $a_i \in U$ $(i = 1, 2, 3, 4, 5)$,

$\delta^3 \delta^1 f(a_1, a_2, a_3, a_4, a_5)$
$= (\rho(a_4, a_5)\rho(a_2, a_3) - \mu(a_3, a_5)\mu(a_2, a_4)E + \lambda(a_3, a_4)\rho(a_2, a_5) - \rho(a_2, (a_3\, a_4\, a_5)))f(a_1)$
$+(\rho(a_4, a_5)\mu(a_1, a_3) - \mu(a_3, a_5)\rho(\varepsilon a_1, a_4) + \lambda(a_3, a_4)\mu(a_1, a_5) - \mu(a_1, (a_3 a_4 a_5)))f(a_2)$
$+(-[\lambda(a_1, a_2), \rho(a_4, a_5)] - \rho((a_2\, \varepsilon a_1\, a_4), a_5) + \rho(a_4, (a_1 a_2 a_5)))f(a_3)$
$+(-\mu(a_3, a_5)\lambda(a_2, \varepsilon a_1) - \lambda(a_1, a_2)\mu(a_3, a_5) + \mu((a_1 a_2 a_3), a_5) + \mu(a_3, (a_1 a_2 a_5)))f(a_4)$
$+(-[\lambda(a_1, a_2), \lambda(a_3, a_4)] + \lambda((a_1 a_2 a_3), a_4) - \lambda(a_3, (a_2\, \varepsilon a_1\, a_4)))f(a_5)$
$+f((a_1 a_2 (a_3 a_4 a_5)) - (a_3 a_4 (a_1 a_2 a_5)) - ((a_1 a_2 a_3) a_4 a_5) + (a_3 (a_2\, \varepsilon a_1\, a_4) a_5))$
$= 0$ from (3.4), (3.9), (3.8), (3.6), (3.7), (3.5) and (U1). □

A mapping $f \in C^3(U, V)$ is called a *cocycle* of order 3 if $\delta^3 f = 0$. We denote by $Z^3(U, V)$ a subspace spanned by cocycles of order 3, and put $B^3(U, V) = \delta^1 C^1(U, V)$. The element of $B^3(U, V)$ is called a *coboundary* of order 3. From the above proposition, $B^3(U, V)$ is a subspace of $Z^3(U, V)$. We denote the factor space $Z^3(U, V)/B^3(U, V)$ by $H^3(U, V)$, and call it a *cohomology space* of order 3 of (U, ε).

Let (U, B_U, ε) and (V, B_V, σ) be FKTA's. A linear mapping φ of U into V is called a *homomorphism* if

$$\varphi(B_U(a, b, c)) = B_V(\varphi(a), \varphi(b), \varphi(c)), \quad \varphi \circ \varepsilon = \sigma \circ \varphi,$$

where $a, b, c \in U$. Moreover, if φ is bijective, φ is called an *isomorphism*.

Proposition 4.2. Let (U, ε), (V, σ) be FKTA's and φ a homomorphism of U onto V.

(1) If I is an (ε-invariant) ideal of (U, ε), then $\varphi(I)$ is a (σ-invariant) ideal of (V, σ).

(2) $\mathrm{Ker}\,\varphi$ is an ε-invariant ideal of (U, ε).

(3) $(U/\mathrm{Ker}\,\varphi, \overline{\varepsilon}) \cong (V, \sigma)$, where $\overline{\varepsilon}$ is an automorphism of $U/\mathrm{Ker}\,\varphi$ induced from ε.

Proof. (1), (2) are clear.

(3) We denote by $(\ ,\ ,\ ), <\ ,\ ,\ >$ the triple products of U, V respectively, and put $N = \mathrm{Ker}\,\varepsilon$. Then $(U/N, \overline{\varepsilon})$ becomes an FKTA with triple product $(\overline{a}\overline{b}\overline{c}) := \overline{(abc)}$, where $\overline{a} = a + N$ $(a \in U)$ ([10] Lemma 3.1). The canonical mapping $\overline{\varphi} : U/N \longrightarrow V$, $\overline{\varphi}(\overline{a}) = \varphi(a)$ is bijective.

