INTUITIONISTIC ASPECTS OF FUZZY TOPOLOGICAL BCH-ALGEBRAS

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ABSTRACT. The purpose of this paper is to generalize the concept of fuzzy topological subalgebras in BCH-algebras, initiated by the second author [9], to the case of intuitionistic fuzzy sets.

1. Introduction

In 1983, Q. P. Hu et al. introduced the notion of a BCH-algebra which is a generalization of a BCK/BCI-algebra (see [5, 6]). In [3], M. A. Chaudhry et al. discussed ideals and filters in BCH-algebras, and studied their properties. Y. B. Jun [7] considered the fuzzification of closed ideals and filters in BCH-algebras, and then he described the relation among fuzzy subalgebras, fuzzy closed ideals and fuzzy filters. After the introduction of fuzzy sets by L. A. Zadeh [10], several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of *intuitionistic fuzzy set* was first published by K. T. Atanassov [1], as a generalization of the notion of fuzzy sets. In [8], as a generalization of the concept of fuzzy closed ideals in BCH-algebras, Y. B. Jun and W. A. Dudek considered the intuitionistic fuzzification of subalgebras and closed ideals, and investigate some of their properties. They established the relation between an intuitionistic fuzzy subalgebra and an intuitionistic fuzzy closed ideal, and stated a condition for an intuitionistic fuzzy subalgebra to be an intuitionistic fuzzy closed ideal. They also gave characterizations of intuitionistic fuzzy closed ideals. The aim of this paper is to generalize the concept of fuzzy topological BCH-algebras, initiated by the second author [9], to the case of intuitionistic fuzzy sets. We show that the homomorphic image and preimage of an intuitionistic fuzzy topological BCH-algebra is an intuitionistic fuzzy topological BCH-algebra.

2. Preliminaries

An algebraic system $(X; *, 0)$ of type (2, 0) is called a *BCH-algebra* if it satisfies the following conditions:

(H1) $x * x = 0$,

(H2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(H3) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$. A nonempty subset S of a BCH-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of BCH-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Definition 2.1. (Atanassov [1]) An *intuitionistic fuzzy set* (IFS for short) A in X is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$$

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where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \gamma_A(x) \le 1$ for each $x \in X$.

For the sake of simplicity, we shall use the notation $A = \langle x, \mu_A, \gamma_A \rangle$ instead of $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$. The IFSs $0_\sim$ and $1_\sim$ are defined to be $0_\sim = \langle x, 0, 1\rangle$ and $1_\sim = \langle x, 1, 0\rangle$, respectively.

Let f be a mapping from a set X to a set Y. If

$$B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle : y \in Y\}$$

is an IFS in Y, then the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\},$$

and if $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ is an IFS in X, then the *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by

$$f(A) = \{\langle y, f_{\sup}(\mu_A)(y), f_{\inf}(\gamma_A)(y)\rangle : y \in Y\},$$

where

$$f_{\sup}(\mu_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \ne \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_{\inf}(\gamma_A)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \gamma_A(x), & \text{if } f^{-1}(y) \ne \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

for each $y \in Y$ (see Çoker [4]).

In [4], Çoker generalized the concept of fuzzy topological space, first initiated by Chang [2], to the case of intuitionistic fuzzy sets as follows.

Definition 2.2. (Çoker [4, Definition 3.1]) An *intuitionistic fuzzy topology* (IFT for short) on a nonempty set X is a family Φ of IFSs in X satisfying the following axioms:

(T1) $0_\sim, 1_\sim \in \Phi$,

(T2) $G_1 \cap G_2 \in \Phi$ for any $G_1, G_2 \in \Phi$,

(T3) $\bigcup\limits_{i \in J} G_i \in \Phi$ for any family $\{G_i : i \in J\} \subseteq \Phi$.

In this case the pair (X, Φ) is called an *intuitionistic fuzzy topological space* (IFTS for short) and any IFS in Φ is called an *intuitionistic fuzzy open set* (IFOS for short) in X.

Definition 2.3. (Çoker [4, Definitions 4.1 and 4.2]) Let (X, Φ) and (Y, Ψ) be two IFTSs. A mapping $f : X \to Y$ is said to be *intuitionistic fuzzy coninuous* if the preimage of each IFS in Ψ is an IFS in Φ; and f is said to be *intuitionistic fuzzy open* if the image of each IFS in Φ is an IFS in Ψ.

