COUNTABLY INFINITE PRODUCTS OF SEQUENTIAL TOPOLOGIES

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ABSTRACT. The limit of an inverse sequence of sequentially compact sequential topologies is sequential of order not greater than the supremum of the sequential orders of the topologies of the sequence +1; if moreover the topologies are α_3, then the order is equal to that supremum. This implies almost countable productivity of the considered properties. This fact is used to construct a compact sequential topology of order 2, the countable power of which is sequential of order 2. On the other hand, the sequential order of the countably infinite power of a space containing a closed subset homeomorphic to the sequential fan, is ω_1.

1. INTRODUCTION

A property $\mathcal{P}$ is said to be *almost countably productive* provided that $\prod_{k=1}^{n} X_k \in \mathcal{P}$ for every $n \in \omega$ implies that $\prod_{k=1}^{\omega} X_k \in \mathcal{P}$. Clearly, if a
property is preserved by limits of inverse sequences, then it is almost countably productive.

In this paper we investigate the almost countable productivity of some properties related to convergent sequences, and more generally the preservation of such properties by the limits of inverse sequences of topological spaces.

Recall that the *sequential order* of a topology is the least ordinal σ for which the σ-th iterate $\mathrm{adh}_{\mathrm{Seq}}^{\sigma}$ of the sequential adherence is idempotent, where $\mathrm{adh}_{\mathrm{Seq}}^{0}\mathrm{A} = \mathrm{A}, \mathrm{adh}_{\mathrm{Seq}}^{1}\mathrm{A} = \mathrm{adh}_{\mathrm{Seq}}\mathrm{A} = \bigcup_{(\mathrm{x_n})\subset \mathrm{A}} \lim(\mathrm{x_n})_{\mathrm{n}}$, and if $\sigma > 1$,

$$\mathrm{adh}_{\mathrm{Seq}}^{\sigma}\mathrm{A} = \mathrm{adh}_{\mathrm{Seq}}\left(\bigcup_{\alpha<\sigma} \mathrm{adh}_{\mathrm{Seq}}^{\alpha}\mathrm{A}\right).$$

A topological space X is *sequential of order* β (in symbols, $\sigma(X) = \beta$) if β is the least ordinal such that $\mathrm{clA} = \mathrm{adh}_{\mathrm{Seq}}^{\beta}\mathrm{A}$ for every subset $A \subset X$.

We show that the property of being sequentially compact [1], α_3 [2], and sequential of order not greater than a given ordinal β, is preserved by the limits of inverse sequences, and thus is almost countably productive. T. Nogura showed that the properties α_3, α_2 and α_1 are preserved by limits of inverse sequences, and that they are countably productive [7].[3] On

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[1] Equivalently, countably compact, because of sequentiality.

[2] Property α_3 and other α_i properties will be defined later on.

[3] The same is true about $\alpha_{1.5}$ spaces.

the other hand, finite products of sequentially compact sequential topologies are sequential.[4] Therefore our result implies that if X_k is sequentially compact, α_3 and sequential for every k, then $\prod_{k=1}^{\omega} X_k$ is sequential and

$$\sigma(\prod_{k=1}^{\omega} X_k) = \sup\{\sigma(\prod_{k=1}^{n} X_k) : n < \omega\}. \tag{1.1}$$

If we do not assume α_3, then we can obtain a slightly weakened form of (1.1), namely

$$\sigma(\prod_{k=1}^{\omega} X_k) \leq \sup\{\sigma(\prod_{k=1}^{n} X_k) + 1 : n < \omega\}. \tag{1.2}$$

This fact refines [2, Theorem 3] of A. I. Bashkirov that the countable product of sequentially compact, sequential topologies, is sequential.

A topological space is *Fréchet* whenever it is sequential of order less than or equal to 1. T. Nogura proved in [6] that the property of being α_3 Fréchet is almost countably productive, so that the assumption of sequential compactness in our result is not needed if $\beta = 1$.