Moreover we have

$$\overline{\varphi}(\overline{a}\overline{b}\overline{c}) = \overline{\varphi}(\overline{(abc)}) = \varphi((abc)) =< \varphi(a)\,\varphi(b)\,\varphi(c) >=< \overline{\varphi}(\overline{a})\,\overline{\varphi}(\overline{b})\,\overline{\varphi}(\overline{c}) >,$$
$$\overline{\varphi}(\overline{\varepsilon}(\overline{a})) = \overline{\varphi}(\overline{\varepsilon a}) = \varphi(\varepsilon a) = \sigma\varphi(a) = \sigma\overline{\varphi}(\overline{a})$$

for all $a, b, c \in U$. Therefore $\overline{\varphi}$ is an isomorphism of $(U/N, \overline{\varepsilon})$ onto (V, σ). □

Definition. Let (V,σ), (W,τ) and (U,ε) be FKTA's over the same base field. (W,τ) is called an *extension* of (U,ε) by (V,σ) if there exists a short exact sequence of FKTA's:

$$\{0\} \longrightarrow (V,\sigma) \xrightarrow{\iota} (W,\tau) \xrightarrow{\pi} (U,\varepsilon) \longrightarrow \{0\}.$$

Two extensions (W,τ) and (W',τ') of (U,ε) by (V,σ) are said to be *equivalent* if there exists an isomorphism φ of (W,τ) onto (W',τ') such that the following diagram is commutative:

$$\begin{array}{ccccccccc} \{0\} & \longrightarrow & (V,\sigma) & \xrightarrow{\iota} & (W,\tau) & \xrightarrow{\pi} & (U,\varepsilon) & \longrightarrow & \{0\} \\ & & \downarrow \mathrm{Id} & & \downarrow \varphi & & \downarrow \mathrm{Id} & & . \\ \{0\} & \longrightarrow & (V,\sigma) & \xrightarrow{\iota'} & (W',\tau') & \xrightarrow{\pi'} & (U,\varepsilon) & \longrightarrow & \{0\} \end{array}$$

An ideal I of an FKTA (U,ε) is said to be *abelian* if $(IIU) = (IUI) = (UII) = 0$. We consider an extension (W,τ) of (U,ε) by (V,σ) such that $\iota(V)$ is an abelian ideal in (W,τ). Such an extension is called an *abelian extension.* Let $\{\ ,\ ,\ \}$ and $(\ ,\ ,\ )$ be the triple products of W and U respectively, and denote the bilinear mappings L, M, R and K of U by L_U, M_U, R_U and K_U respectively. Since $\iota(V)$ is the abelian ideal of (W,τ), we can define bilinear mappings λ, μ and ρ of U into $\mathrm{End}(V)$ by the following formulas:

$$\begin{aligned} \lambda(a,b)x &:= \iota^{-1}(\{s\,t\,\iota(x)\}) = \iota^{-1}L_W(s,t)\iota(x), \\ \mu(a,b)x &:= \iota^{-1}(\{s\,\iota(x)\,t\}) = \iota^{-1}M_W(s,t)\iota(x), \\ \rho(a,b)x &:= \iota^{-1}(\{\iota(x)\,s\,t\}) = \iota^{-1}R_W(s,t)\iota(x), \end{aligned} \tag{4.6}$$

where $a, b \in U$, $x \in V$ and $s, t \in W$ such that $\pi(s) = a$, $\pi(t) = b$. Then (λ,μ,ρ) becomes a representation of (U,ε) into (V,τ) since (L_W, M_W, R_W) is the representation of (W,τ) into itself. Let (W',τ') be another abelian extension of (U,ε) by (V,σ) which is equivalent to (W,τ). Then we shall show that the representation (λ',μ',ρ') defined by (4.6) coincides with (λ,μ,ρ). Let φ be an isomorphism of (W,τ) onto (W',τ'). For $a, b \in U$, choose $s, t \in W$ such that $\pi(s) = a$, $\pi(t) = b$. Then, since $\pi'(\varphi(s)) = a$, $\pi'(\varphi(t)) = b$, $\lambda'(a,b)x = \iota'^{-1}(L_{W'}(\varphi(s),\varphi(t))\iota'(x)) = \iota'^{-1}(\varphi L_W(s,t)\iota(x)) = \iota^{-1}(L_W(s,t)\iota(x)) = \lambda(a,b)x$. Similarly we have that $\mu' = \mu$, $\rho' = \rho$. For the simplicity, we identify V with its image $\iota(V)$ by the injection ι hereafter. Let l be a linear mapping of U into W such that $\pi \circ l = \mathrm{Id}$ and $\tau \circ l = l \circ \varepsilon$. Such a mapping l is called a *section.* (W,τ) is called a *modularly split extension* if there exists a section l. Put

$$f(a,b,c) = \{l(a)\,l(b)\,l(c)\} - l((abc)) \tag{4.7}$$

for $a, b, c \in U$, then f is a trilinear mapping of $U \times U \times U$ into V. We shall verify that $f \in C^3(U,V)$. Obviously, f satisfies the condition (4.2). From (4.7), we have

(4.8) $l((abc)) = L_W(l(a),l(b))l(c) - f(a,b,c),$

(4.9) $l(K_U(a,b)c) = K_W(l(a),l(b))l(c) - f(a,c,b) + f(b,c,a)$

for all $a, b, c \in U$. Using these identities and the condition $l \circ \varepsilon = \tau \circ l$, we obtain