3. Intuitionistic fuzzy topological BCH-algebras

Definition 3.1. Let A be an IFS in an IFTS (X, Φ). Then the *induced intuitionistic fuzzy topology* (IIFT for short) on A is the family of IFSs in A which are the intersection with A of IFOSs in X. The IIFT is denoted by Φ_A, and the pair (A, Φ_A) is called an *intuitionistic fuzzy subspace* of (X, Φ).

Definition 3.2. Let (A, Φ_A) and (B, Ψ_B) be intuitionistic fuzzy subspaces of IFTSs (X, Φ) and (Y, Ψ), respectively, and let $f : X \to Y$ be a mapping. Then f is a mapping of A into B if $f(A) \subset B$. Furthermore, f is said to be *relatively intuitionistic fuzzy continuous* if for each IFS V_B in Ψ_B, the intersection $f^{-1}(V_B) \cap A$ is an IFS in Φ_A; and f is said to be *relatively intuitionistic fuzzy open* if for each IFS U_A in Φ_A, the image $f(U_A)$ is an IFS in Ψ_B.

Proposition 3.3. *Let (A, Φ_A) and (B, Ψ_B) be intuitionistic fuzzy subspaces of IFTSs (X, Φ) and (Y, Ψ) respectively, and let f be an intuitionistic fuzzy continuous mapping of X into Y such that $f(A) \subset B$. Then f is relatively intuitionistic fuzzy continuous mapping of A into B.*

Proof. Let V_B be an IFS in Ψ_B. Then there exists $V \in \Psi$ such that $V_B = V \cap B$. Since f is intuitionistic fuzzy continuous, it follows that $f^{-1}(V)$ is an IFS in Φ. Hence

$$f^{-1}(V_B) \cap A = f^{-1}(V \cap B) \cap A = f^{-1}(V) \cap f^{-1}(B) \cap A = f^{-1}(V) \cap A$$

is an IFS in Φ_A. This completes the proof. □

Definition 3.4. Let X be a BCH-algebra. An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in X is called an *intuitionistic fuzzy BCH-algebra* if it satisfies:

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \gamma_A(x * y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$$

for all $x, y \in X$.

Example 3.5 Consider a proper BCH-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	0	d
c	c	c	c	0	d
d	d	d	d	d	0

Let $A = \langle x, \mu_A(x), \gamma_A(x) \rangle$ be an IFS in X defined by

$$\mu_A(0) = \mu_A(a) = \mu_A(b) = \mu_A(d) = 0.6 > 0.3 = \mu_A(c)$$

and

$$\gamma_A(0) = \gamma_A(a) = \gamma_A(b) = \gamma_A(d) = 0.03 < 0.06 = \gamma_A(c).$$

Then $A = \langle x, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy BCH-algebra.

For any BCH-algebra X and any element $a \in X$ we use a_r denote the selfmap of X defined by $a_r(x) = x * a$ for all $x \in X$.

Definition 3.6. Let X be a BCH-algebra, Φ an IFT on X and A an intuitionistic fuzzy BCH-algebra with IIFT Φ_A. Then A is called an *intuitionistic fuzzy topological BCH-algebra* if for each $a \in X$ the mapping $a_r : (A, \Phi_A) \to (A, \Phi_A)$, $x \mapsto x * a$, is relatively intuitionistic fuzzy continuous.

Theorem 3.7. *Given BCH-algebras X and Y, and a homomorphism $f : X \to Y$, let Φ and Ψ be the IFTs on X and Y respectively such that $\Phi = f^{-1}(\Psi)$. If B is an intuitionistic fuzzy topological BCH-algebra in Y, then $f^{-1}(B)$ is an intuitionistic fuzzy topological BCH-algebra in X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}\mu_{f^{-1}(B)}(x*y) &= \mu_B(f(x*y)) = \mu_B(f(x)*f(y)) \\ &\geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\ &= \min\{\mu_{f^{-1}(B)}(x), \mu_{f^{-1}(B)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\gamma_{f^{-1}(B)}(x*y) &= \gamma_B(f(x*y)) = \gamma_B(f(x)*f(y)) \\ &\leq \max\{\gamma_B(f(x)), \gamma_B(f(y))\} \\ &= \max\{\gamma_{f^{-1}(B)}(x), \gamma_{f^{-1}(B)}(y)\}.\end{aligned}$$

