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RINGLIKE IDEAL MONOIDS*

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ABSTRACT. In this paper *commutative algebraic multiplication m-lattices* are investigated. They are closely related to the Dedekind *ideal structures* of *commutative rings with identity* satisfying $\mathfrak{a} \supseteq \mathfrak{b} \Longrightarrow \mathfrak{a} | \mathfrak{b}$. Rings of this type were introduced as *multiplication rings* by WOLFGANG KRULL and studied by SHINZIRO MORI over a period of 25 years. Modifying notions of classical ideal theory, we succeed in carrying over classical ideal results to algebraic m-lattices satisfying for at least one *generating system of compact elements* the implication $a \cdot U = a \Longrightarrow a \cdot U^* = 0 \ (\exists U^* \perp U)$. In particular we study the *Prüfer property* $a_1 + \ldots + a_n \supseteq B \Longrightarrow a_1 + \ldots + a_n \mid B$, *the archimedean property* $A^n \supseteq B \ (\forall n \in \mathbf{N}) \Longrightarrow AB = B$, the *kernel property* $\ker A = A$, *and the* multiplication property, *that is* $A \supseteq B \Longrightarrow A \mid B$.

1. INTRODUCTION

In this note multiplication is always commutative.

Recall: A *partially ordered set* $\mathfrak{V} := (V, \leq)$ is called a *complete lattice* if each nonempty subset A has a *supremum* $\bigvee A$, and an *infimum* $\bigwedge A$. An element a of a complete lattice is called *compact* if it satisfies $a \leq \bigvee b_i \ \ (i \in I) \Longrightarrow a \leq b_{i_1} \vee \ldots \vee b_{i_n} \ \ (\exists i_k \in I, 1 \leq k \leq n)$. A complete lattice is called *algebraic* if each element is the supremum of some set of compact elements. Finally, a complete lattice is called a *multiplicative lattice* if a multiplication is defined satisfying $X \cdot (\bigvee A_i) \cdot Y = \bigvee (XA_iY)$ and $1 \cdot X = X = X \cdot 1$, where 1 is the *lattice-maximum.* In particular this means $A \supseteq B \Longrightarrow AX \supseteq BX$ and thereby $0X = 0$ for the *lattice minimum* 0, henceforth called the *zero element.*

By definition the zero element of a complete lattice is always compact, however, the identity element need not be compact, consider for instance the *ideal structure* of an infinite Boolean ring.

The fundamental structure of this paper is that of an *algebraic multiplicative lattice* abbreviated by AML. Any AML is embeddable in an AML with a compact identity – add some new maximum, if necessary.

We agree that in this paper the identity element 1 be always compact, unless the opposite is emphasized. We say that $\mathfrak{A}$ is *generated* by the subset B if any A of $\mathfrak{A}$ is the supremum of some subset of B.

Clearly, apart from extreme situations in an AML there exist different *generating subsets of compact elements.* For instance, in an arbitrary ring with identity one adequate generating system of its *ideal structure* is the set of all *principal ideals,* another one the set of all *finitely generated* ideals. One fundamental difference: principal ideals are *divisors,*

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*This paper belongs to a series of articles on abstract monoid ideal theory, along the lines of the monograph on Multiplicative Theory of (Ring) Ideals by LARSEN/MCCARTHY, [18], which had an initiating and inspiring influence on the author's work.

that is they satisfy $\langle a\rangle \supseteq \mathfrak{b} \Longrightarrow \langle a\rangle \,\big|\, \mathfrak{b}$, whereas finitely generated ideals need not have this property.

Special ideal monoids are the *Dedekind ideal structure* of a *ring with identity,* the *Rees ideal structure* of a *monoid with zero,* the *filter structure* of a *bounded distributive lattice,* and the *filter structure* of an *ℓ-group cone with zero.* Hence, studying AML-theory includes studying ideal theory from a general point of view. So it makes sense to replace $\geq$ by $\supseteq$, $\wedge$ by $\cap$, $\bigwedge$ by $\bigcap$, $\vee$ by $+$, and $\bigvee$ by Σ.