T. Nogura and Y. Tanaka in [8, Theorem 2.8] showed, under the hypothesis that each singleton is G_δ, that if $\prod_{k=1}^{\omega} X_k$ is sequential, then either $\prod_{k=1}^{\omega} X_k$ is (strongly) Fréchet or X_n is sequentially compact for all
but finitely many n.

In general, the fact that $\prod_{k=1}^{\omega} X_k$ is sequential but not Fréchet does not imply that almost all X_k are sequentially compact (for example, Corollary 3.6).

Using Martin's Axiom, K. Tamano showed in [9] that there is a compact Hausdorff space, each finite power of which is Fréchet, but its infinite countable power is not Fréchet. In view of the mentioned theorem by T. Nogura [6], this power is not α_3, and by (1.2) its sequential order is 2.

We also prove in this paper that each finite power of a MAD compact topology [5] is sequential of order 2. As this topology is α_1 and sequentially compact it follows that the sequential order of the infinitely countable power of a MAD compact topology is 2, which is an unexpected result.[6]

On the other hand, we prove that if a topological space contains an antitransverse transversally closed bisequence, then its countably infinite power is not sequential and its sequential order is ω_1, which is the greatest sequential order possible for a topological space.

The topological spaces considered in this paper are Hausdorff.

2. Preliminaries

Recall [1] that a topological space is respectively α_3 (in our terminology, *cofinal-cofinal*), α_2 (*eventual-cofinal*), $\alpha_{1.5}$ (*cofinal-eventual*) and α_1 (*eventual-eventual*) if whenever $\lim_k x_{n,k} = x$ for every $n \in \omega$, then there exists a subset B of $\{(n,k) : n, k \in \omega\}$ such that (the cofinite

[4] Actually of A. I. Bashkirov proved in [2, Theorem 3] that the countable product of sequential sequentially compact spaces is sequential.

[5] Called often the Alexandroff compactification of a *Mrówka space*, or a *Franklin space*, or an *Isbell space* or *a* Ψ*-space*.

[6] Professor D. Shakhmatov (Ehime University, Matsuyama) conjectured that if every finite power is sequential of order greater than 1, then the sequential order of its countably infinite power is uncountable. We are grateful to him for having communicated us that conjecture.

filter on) $\{x_{n,k} : (n,k) \in B\}$ converges to x, and for which, respectively,

$$(\alpha_3) \qquad |\{n : |\{k : (n,k) \in B\}| = \infty\}| = \infty;$$

$$(\alpha_2) \qquad |\{n : |\{k : (n,k) \in B\}| < \infty\}| < \infty;$$

$$(\alpha_{1.5}) \qquad |\{n : |\{k : (n,k) \notin B\}| < \infty\}| = \infty;$$

$$(\alpha_1) \qquad |\{n : |\{k : (n,k) \notin B\}| = \infty\}| < \infty.$$

Such a subset B is called respectively *cofinal-cofinal*, *eventual-cofinal*, *cofinal-eventual* and *eventual-eventual.*

Recall [3][5] that an inversely well-founded tree with the least element is a called a *sequential cascade* if it is of countable rank and if the set of immediate successors of every non maximal element is countably infinite. The rank of maximal elements of T is null, and if $t \in T \setminus \max T$, then the rank of t is defined as $r(t) = \sup\{r(t,n) + 1 : n \in \omega\}$; the *rank* $r(T)$ of T is by definition the rank of its least element $\varnothing = \varnothing_T$. A cascade T is *monotone* if for every $t \in T \setminus \max T$, the sequence $(r(t,n))_n$ is increasing.