Hence $f^{-1}(B)$ is an intuitionistic fuzzy BCH-algebra in X. Now let $a \in X$ and let U be an IFS in $\Phi_{f^{-1}(B)}$. Since f is an intuitionistic fuzzy continuous mapping of (X, Φ) into (Y, Ψ), it follows from Proposition 3.3 that f is a relatively intuitionistic fuzzy continuous mapping of $(f^{-1}(B), \Phi_{f^{-1}(B)})$ into (B, Ψ_B). Note that there exists an IFS V in Ψ_B such that $f^{-1}(V) = U$. Then

$$\begin{aligned}\mu_{a_r^{-1}(U)}(x) &= \mu_U(a_r(x)) = \mu_U(x*a) = \mu_{f^{-1}(V)}(x*a) \\ &= \mu_V(f(x*a)) = \mu_V(f(x)*f(a))\end{aligned}$$

and

$$\begin{aligned}\gamma_{a_r^{-1}(U)}(x) &= \gamma_U(a_r(x)) = \gamma_U(x*a) = \gamma_{f^{-1}(V)}(x*a) \\ &= \gamma_V(f(x*a)) = \gamma_V(f(x)*f(a)).\end{aligned}$$

Since B is an intuitionistic fuzzy topological BCH-algebra in Y, the mapping

$$b_r : (B, \Psi_B) \to (B, \Psi_B),\ y \mapsto y*b$$

is relatively intuitionistic fuzzy continuous for every $b \in Y$. Hence

$$\begin{aligned}\mu_{a_r^{-1}(U)}(x) &= \mu_V(f(x)*f(a)) = \mu_V(f(a)_r(f(x))) \\ &= \mu_{f(a)_r^{-1}(V)}(f(x)) = \mu_{f^{-1}(f(a)_r^{-1}(V))}(x)\end{aligned}$$

and

$$\begin{aligned}\gamma_{a_r^{-1}(U)}(x) &= \gamma_V(f(x)*f(a)) = \gamma_V(f(a)_r(f(x))) \\ &= \gamma_{f(a)_r^{-1}(V)}(f(x)) = \gamma_{f^{-1}(f(a)_r^{-1}(V))}(x).\end{aligned}$$

Therefore $a_r^{-1}(U) = f^{-1}(f(a)_r^{-1}(V))$, and so

$$a_r^{-1}(U) \cap f^{-1}(B) = f^{-1}(f(a)_r^{-1}(V)) \cap f^{-1}(B)$$

is an IFS in $\Phi_{f^{-1}(B)}$. This completes the proof. □

Theorem 3.8. *Given BCH-algebras X and Y, and an isomorphism f of X to Y, let Φ and Ψ be the IFTs on X and Y respectively such that $f(\Phi) = \Psi$. If A is an intuitionistic fuzzy topological BCH-algebra in X, then $f(A)$ is an intuitionistic fuzzy topological BCH-algebra in Y.*

Proof. Let $A = (x, \mu_A, \gamma_A)$ be an intuitionistic fuzzy topological BCH-algebra in X and let $y_1, y_2 \in Y$. Noticing that

$$\{x_1 * x_2 : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X : x \in f^{-1}(y_1 * y_2)\},$$