In this paper by an ideal monoid we mean an AML $\mathfrak{A} := (\mathcal{A}, \mathcal{A}_c, \Sigma, \cap, \cdot)$ where $\mathcal{A}_c$ denotes a fixed generating submonoid of compact divisors, containing the zero element 0.

The elements of that AML will in general be symbolized by capitals, however by lower case italics, whenever we wish to emphasize that an element belongs to $\mathcal{A}_c$, and by lower case roman letters, whenever we wish to emphasize the compactness of an element.

As is easily seen, notions and rules of ideal arithmetic carry over in a most natural manner from rings with identity to ideal monoids. In particular central notions like *prime ideal, primary ideal, radical* etc. may be assumed to be understood by LARSEN/MCCARTHY. However, in this paper we will write $A * B$ instead of $B : A$, in order to emphasize the general situation which includes, of course, also complete *Brouwerian lattices.*

The essential: Inspired by WOLFGANG KRULL, [17], this paper is worked out as an element-free contribution to abstract ring ideal theory. Principal ideals are considered as compact generators, satisfying $a \in A \Longleftrightarrow \langle a\rangle \subseteq A$. So, $a+b$ will mean the ideal, *generated* by the set theoretic union of the principal ideals $\langle a\rangle, \langle b\rangle$, which is usually symbolized by $\langle a, b\rangle$. As a paper on questions of this kind we mention an article of ISIDORE FLEISCHER, compare [9].

For a ring $\mathfrak{S}$ of numbers two questions are most important, namely: Does $\mathfrak{S}$ have the *Prüfer property*

$$\text{(P)} \qquad \langle a_1, \dots , a_n\rangle \supseteq \mathfrak{b} \quad\Longrightarrow\quad \langle a_1, \dots , a_n\rangle \,\Big|\, \mathfrak{b}$$

or does $\mathfrak{S}$ even have the *multiplication property*

$$\text{(M)} \qquad \mathfrak{a} \supseteq \mathfrak{b} \quad\Longrightarrow\quad \mathfrak{a} \,\Big|\, \mathfrak{b}\,.$$

Whereas the Prüfer property guarantees the best possible GCD-arithmetic in the sense of KRONECKER, consult [17], the multiplication property is most important above all in commutative algebra, since it characterizes Dedekind domains.

Multiplication rings, briefly *M-rings,* that is rings whose ideals satisfy condition (M), were introduced by WOLFGANG KRULL in [14] and taken up again in [16], but it was SHINZIRO MORI who started a general theory in [19] and contributed again and again to this theory over a period of 25 years, cf. [20] through [23].

These papers meanwhile are outdated, of course, by articles of – alphabetically (!) – ALARCON/ANDERSON/JAYARAM, [1], D. D. ANDERSON, [2], GILMER/MOTT [10], GRIFFIN, [11], and MOTT, [24], [25]. But some of KRULL's and MORI's ideas are still relevant.

Ideal monoids satisfying the Prüfer property (P) are studied in [6], ideal monoids satisfying the multiplication property (M) are studied in [3]. In the present paper we are concerned with a situation closely related to that in commutative rings. That is, a central requirement will be developed from

$$\text{(RL)} \qquad \langle a\rangle \subseteq B + \langle a\rangle\, C \quad\Longrightarrow\quad \langle a\rangle * B + C = \langle 1\rangle$$

which is easily verified for commutative rings with identity.

Finally, as to the symbols: $\Longrightarrow$ will mean *"if .. then"* whereas $\rightsquigarrow$ stands for *"it holds ... consequently it holds, too"*.

2. Some crucial Rules

We consider an arbitrary AML $\mathfrak{A} = (\mathcal{A}, \mathcal{A}_c, \Sigma, \cap, \cdot)$.

Again: lower case italics symbolize generators, lower case roman letters symbolize compact elements.

A is called a *divisor* if A satisfies $A \supseteq B \Longrightarrow B = AX \;\; (\exists\, X) \Longleftrightarrow: A \,\big|\, B$. Hence P is a *prime divisor* if it is both, prime and a divisor. An element $M \neq 1$ is called *maximal* or an *atom* if there lies no element strictly between M and 1.