(4.10) $\{l(K_U(a_1,a_2)a_3)\,l(a_5)\,l(a_4)\}$

$$= \{(K_W(l(a_1), l(a_2))l(a_3))\, l(a_5)\, l(a_4)\} - \rho(a_5, a_4)(f(a_1, a_3, a_2) - f(a_2, a_3, a_1)),$$

$$\{l(a_4)\, l(a_5)\, l(K_U(a_1, a_2)a_3)\} \tag{4.11}$$
$$= \{l(a_5)\, l(a_4)\, (K_W(l(a_1), l(a_2))l(a_3))\} - \lambda(a_4, a_5)(f(a_1, a_3, a_2) - f(a_2, a_3, a_1)),$$

$$\{l(a_1)\, l((a_3 a_4 a_5))\, l(a_2)\} = \{l(a_1)\, (L_W(l(a_3), l(a_4))l(a_5))\, l(a_2)\} - \mu(a_1, a_2) f(a_3, a_4, a_5), \tag{4.12}$$

where $a_i \in U$ $(i = 1, 2, 3, 4, 5)$. From these identities and the conditions $l \circ \varepsilon = \tau \circ l$ and (U2), we get

$$\begin{aligned}
&f(K_U(a_1, a_2)a_3, a_5, a_4) - f(a_4, a_5, K_U(a_1, a_2)a_3) - f(a_4, a_3, K_U(a_1, a_2)a_5)\\
&-f(a_1, (a_3\, \varepsilon a_4\, a_5), a_2) + f(a_2, (a_3\, \varepsilon a_4\, a_5), a_1)\\
&= \{l(K_U(a_1, a_2)a_3)\, l(a_5)\, l(a_4)\} - \{l(a_4)\, l(a_5)\, l(K_U(a_1, a_2)a_3)\} - \{l(a_4)\, l(a_3)\, l(K_U(a_1, a_2)a_5)\}\\
&-\{l(a_1)\, l((a_3\, \varepsilon a_4\, a_5))\, l(a_2)\} + \{l(a_2)\, l((a_3\, \varepsilon a_4\, a_5))\, l(a_1)\}\\
&= \{(K_W(l(a_1), l(a_2))l(a_3))\, l(a_5)\, l(a_4)\} - \rho(a_5, a_4)(f(a_1, a_3, a_2) - f(a_2, a_3, a_1))\\
&-\{l(a_4)\, l(a_5)\, (K_W(l(a_1), l(a_2))l(a_3))\} + \lambda(a_4, a_5)(f(a_1, a_3, a_2) - f(a_2, a_3, a_1))\\
&-\{l(a_4)\, l(a_3)\, (K_W(l(a_1), l(a_2))l(a_5))\} - \lambda(a_4, a_3)(f(a_1, a_5, a_2) - f(a_2, a_5, a_1))\\
&-\{l(a_1)\, (L_W(l(a_3), \tau l(a_4))l(a_5))\, l(a_2)\} + \mu(a_1, a_2) f(a_3, \varepsilon a_4, a_5)\\
&+\{l(a_2)\, (L_W(l(a_3), \tau l(a_4))l(a_5))\, l(a_1)\} - \mu(a_2, a_1) f(a_3, \varepsilon a_4, a_5)\\
&= -\phi(a_5, a_4)(f(a_1, a_3, a_2) - f(a_2, a_3, a_1)) + \lambda(a_4, a_3)(f(a_1, a_5, a_2) - f(a_2, a_5, a_1))\\
&+\kappa(a_1, a_2) f(a_3, \varepsilon a_4, a_5)
\end{aligned}$$

Hence f satisfies the condition (4.3).

Now we identify $V \times U$ and W as vector spaces by $(x, a) \mapsto x + l(a)$. An element (x, a) of $V \times U$ is also denoted as $\begin{pmatrix} x \\ a \end{pmatrix}$ in column vector form. In the FKTA (W, τ), it holds that

$$\begin{aligned}
&\{x + l(a) \quad y + l(b) \quad z + l(c)\}\\
&\qquad = \{x\, l(b)\, l(c)\} + \{l(a)\, y\, l(c)\} + \{l(a)\, l(b)\, z\} + f(a, b, c) + l((abc))
\end{aligned}$$

for all $x, y, z \in V,\ a, b, c \in U$. From this we can define a triple product of $V \times U$ by

$$\left\{\begin{pmatrix} x \\ a \end{pmatrix}\begin{pmatrix} y \\ b \end{pmatrix}\begin{pmatrix} z \\ c \end{pmatrix}\right\} := \begin{pmatrix} \rho(b, c)x + \mu(a, c)y + \lambda(a, b)z + f(a, b, c) \\ (abc) \end{pmatrix}. \tag{4.13}$$