Each non maximal element can be identified with a free sequence (cofinite filter on the set of its immediate successors). The elements of a cascade are called *multiindices.*

If T is a sequential cascade, then a mapping $\Phi : \max T \to A$ is called a *multisequence on* A. The rank of a multisequence is by definition the *rank* of its cascade; a multisequence is *monotone* if its cascade is monotone. A multisequence $\Psi : \max S \to A$ is a *submultisequence* of Φ provided that there is a map $g : S \to T$ such that $g(\varnothing) = \varnothing$ and $\{g(s,n) : n \in \omega\}$ is an infinite subset of $\{(g(s),n) : n \in \omega\}$ for every $s \in S \setminus \max S$ and such that $\Phi(s) = \Psi(g(s))$ for every $s \in \max S$. The rank of a submultisequence of Ψ is less than or equal to the rank Ψ.

If X is a Hausdorff space, then a multisequence Φ on X *converges* to x provided that there is an extension of Φ to a map $\Phi : T \to A$ such that $x = \Phi(\varnothing)$ and $\Phi(t) = \lim_n \Phi(t,n)$ for every non maximal t. If a multisequence converges to x, then its every submultisequence converges to x. An extended convergent multisequence $\Phi : T \to Y$ is called *antitransverse, transversally closed* if $\lim \Phi(t_k) = \varnothing$ for every $t \in T$ and every sequence (t_k) in T such that $t_k \sqsupset (t, n_k)$ where (n_k) tends to infinity; it is *free* if for every $t \in T \setminus \max T$, the sequence $(\Phi(t,n))_n$ is free [4].

In various arguments concerning sequential order we shall use without explicit mention the following fundamental fact from [5, Theorem 1.3]:

Lemma 2.1 *If $x \in \mathrm{adh}^{\alpha}_{\mathrm{Seq}}\mathrm{A}$, then there exists on A a monotone multisequence of rank not greater than α that converges to x. If a topology contains an extended convergent antitransverse, free, transversally closed multisequence of rank β, then its sequential order is not less than β.*

It follows that a topology is sequential (of rank not greater than α) if and only if $x \in \mathrm{clA}$ implies the existence of a multisequence (of rank not greater than α) on A that converges to x.

Lemma 2.2 *A multisequence in a sequentially compact space admits a convergent monotone submultisequence.*

Proof. Let Ψ be a multisequence. We induce on the rank of Ψ. For the rank 0 and 1, the fact amounts to the definition of sequential compactness. Suppose that the rank $r(\Psi) = \beta$ for a countable ordinal $\beta > 1$, and that the hypothesis holds for every $\alpha < \beta$. Let $\Psi_n(t) = \Psi(n,t)$ for every $n < \omega$. There exists a sequence $(n_k)_k$ such that the sequence $(r(\Psi_{n_k}))_k$ increasingly converges to $r(\Psi) = \beta$. By inductive assumption, there exist convergent monotone submultisequences Φ_k of Ψ_{n_k}. By sequential compactness, there exists a sequence $(k_p)_p$ such that $(\Phi_{k_p}(\varnothing))_p$ converges to an element x. The multisequence Φ

defined by $\Phi(\varnothing) = x$, and $\Phi(p, s) = \Phi_{k_p}(s)$ for every s in the domain of Φ_{k_p} is a convergent extension of a monotone submultisequence of Ψ. $\square$

Lemma 2.3 *If X is sequentially compact, $A \subset X$, and if $f : X \to Y$ is surjective and continuous, then for every convergent multisequence Ψ on $f(A)$ there exists a convergent multisequence Φ on A such that $f \circ \Phi$ is a submultisequence of Ψ.*

Proof. Let $\Psi : \max T \to Y$ be a multisequence on $f(A)$ that converges to y. Then for every $t \in \max T$ there is $\Phi^0(t) \in A$ such $\Psi(t) = f(\Phi^0(t))$. Because X is sequentially compact, by Lemma 2.2, there exists a submultisequence Φ of Φ^0 that converges to an element x of X. Then $f \circ \Phi$ is a submultisequence of Ψ. $\square$

Each sequential cascade carries its natural topology (the finest topology for which $\lim_n(t, n) = t$ for every non maximal t). A subset of T is called *eventual* if it is a neighborhood of $\varnothing$; *cofinal* (or *frequent*) if $\varnothing$ belongs to its closure.