We point out some fundamental arithmetic for the sake of referring. Nearly by definition it results that the multiplication is isotone since

$$A \supseteq B \Longrightarrow AX = (A+B)X = AX + BX \Longrightarrow AX \supseteq BX\,. \tag{4}$$

In particular (4) implies $A = A \cdot 1 \supseteq A \cdot B$. As a further crucial rule will turn out:

$$A + BC = A + AC + BC = A + (A+B) \cdot C\,. \tag{5}$$

Putting $A \perp B :\Longleftrightarrow A + B = 1$, equation (5) leads immediately to

$$A \supseteq BC \;\&\; A \perp C \Longrightarrow A \supseteq B \tag{6}$$

and by induction it leads to

$$A + B^n \quad \supseteq \quad (A+B)^n\,. \tag{7}$$

Furthermore, it holds $A \cap B \supseteq AB \supseteq (A+B)(A \cap B)$ and thereby

$$A \perp B \Longrightarrow AB = A \cap B\,. \tag{8}$$

Recall now $A * B := \sum\{x \,\big|\, Ax \subseteq B\}$. By $A \cdot x \subseteq B \Longleftrightarrow x \subseteq A * B$ this implies

$$AB * C = B * (A * C)\,. \tag{9}$$

Next we get straightforwardly

$$A \supseteq B \Longrightarrow A * C \subseteq B * C \;\&\; C * A \supseteq C * B \tag{10}$$

meaning in particular

$$(A+B) * C = A * C \cap B * C \quad \& \quad (A \cap B) * C \supseteq A * C + B * C\,. \tag{11}$$

As a divisor criterion we formulate:

$$A \,\big|\, B \Longleftrightarrow A(A * B) = A \cap B\,. \tag{12}$$

An AML is called *lattice distributive* or briefly *distributive* if its lattice satisfies

$$A \cap (B + C) \quad = \quad (A \cap B) + (A \cap C). \tag{D}$$

In [6] it is shown that it suffices to verify (D) for finitely generated elements B, C. Recall that (D) is equivalent to

$$A + (B \cap C) \quad = \quad (A+B) \cap (A+C). \tag{D+}$$

Furthermore, recall that (D) is equivalent to the fundamental implication:

$$A + X = A + Y \;\&\; A \cap X = A \cap Y \quad \Longrightarrow \quad X = Y\,. \tag{CP}$$

Next we repeat, compare [3]:

Lemma 2.1. *Let $\mathfrak{A}$ be an AML and suppose $b\cdot X \subset b$. Then there exists a maximal $P \supseteq X$ satisfying $b \cdot P \subset b$ and this P is prime and satisfies in addition $b * b \cdot P = P$.*

Proof. Let $X_i \ \ (i \in I)$ be an ascending $\supseteq$ −chain satisfying $X_i \supseteq X$ and $b \cdot X_i \subset b$. Then it follows $\sum_{i\in I} (b \cdot X_i) = b \cdot \sum_{i\in I} X_i \subset b$. Therefore the ascending chain $X_i \ (i \in I)$ is bounded by $\sum_{i\in I} X_i$. So by ZORN's lemma there exists a maximal P with $P \supseteq X$ and $b \cdot P \subset b$.

Assume now $P \supseteq U \cdot V$ but $P \not\supseteq U$, $P \not\supseteq V$. Then it results

$$b \cdot P \supseteq b \cdot (U + P) \cdot (V + P) \quad = \quad b \cdot (V + P) = b\,,$$

a contradiction. Hence P is prime.

It remains to show $b * b \cdot P = P$. Here first we get $b * b \cdot P =: Y \neq 1$ since $b * b \cdot P = 1$ would imply $b = b \cdot P$. This leads to $1 \neq Y \supseteq P$ and thereby to $Y = P$, recall P was chosen maximal with respect to $C \supseteq X$ and $b \cdot C \subset b$. □

