HILBERT-SCHMIDT INTERPOLATION FOR VECTORS IN TRIDIAGONAL ALGEBRA

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Abstract. Given vectors x and y in a Hilbert space, an interpolating operator is a bounded operator T such that $Tx = y$. An interpolating operator for n vectors satisfies the equation $Tx_i = y_i$, for $i = 1, 2, \cdots, n$. In this article, we investigate Hilbert-Schmidt interpolation problems for vectors x and y in tridiagonal algebras.

1. Introduction

Let $\mathcal{A}$ be a subalgebra the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. An *interpolation question* for $\mathcal{A}$ asks for which x and y is there a bounded operator $A \in \mathcal{A}$ such that $Ax = y$. A variation, the 'n-vector interpolation problem', asks for an operator A such that $Ax_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison[9]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance[10]: his result was extended by Hopenwasser[5] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[11] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[6] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors, although necessity was not proved in that paper.

Roughly speaking, when an operator maps one thing to another, we think of the operator as the interpolating operator and the equation representing the mapping as the interpolation equation.

In this article, we investigate Hilbert-Schmidt interpolation problem for vectors in tridiagonal algebras.

First, we establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection

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invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The algebra Alg$\mathcal{L}$ is the set of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $< x, y >$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let N be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}], [e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson[3]. These algebras have been found to be useful counterexample to a number of plausible conjectures. Recently, such algebras have been found to be of use in physics, in electrical engineering and in general system theory.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$. Let D=$\{A : A$ is a diagonal operator acting on $\mathcal{H}\}$. Then D is a masa of Agl$\mathcal{L}$ and $\mathcal{D}$=(Alg$\mathcal{L}$)$\cap$ (Alg$\mathcal{L}$)*, where (Alg$\mathcal{L}$)$^* = \{A^* : A \in$Alg$\mathcal{L}\}$.

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators acting on $\mathcal{H}$.

In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

The following theorem is well-known.

Theorem 1 [4]**.** *Let A be a diagonal operator in $\mathcal{B}(\mathcal{H})$ with diagonal $\{a_n\}$. Then A is a Hilbert-Schmidt operator if and only if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.*

Theorem 2. *Let $x = (x_i)$ and $y = (y_i)$ be two vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots$. Then the following statements are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, A is Hilbert-Schmidt and every E in $\mathcal{L}$ reduces A.

(2) $\sup \left\{ \dfrac{\| \sum_{k=1}^{l} \alpha_k E_k y \|}{\| \sum_{k=1}^{l} \alpha_k E_k x \|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L} \right\} < \infty$ *and* $\displaystyle\sum_{n=1}^{\infty} |y_n|^2 |x_n|^{-2} < \infty$.

Proof. (2) $\Rightarrow$ (1). If $\sup \left\{ \dfrac{\| \sum_{k=1}^{l} \alpha_k E_k y \|}{\| \sum_{k=1}^{l} \alpha_k E_k x \|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L} \right\} < \infty$, then, without loss of generality, we may assume that

$\sup\left\{\dfrac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} = 1$. So $\|\sum_{k=1}^{l}\alpha_k E_k y\| \leq \|\sum_{k=1}^{l}\alpha_k E_k x\|$, $l \in N, \alpha_k \in \mathbb{C}$ and $E_k \in \mathcal{L} \cdots (*)$. Let $\mathcal{M} = \left\{\sum_{k=1}^{l}\alpha_k E_k x : l \in N,\ \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\}$. Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \longrightarrow \mathcal{H}$ by $A(\sum_{k=1}^{l}\alpha_k E_k x) = \sum_{k=1}^{l}\alpha_k E_k y$. Then A is well-defined. For, if $\sum_{k=1}^{l}\alpha_k E_k x = \sum_{j=1}^{t}\beta_j E_j x$, then $\sum_{k=1}^{l}\alpha_k E_k x + \sum_{j=1}^{t}(-\beta_j)E_j x = 0$. So $\|\sum_{k=1}^{l}\alpha_k E_k x + \sum_{j=1}^{t}(-\beta_j)E_j x\| = 0$ and hence $\|\sum_{k=1}^{l}\alpha_k E_k y + \sum_{j=1}^{t}(-\beta_j)E_j y = 0$ by $(*)$. Thus $\sum_{k=1}^{l}\alpha_k E_k y = \sum_{j=1}^{t}\beta_j E_j y$ and hence A is well-defined. Extend A to $\overline{\mathcal{M}}$ by continuity and define $A|_{\overline{\mathcal{M}}^{\perp}} = 0$. Clearly, $Ax = y$ and $\|A\| \leq 1$. Since $EA(\sum_{k=1}^{l}\alpha_k E_k x) = \sum_{k=1}^{l}\alpha_k EE_k y$, $AE(\sum_{k=1}^{l}\alpha_k E_k x) = A(\sum_{k=1}^{l}\alpha_k EE_k x) = \sum_{k=1}^{l}\alpha_k EE_k y$, $EAg = E0 = 0$ and $AEg = 0$ $(< Eg, \sum_{k=1}^{l}\alpha_k E_k x > = < g, \sum_{k=1}^{l}\alpha_k EE_k x > = 0)$ for g in $\overline{\mathcal{M}}^{\perp}$, every E in $\mathcal{L}$ reduces A. Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $A = (a_{ii})$ is diagonal and $Ax = y$, $a_{ii}x_i = y_i$ for all $i = 1, 2, \cdots$. Since $\sum_{n=1}^{\infty}|y_n|^2|x_n|^{-2} < \infty$, A is Hilbert-Schmidt.