Using the conditions $\pi \circ \tau = \varepsilon \circ \pi,\ l \circ \varepsilon = \tau \circ l$, we see that $\sigma \times \varepsilon : (x, a) \mapsto (\sigma x, \varepsilon a)$ is an automorphism of $V \times U$ corresponding to the automorphism τ of W. For $X_i = (x_i, a_i) \in V \times U$ $(i = 1, 2, 3, 4, 5)$, put

$$(x, a) = ([L(X_1, X_2), L(X_3, X_4)] - L(L(X_1, X_2)X_3, X_4) + L(X_3, L(X_2, (\sigma \times \varepsilon)X_1)X_4))X_5,$$

where $L(X, Y)$ is the left multiplication of $V \times U$. Using (3.5), (3.6), (3.7), (3.8), (3.9) and (U2) (see the proof of Proposition 3.1), we have $a = 0$ and

$$\begin{aligned}
x = &-\rho(a_4, a_5) f(a_1, a_2, a_3) + \mu(a_3, a_5) f(a_2, \varepsilon a_1, a_4) + \lambda(a_1, a_2) f(a_3, a_4, a_5)\\
&-\lambda(a_3, a_4) f(a_1, a_2, a_5) - f((a_1 a_2 a_3), a_4, a_5) + f(a_3, (a_2\, \varepsilon a_1\, a_4), a_5)\\
&-f(a_3, a_4, (a_1 a_2 a_5)) + f(a_1, a_2, (a_3 a_4 a_5))\\
&= \delta^3 f(a_1, a_2, a_3, a_4, a_5).
\end{aligned}$$

This means that f is a cocycle of order 3, that is, $f \in Z^3(U, V)$. Assume that there exists another section l'. Put $g(a) = l'(a) - l(a)$, then $g(a) \in V$ and $g(\varepsilon a) = \tau g(a)$, therefore $g \in C^1(U, V)$. Since V is abelian,

$$\begin{aligned}
f'(a,b,c) &= \{l'(a)\, l'(b)\, l'(c)\} - l'((abc)) \\
&= \{l(a)\, l(b)\, l(c)\} + \{l(a)\, l(b)\, g(c)\} + \{l(a)\, g(b)\, l(c)\} + \{g(a)\, l(b)\, l(c)\} \\
&\quad -l((abc)) - g((abc)) \\
&= f(a,b,c) + \lambda(a,b)g(c) + \mu(a,c)g(b) + \rho(b,c)g(a) - g((abc)) \\
&= f(a,b,c) - \delta^1 g(a,b,c)
\end{aligned}$$

for all $a, b, c \in U$. Therefore the cohomology class of f does not depend on the choice of the section l, hence the modularly split extention of (U, ε) by abelian (V, σ) which has the section l determines uniquely an element of $H^3(U, V)$. Two equivalent extensions define the same element of $H^3(U, V)$.

Conversely, let (V, σ) be an abelian FKTA, and (λ, μ, ρ) a representation of an FKTA (U, ε) into (V, σ). Let f be a cocycle of order 3. We define a triple product on a vector space $W = V \times U$ by (4.13). Then $\tau = \sigma \times \varepsilon$ is an automorphism of the triple system W, and (W, τ) becomes an FKTA. Next, we define the short exact sequence

$$\{0\} \longrightarrow (V, \sigma) \xrightarrow{\iota} (W, \tau) \xrightarrow{\pi} (U, \varepsilon) \longrightarrow \{0\}.$$

by $\iota(x) = (x, 0)$ and $\pi(x, a) = a$ $(x \in V,\ a \in U)$. It is clear that ι and τ are homomolphisms. Therefore (W, τ) is an extension of (U, ε) by (V, σ). Moreover it is easy to see that V is abelian ideal in (W, τ). We define a linear mapping l of U into W by $l(a) = (0, a)$. Then we have

$$\{l(a)\, l(b)\, l(c)\} - l((abc)) = (f(a,b,c), 0), \quad l(\varepsilon a) = \tau l(a)$$

for $a, b, c \in U$. This means that f is a cocycle defined by this extension. Therefore to each element of $Z^3(U, V)$ corresponds an extension of (U, ε) by abelian (V, σ).

Summarizing the above results, we have

Theorem 4.3. To each equivalent class of modularly split extensions (W, τ) of an FKTA (U, ε) by abelian (V, σ) corresponds an element of $H^3(U, V)$. Let (λ, μ, ρ) be a representation of an FKTA (U, ε) into a vector space V with a non-singular endomorphism E of V, then there exists an extension of (W, τ) of (U, ε) by (V, E) such that (V, E) is abelian in (W, τ).

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