For the sake of completeness we repeat the fundamental

Lemma 2.2. *Let $\mathfrak{A}$ be not necessarily ringlike but lattice distributive. Let moreover A be a divisor. Then it holds the implication:*

$$(A + B) \cdot U = A + B \quad\Longrightarrow\quad B \cdot U = B\,.$$

Proof. Consult [6] OR: Start with an arbitrary A and assume $(A+B)U = A+B$. Then it follows:

$$\begin{array}{rcl} A * BU &=& (A + BU) * BU \\ &\supseteq& (A + B) * BU \\ &=& (A + B)U * BU \\ &=& (A + B) * (U * BU) \\ &\supseteq& A * B \end{array}$$

and thereby for divisors A

$$\begin{array}{rcl} A + BU &=& A + (A + B)U \\ &=& A + B \\ &\&& \\ A \cap BU &=& A(A * BU) \\ &=& A(A * B) \\ &=& A \cap B\,. \end{array}$$

This means $BU = B$ because of distributivity, recall (CP). □

We say that A *covers* B, in symbols $A \succ B$, if A is different from B and $A \supset X \supseteq B \Longrightarrow X = B$. We write $A \succeq B$ if no element lies strictly between A and B, that is if $A = B$ or $A \succ B$. By definition, for instance, the maximal elements cover the identity element 1 . Applying this relation we are led to:

(16) $\quad A^m \supseteq B \supseteq A^{m+p}$ & $A^n \succeq A^{n+1}$ $(\forall n:\, m \leq n \leq m+p) \Longrightarrow B = A^{m+\ell}$ $(\exists \ell \leq p)$.

For, assume $A^k \supseteq B$ & $A^{k+1} \not\supseteq B$ & $B \not\supseteq A^{k+\ell-1}$ & $B \supseteq A^{k+\ell}$. Then it results:

$$\begin{array}{rcll} B &=& B + A^k \cdot A^{\ell-1} \cdot A^1 & \\ &=& B + (B + A^{k+\ell-1}) \cdot A^1 & \text{(by (5))} \\ &=& B + A^{k+\ell-2} \cdot A^1 & \\ &=& B + A^{k+\ell-1} & \\ &=& B + A^k & \text{(by induction)} \\ &=& A^k\,. & \end{array}$$

3. Ringlike Prüfer Ideal Monoids

Again $\mathfrak{A}$ denotes (at least) an AML.

In [8] Robert P. Dilworth completed his investigations of abstract commutative ideal theory, initiated by Morgan Ward in [27] and continued by Ward and Dilworth himself in [28]. He succeeded in an abstract proof of the celebrated Theorem of Lasker*, by creating the notion of a *principal element,* which in polynomial rings over fields coincides with the notion of a principal ideal. More precisely:

Definition 3.1. By a *principal element,* according to Dilworth, we mean an element A satisfying:

$$\text{(MP)} \qquad A \supseteq B \implies A \,|\, B$$

$$\text{(JP)} \qquad A * (B + AC) = A * B + C.$$

The verification of (MP) and (JP) for principal ideals in commutative rings with identity may be left to the reader. Obviously (JP) is equivalent to $AX \subseteq B + AC \Longrightarrow X \subseteq A * B + C$. Hence, putting $X = 1$ we get

$$\text{(RL)} \qquad A \subseteq B + AC \implies A * B \perp C\,,$$

and putting $B = 0$ and $C = U$ – in (RL) – it results

$$\text{(HY)} \qquad AU = A \implies AU^* = 0 \ (\exists\, U^* \perp U)\,.$$

Definition 3.2. By a *Dilworth AML,* we mean an AML that is generated by a submonoid of compact principal elements, and which is thereby a fortiori an ideal monoid.

An AML $\mathfrak{A} = (\mathcal{A}, \mathcal{A}_c, \Sigma, \cap\,, \cdot)$ satisfying (RL) with respect to $\mathcal{A}_c$ is called *ringlike,* here, and it is called *hypernormal* if it satisfies (HY) with respect to $\mathcal{A}_c$.

A hypernormal AML need not have the Dilworth property, not even in the distributive case. But if $\mathfrak{A}$ has the Prüfer property then $\mathfrak{A}$ is already Dilworth if only (HY) is satisfied. This is shown in [5].