we have

$$\begin{aligned}
& f_{\sup}(\mu_A)(y_1 * y_2) \\
= \ & \sup\{\mu_A(x) : x \in f^{-1}(y_1 * y_2)\} \\
\geq \ & \sup\{\mu_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\
\geq \ & \sup\Big\{\min\{\mu_A(x_1), \mu_A(x_2)\} : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\Big\} \\
= \ & \min\Big\{\sup\{\mu_A(x_1) : x_1 \in f^{-1}(y_1)\}, \sup\{\mu_A(x_2) : x_2 \in f^{-1}(y_2)\}\Big\} \\
= \ & \min\{f_{\sup}(\mu_A)(y_1), f_{\sup}(\mu_A)(y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
& f_{\inf}(\gamma_A)(y_1 * y_2) \\
= \ & \inf\{\gamma_A(x) : x \in f^{-1}(y_1 * y_2)\} \\
\leq \ & \inf\{\gamma_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\
\leq \ & \inf\Big\{\max\{\gamma_A(x_1), \gamma_A(x_2)\} : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\}\Big\} \\
= \ & \max\Big\{\inf\{\gamma_A(x_1) : x_1 \in f^{-1}(y_1)\}, \inf\{\gamma_A(x_2) \mid x_2 \in f^{-1}(y_2)\}\Big\} \\
= \ & \max\{f_{\inf}(\gamma_A)(y_1), f_{\inf}(\gamma_A)(y_2)\}.
\end{aligned}$$

Hence $f(A) = (y, f_{\sup}(\mu_A), f_{\inf}(\gamma_A))$ is an intuitionistic fuzzy BCH-algebra in Y. Now we show that the mapping

$$b_r : (f(A), \Psi_{f(A)}) \to (f(A), \Psi_{f(A)}),\ y \mapsto y * b$$

is relatively intuitionistic fuzzy continuous for each $b \in Y$. Let U_A be an IFS in Φ_A. Then there exists an IFS U in Φ such that $U_A = U \cap A$. Since f is one-one, it follows that

$$f(U_A) = f(U \cap A) = f(U) \cap f(A)$$

which is an IFS in $\Psi_{f(A)}$. This shows that f is relatively intuitionistic fuzzy open. Let $V_{f(A)}$ be an IFS in $\Psi_{f(A)}$. The sujectivity of f implies that for each $b \in Y$ there exists $a \in X$ such that $b = f(a)$. Hence

$$\begin{aligned}
\mu_{f^{-1}(b_r^{-1}(V_{f(A)}))}(x) &= \mu_{f^{-1}(f(a)_r^{-1}(V_{f(A)}))}(x) = \mu_{f(a)_r^{-1}(V_{f(A)})}(f(x)) \\
&= \mu_{V_{f(A)}}(f(a)_r(f(x))) = \mu_{V_{f(A)}}(f(x) * f(a)) \\
&= \mu_{V_{f(A)}}(f(x * a)) = \mu_{f^{-1}(V_{f(A)})}(x * a) \\
&= \mu_{f^{-1}(V_{f(A)})}(a_r(x)) = \mu_{a_r^{-1}(f^{-1}(V_{f(A)}))}(x)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{f^{-1}(b_r^{-1}(V_{f(A)}))}(x) &= \gamma_{f^{-1}(f(a)_r^{-1}(V_{f(A)}))}(x) = \gamma_{f(a)_r^{-1}(V_{f(A)})}(f(x)) \\
&= \gamma_{V_{f(A)}}(f(a)_r(f(x))) = \gamma_{V_{f(A)}}(f(x) * f(a)) \\
&= \gamma_{V_{f(A)}}(f(x * a)) = \gamma_{f^{-1}(V_{f(A)})}(x * a) \\
&= \gamma_{f^{-1}(V_{f(A)})}(a_r(x)) = \gamma_{a_r^{-1}(f^{-1}(V_{f(A)}))}(x).
\end{aligned}$$

Therefore $f^{-1}(b_r^{-1}(V_{f(A)}) = a_r^{-1}(f^{-1}(V_{f(A)}))$. By hypothesis, the mapping

$$a_r : (A, \Phi_A) \to (A, \Phi_A),\ x \mapsto x * a$$

is relatively intuitionistic fuzzy continuous and f is a relatively intuitionistic fuzzy continuous map: $(A, \Phi_A) \to (f(A), \Psi_{f(A)})$. Thus

$$f^{-1}(b_r^{-1}(V_{f(A)})) \cap A = a_r^{-1}(f^{-1}(V_{f(A)})) \cap A$$

is an IFS in Φ_A. Since f is relatively intuitionistic fuzzy open,

$$f(f^{-1}(b_r^{-1}(V_{f(A)})) \cap A) = b_r^{-1}(V_{f(A)}) \cap f(A)$$

is an IFS in $\Psi_{f(A)}$. This completes the proof. □

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