(1) $\Rightarrow$ (2). Since $Ax = y$ and every E in $\mathcal{L}$ reduces A, $AEx = Ey$ for every E in $\mathcal{L}$. Then $A(\sum_{k=1}^{l}\alpha_k E_k x) = \sum_{k=1}^{l}\alpha_k E_k y$ for every $l \in N$, every $\alpha_k \in \mathbb{C}$ and every $E_k \in \mathcal{L}$. Thus $\|\sum_{k=1}^{l}\alpha_k E_k y\| \leq \|A\|\|\sum_{k=1}^{l}\alpha_k E_k x\|$. If $\|\sum_{k=1}^{l}\alpha_k E_k x\| \neq 0$, then $\dfrac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} \leq \|A\|$.
$\sup\left\{\dfrac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$. Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $Ax = y$, $y_i = a_{ii}x_i$ and hence $a_{ii} = y_i {x_i}^{-1}$ for all $i = 1, 2, \cdots$. Since A is a Hilbert-Schmidt operator, $\sum_{n=1}^{\infty}|y_n|^2|x_n|^{-2} < \infty$.

Theorem 3. *Let $x_p = (x_{pi})$ and $y_p = (y_{pi})$ be vectors in $\mathcal{H}$ such that $x_{qi} \neq 0$ for some fixed q and all $i = 1, 2, \cdots$. Then the following statements are equivalent.*

(1) There exists an operator A in $Alg\mathcal{L}$ such that $Ax_p = y_p$ $(p = 1, \cdots, n)$, every E in $\mathcal{L}$ reduces A and A is Hilbert-Schmidt.

(2) $\sup\left\{\dfrac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in N, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$ and $\sum_{i=1}^{\infty}|y_{qi}|^2|x_{qi}|^{-2} < \infty$.

Proof. If we assume that (1) holds, then $AEx_p = Ey_p$ since $Ax_p = y_p$ and every E in $\mathcal{L}$ reduces A. So $A(\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p) = \sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p$, $m_p \in N$, $l \leq n$, $E_{k,p} \in \mathcal{L}$ and $\alpha_{k,p} \in \mathbb{C}$. Thus $\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\| \leq \|A\|\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|$. If $\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\| \neq 0$, then $\dfrac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} \leq \|A\|$. Hence $\sup\left\{\dfrac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in N, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$. Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $Ax_p = y_p$, $y_{pi} = a_{ii}x_{pi}$ $(p = 1, 2, \cdots, n$ and $i = 1, 2, \cdots)$. Since $x_{qi} \neq 0$, $a_{ii} = y_{qi}{x_{qi}}^{-1}$ $(i = 1, 2, \cdots)$. Since A is a Hilbert-Schmidt operator, $\sum_{i=1}^{\infty} |y_{qi}|^2 |x_{qi}|^{-2} < \infty$. Conversely, if we suppose that (2) holds, then without loss of generality, we may assume that

$$\sup\left\{\frac{\| \sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} y_p \|}{\| \sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} x_p \|} : m_p \in N, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} = 1. \text{ Then}$$

$$\| \sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} y_p \| \leq \| \sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} x_p \|,\ m_p \in N, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \cdots (*).$$

Let $\mathcal{M} = \left\{ \sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} x_p : m_p \in N, l \leq n, \alpha_{k,p} \in \mathbb{C} \text{ and } E_{k,p} \in \mathcal{L} \right\}$.

Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \longrightarrow \mathcal{H}$ by $A(\sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} x_p) = \sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} y_p$. Then A is well-defined by $(*)$. Extend A to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^{\perp}} = 0$. $Ax_p = y_p$ $(p = 1, 2, \cdots, n)$ and $\|A\| \leq 1$. By an argument similar to that of proof of Theorem 2, every E in $\mathcal{L}$ reduces A. So A is a diagonal operator. Let $A = (a_{ii})$. Since $y_p = Ax_p$, $y_{pi} = a_{ii} x_{pi}$ $(p = 1, 2, \cdots, n$ and $i = 1, 2, \cdots)$. Since $\sum_{i=1}^{\infty} |y_{qi}|^2 |x_{qi}|^{-2} < \infty$, A is Hilbert-Schmidt.

If we modify the proof of Theorem 3, then we can get the following theorem.

Theorem 4. *Let $x_p = (x_{pi})$ and $y_p = (y_{pi})$ be vectors in $\mathcal{H}(p = 1, 2, \cdots)$ such that $x_{qi} \neq 0$ for all i and for some fixed q. Then the following statements are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ $(p = 1, \cdots)$, every E in $\mathcal{L}$ reduces A and A is Hilbert-Schmidt.

(2) $\sup\left\{\frac{\| \sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} y_p \|}{\| \sum_{k=1}^{m_p} \sum_{p=1}^{l} \alpha_{k,p} E_{k,p} x_p \|} : m_p,\ l \in N, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$ *and* $\sum_{i=1}^{\infty} |y_{qi}|^2 |x_{qi}|^{-2} < \infty$.

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