Definition 3.3. An AML is called *normal* if it satisfies

$$\text{(n)} \qquad a * b + b * a = 1 \ (a, b \in \mathcal{A}_c)\,.$$

$A \in \mathcal{A}$ is called $n-$generated if it satisfies $A = a_1 + \ldots + a_n \ \ (\exists\, a_i \in \mathcal{A}_c \ \ (1 \le i \le n))$ but not $A = b_1 + \ldots + b_{n-1} \quad (\exists\, b_j \in \mathcal{A}_c \ \ (1 \le i \le n-1))$. This means in particular, that the elements a_i are assumed to be pairwise different.

Proposition 3.4. Let $\mathfrak{A}$ be normal and let A, B be finitely generated. Then it holds

$$\text{(N)} \qquad A * B + B * A = 1\,.$$

Proof. For $1-$generated elements the assertion holds by (n). Suppose now that it is verified for all at most $k-$generated elements with $1 \le k \le n-1$, and suppose moreover that A and B are at most $(n-1)-$generated. Then it follows

$$\text{(23)} \qquad \begin{array}{ll} & (A+a) * B + B * (A+a) \\ \supseteq & ((A * B) \cap (a * B)) + ((B * A) + (B * a)) \qquad ((11), (10)) \\ \supseteq & (A * B) \cdot (a * B) + \big((B * A) + (B * a)\big) \\ \supseteq & (A * B + B * A + B * a) \cdot (a * B + B * A + B * a) \quad ((11), (5)) \\ \supseteq & 1\,. \end{array}$$

Thus (N) results from (n) by induction. ∎

*to say it fair, the importance of Lasker's contribution results above all from its constructive methods

Now we are in the position to prove:

Lemma 3.5. *Any normal AML is distributive, that is satisfies the equation*

$$\text{(D)} \qquad A \cap (B + C) \quad = \quad (A \cap B) + (A \cap C)\,.$$

Proof. We have to show $A \cap (B+C) \subseteq (A \cap B) + (A \cap C)$. This is equivalent to the equation $(A \cap (B+C)) * ((A \cap B) + (A \cap C)) = 1$. As remarked above it suffices to prove condition (D) for finitely generated elements B, C. So let B, C be finitely generated. Then it follows:

$$\begin{array}{ll} & (A \cap (B+C)) * ((A \cap B) + (A \cap C)) \\ \supseteq & ((A \cap (B+C)) * (A \cap B)) + ((A \cap (B+C)) * (A \cap C)) \\ = & (A \cap (B+C)) * B + (A \cap (B+C)) * C \qquad (11) \\ \supseteq & (B+C) * B + (B+C) * C \qquad (11) \\ = & C * B + B * C = 1\,. \end{array}$$

□

The next result is based on the divisor property of the generators.

Lemma 3.6. *Let $\mathfrak{A}$ be a normal ideal monoid. Then $\mathfrak{A}$ is a Prüfer ideal monoid, that is any finitely generated element is a divisor.*

Proof. By assumption all $1-$generated elements, that is all generators are divisors. Suppose now that all at most $n-$generated elements are divisors and assume that A, B are at most $n-$generated. Then it results

$$\begin{aligned} (A+B)(A*B) &= A(A*B) + B(A*B) \\ &= B(B*A) + B(A*B) \\ &= B(A*B + B*A) \\ &= B\,. \end{aligned}$$

Thus, by induction, all finitely generated elements are divisors. □

Clearly a ringlike AML need not be normal, since otherwise any commutative ring with identity would be *arithmetical* that is have a distributive ideal lattice, consult for instance LARSEN/MCCARTHY.

But condition (HY) implies $a \cdot U = a \Longrightarrow a \cdot U^* = 0\ (\exists\, U^* \bot U) \Longrightarrow a \cdot (U^* + a * b) = a \cap b \rightsquigarrow a * b = U^* + a * b\ (\exists\, U^*)$. Hence under the assumption of (HY) condition (n) holds, if only $a \cdot (a * b + b * a) = a$ is guaranteed. Therefore

Proposotion 3.7. *A ringlike ideal monoid is a Prüfer monoid if and only if it is normal.*

Finally we remark that in a distributive $\mathfrak{A}$ – according to 2.2 – condition (HY) is carries over to all compact elements $A := a_1 + \ldots + a_n\ (a_i \in \mathcal{A}_c)$. Observe:

$$A \cdot U = A \quad \Longrightarrow \quad a_i \cdot U = a_i\ \ (1 \le i \le n),$$

which leads to $U \bot a_i * 0\ (1 \le i \le n)$ and hence by (4) to $U \bot U^* := \prod_1^n (a_i * 0)$ with $A \cdot U^* = 0$.