If (X_m) is a sequence of topological spaces such that for every $l \le m$ there exists a continuous map $\pi_l^m : X_m \to X_l$ so that π_m^m is the identity map on X_m, and $\pi_l^m \circ \pi_k^l = \pi_k^m$ provided that $k \le l \le m$, then the subspace of $\prod_{m<\omega} X_m$ consisting of all those sequences (x_m) for which $x_l = \pi_l^m(x_m)$ is called the *limit of the inverse sequence* and is denoted by $\varprojlim_m X_m$. Let $\pi_l : \varprojlim_m X_m \to X_l$ be defined by $\pi_l((x_m)_m) = x_l$. Then the induced topology of $\varprojlim_m X_m$ coincides with the final topology with respect to $(\pi_m)_{m<\omega}$.

3. The results

Theorem 3.1 *If X_m is sequentially compact and sequential for every $m < \omega$, then*

$$\sigma(\varprojlim_m X_m) \le \sup\{\sigma(X_m) + 1 : m < \omega\}. \tag{3.1}$$

If moreover X_m is α_3 for every $m < \omega$, then

$$\sigma(\varprojlim_m X_m) = \sup\{\sigma(X_m) : m < \omega\} \tag{3.2}$$

Proof. Let $X = \varprojlim_m X_m$ and suppose that $x \in \mathrm{clA}$ for $A \subset X$. Then $\pi_m(x) \in \mathrm{cl}\pi_\mathrm{m}(\mathrm{A})$ for every m. Consequently there exists on $\pi_m(A)$ a multisequence Ψ_m of rank not greater than $\sigma(X_m)$ that converges to $\pi_m(x)$. By Lemma 2.3, there exists on A a multisequence Φ_m that converges to an element x^m of X and such that $\pi_m \circ \Phi_m$ is a submultisequence of Ψ_m. Of course, $r(\Phi_m) \le r(\Psi_m)$ and by continuity, $\pi_m(x) = \pi_m(x^m)$ for every $m < \omega$, hence (x^m) converges to x.

Then Φ defined by $\Phi(\varnothing) = x, \Phi(m) = \Phi_m(\varnothing) = x^m$ and $\Phi(m, t) = \Phi_m(t)$ for each t in the domain of Φ_m is an extended multisequence on A that converges to x. Its rank is not greater than $\sup\{\sigma(X_m) + 1 : m < \omega\}$.

Of course, $\pi_l \circ \Phi_m$ converges to $\pi_l(x)$ for every $l \le m$. If now each X_m is α_3, then there exists cofinal-cofinal subsets B_l of $\omega \times \omega$ such that $B_l \subset B_{l+1}$, and for every l, the cofinite filter of $\{\pi_l \circ \Phi_m(n) : (m, n) \in B_l, m \ge l\}$ converges to $\pi_l(x)$. Hence there exists a cofinal-cofinal subset B_∞ of $\omega \times \omega$ such that the cofinite filter of $\{\pi_l \circ \Phi_m(n) : (m, n) \in B_\infty, m \ge l\}$ converges to $\pi_l(x)$ for every $l < \omega$. Therefore if $m_k \ge k$ and n_k is such that $(m_k, n_k) \in B_\infty$, then the sequence $(\pi_l \circ \Phi_{m_k}(n_k))_k$ converges to $\pi_l(x)$ for every $l < \omega$, thus $(\Phi_{m_k}(n_k))_k$ converges to x. We define now an extended multisequence to the effect that $\Phi(\varnothing) = x$ and $\Phi(k, t) = \Phi_{m_k}(n_k, t)$ for every k and t in the domain of the extended multisequence Φ_{m_k}. The rank of Φ is not greater than the supremum $r(\Phi_{m_k})$ over $k < \omega$. This establishes the inequality $\le$ in (3). The equality holds, because each X_m is closed in $\varprojlim_m X_m$. $\square$

We conclude that α_3, sequentially compact, sequential spaces of given order are almost countably productive:

Corollary 3.2 *If for every n, the space X_n is α_3 sequentially compact and if $\prod_{k=1}^{n} X_k$ is sequential of order not greater than β, then $\prod_{k=1}^{\infty} X_k$ is sequential of order not greater than β.*

On the other hand, it follows from the first part of Theorem 3.1 that

Corollary 3.3 *The countably infinite product of sequentially compact sequential spaces X_k is sequential of order fulfilling (1.2).*

This refines the already mentioned theorem of A. I. Bashkirov [2, Theorem 3]. By [9] of K. Tamano, the estimate (1.2) cannot be improved.

The following lemma is a special case of a result to appear in our future paper.

Lemma 3.4 *The product of a regular, locally sequentially compact sequential space of order σ, and of a Fréchet α_2 space, is sequential of order σ.*

Theorem 3.5 *For every $k < \omega$, let X_k be a closed subset of $K_k \times L_k$ where K_k is regular, sequentially compact, sequential and L_k is first-countable, then $\prod_{k=1}^{\omega} X_k$ is sequential and $\sigma(\prod_{k=1}^{\omega} X_k) \leq \sup\{\sigma(\prod_{k=1}^{n} K_k) + 1 : n < \omega\}$.*

Proof. By Corollary 3.3, $\prod_{k=1}^{\omega} K_k$ is sequentially compact and sequential of order not greater than $\sup\{\sigma(\prod_{k=1}^{n} K_k) + 1 : n < \omega\}$. On the other hand, $\prod_{k=1}^{\omega} L_k$ is first-countable, hence Fréchet and α_2. By Lemma 3.4, $\prod_{k=1}^{\omega}(K_k \times L_k)$ is sequential of order not greater than $\sup\{\sigma(\prod_{k=1}^{n} K_k) + 1 : n < \omega\}$, so is $\prod_{k=1}^{\omega} X_k$ as its closed subspace. □

Corollary 3.6 *If X is a sequentially compact, sequential space with infinitely many isolated points, and if D is discrete, then $(X \oplus D)^{\omega}$ is sequential; if moreover X is regular, then $\sigma((X \oplus D)^{\omega}) = \sigma(X^{\omega})$.*

Proof. As X has infinitely many isolated points, then X is homeomorphic to $X \oplus \{x_0\}$. On the other hand, $X \oplus D$ is homeomorphic to a closed subset of $(X \oplus \{x_0\}) \times D$, namely to $(X \oplus \{x_0\}) \cup (\{x_0\} \times D)$. It is enough to apply Theorem 3.5. □

Let X be a *MAD* compact space: N is an infinite countable set, $\mathfrak{R}$ is an infinite maximal almost disjoint family of subsets of N. In $N \cup \mathfrak{R}$ a set W is a neighborhood of $A \in \mathfrak{R}$ whenever it contains A as an element of $\mathfrak{R}$ and if $A \setminus W$ is finite when A is considered as a subset of N; the Alexandroff compactification $N \cup \mathfrak{R} \cup \{\infty\}$ of $N \cup \mathfrak{R}$ is a *MAD* compact. This space is sequentially compact and α_1.

Theorem 3.7 *Each finite power of a* MAD *compact topology is sequential of order* 2.

Proof. Let $X = N \cup \mathfrak{R} \cup \{\infty\}$ be a *MAD* compact topology. It is enough to prove that if $A \subset N^m \subset X^m$ and $x = (\infty_1, \infty_2, \ldots, \infty_m) \in \text{clA}$, then there exists on A a bisequence that converges to x. Let p_j denote the projection on the j-th component.

There is a sequence $(z_k)_k$ in A such that for every $1 \leq j \leq m$, all the terms of the sequence $(p_j(z_k))_k$ are distinct. Indeed, there is $z_0 \in A$ with the property above, because A is not empty; if $\{z_0, z_1, \ldots, z_k\}$ with this property have been already found, then there exists a closed neighborhood W of ∞ such that $W \cap \{p_j(z_i) : 0 \leq i \leq k\} = \varnothing$ for $0 \leq j \leq m$. An element z_{k+1} of $W^m \cap A$ fulfills $p_j(z_{k+1}) \neq p_j(z_i)$ for each $0 \leq i \leq k$ and every $1 \leq j \leq m$.

Therefore, by the maximality of $\mathfrak{R}$, there exists on A a sequence $(x_{0,k})_k$ that converges to an element x_0 of $\mathfrak{R}^m$. If we have already constructed elements $x_0, x_1, \ldots, x_n$ of $\mathfrak{R}^m$ such that $p_j(x_0), p_j(x_1), \ldots, p_j(x_n)$ are all distinct and sequences $(x_{j,k})_k \to_k x_j$ for every $1 \leq j \leq n$, then there exists a closed neighborhood W of ∞ such that $x_j \notin W^m$ and $x_{j,k} \notin W^m$ for

$j \leq n$ and for every k. As $x \in \mathrm{cl}(\mathrm{A} \cap \mathrm{W}^{\mathrm{m}})$, there exists on $A \cap W^m$ a sequence $(x_{n+1,k})_k$ that converges to an element x_{n+1} of $\mathfrak{R}^m$. The free sequence $(x_n)_n$ converges to x, hence $x_{n,k} \to_k x_n \to_n x$. □

Theorem 3.7 and Corollary 3.2 imply that

Corollary 3.8 *The countable power of a* MAD *compact topology is sequential of order* 2. *The countable power of the simple sum of a* MAD *compact space and of a discrete space, is sequential of order* 2.

We shall now consider situations in which the sequential order of countably infinite products of sequential topologies of finite order (in particular, of Fréchet topologies) is ω_1. This explosion of sequential order is due to the presence of antitransverse, transversally closed multisequences. This is the case when a space includes a closed subspace homeomorphic to the sequential fan S_ω or to the bisequence S_2 .

It follows from [5] that the sequential order of the n-th power of the sequential fan is n. It can be easily checked that the sequential order of the n-th power of the canonical bisequence is $n+1$.

Proposition 3.9 *If a topology contains an antitransverse, transversally closed multisequence of rank greater than* 1*, then the sequential order of its countably infinite product is* ω_1.

Proof. First let us prove that if X contains a free, antitransverse, transversally closed multisequence Φ of rank β, and Y contains a free, antitransverse, transversally closed multisequence Ψ of rank 2, then $X \times Y$ contains a free, antitransverse, transversally closed multisequence of rank $\beta + 1$. Indeed, define $\Omega(\varnothing) = (\Phi(\varnothing), \Psi(\varnothing)), \Omega(n) = (\Phi(\varnothing), \Psi(n))$ and $\Omega(n,k,s) = (\Phi(k,s), \Psi(n,k))$ for every s such that (n,k,s) is in the domain of Φ. Then $\Omega : T \to X \times Y$ is a free multisequence of rank $\beta+1$;it is also antitransverse and transversally closed. In fact, if $t_m \sqsupset n_m$ and (n_m) tends to infinity, then $t_m = (n_m, k_m, s_m)$ (where s_m may be of length 0), hence $\Psi(n_m, k_m)$ does not converge, because Ψ is antitransverse and transversally closed; if $t_m \sqsupset (n, k_m)$ and (k_m) tends to infinity, then $t_m = (n, k_m, s_m)$ and $\Phi(k_m, s_m)$ does not converge, because Φ is antitransverse and transversally closed. It follows that the sequential order of $X \times Y$ is greater than or equal to $\beta + 1$.

If X contains a free antitransverse, transversally closed multisequence of rank greater than 1, then by what we have just proved $\sigma(X^2) \geq 2$. If $\sigma(X^\omega) = \beta < \omega_1$, then because X^ω is homeomorphic to $X^\omega \times X$, $\sigma(X^\omega) \geq \beta + 1$. Therefore $\sigma(X^\omega) = \omega_1$. □

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