In [6] it is shown, that $A \cdot (B \cap C) = A \cdot B \cap A \cdot C$ holds if only $a(b \cap c) = ab \cap ac$ is satisfied. But, as an immediate consequence of (n), we get $(ab \cap ac) * a(b \cap c) \supseteq ab * a(b \cap c) + ac * a(b \cap c) \supseteq b * c + c * b = 1$. Hence

Lemma 3.8. *Any normal AML satisfies the equation*

$$\text{(K)} \qquad A \cdot (B \cup C) \quad = \quad A \cdot B \cap A \cdot C\,.$$

4. Archimedean Prüfer Ideal Monoids

We turn to Prüfer ideal monoids, that is ideal monoids satisfying (P).

We start with:

Lemma 4.1. *Let $\mathfrak{A}$ be a Prüfer ideal monoid. Then prime elements P satisfy:*

$$A \supset P \quad \Longrightarrow \quad AP = P\,. \tag{I}$$

Proof. From $A \supseteq a$ & $P \not\supseteq a$ & $P \supseteq p$ by (P) we get $(a+p)(a*p) = p$ with $P \supseteq a*p$. This implies $AP \supseteq A(a*p) \supseteq p \supseteq (a+p)(a*p) = p$ which leads to $AP = P$. □

This lemma – together with (16) – implies

Proposition 4.2. *Let $\mathfrak{A}$ be a Püfer ideal monoid, and let P be prime in $\mathfrak{A}$. Then in case of $P^n \succ P^{n+1}$ $(\forall\, n \in \mathbf{N})$ we obtain that $Q := \bigcap P^n$ $(n \in \mathbf{N})$ is prime.*

Proof. Suppose $Q \supseteq ab$ and $Q \not\supseteq a$ & $Q \not\supseteq b$. Then there exist maximal exponents $0 \leq k \in \mathbf{N}, 0 \leq \ell \in \mathbf{N}$ with $P^k \supseteq a$ but $P^{k+1} \not\supseteq a$ and $P^\ell \supseteq b$ but $P^{\ell+1} \not\supseteq b$. So by (16) and 4.1 it results $P^{k+\ell+1} \supseteq ab \Longrightarrow P^{k+\ell+1} \supseteq (P+a)^{k+\ell+1} \cdot (P+b)^{k+\ell+1} \supseteq P^{k+\ell}$. □

Next it holds:

Lemma 4.3. *Let $\mathfrak{A}$ be a Prüfer ideal monoid and suppose $a, b \subseteq P^n$ but $a, b \not\subseteq P^{n+1}$. Then it follows $(P^{n+1} + (a+b)) \cdot (P + a*b) = P^{n+1} + b$.*

Proof. By $a*b = (a+b)*b \not\subseteq P$ it holds $P + a*b \supset P$. Hence by 4.1 we get

$$\begin{aligned}(P^{n+1} + (a+b)) \cdot (P + a*b) &= P^{n+1}(P + a*b) + (a+b)P + (a+b) \cdot (a*b) \\ &= P^{n+1} + b\,.\end{aligned}$$ □

Corollary 4.4. *Let $\mathfrak{A}$ be a Prüfer ideal monoid. Then each maximal M satisfies $M^n \succeq M^{n+1}$ $(\forall\, n \in \mathbf{N})$.*

Proof. This follows by the proof of the preceding lemma, observe that $a*b \not\subseteq M$ implies $M + a*b = 1$ which in case of $M^n \supseteq a, X$ and $X \supseteq b, M^{n+1}$ but $X \not\supseteq a$ leads to $M + a = M + b$, a contradiction. □

Corollary 4.5. *Let $\mathfrak{A}$ be a Prüfer ideal monoid. Then the powers of maximal elements are primary.*

Proof. Suppose $M^n \supset ab$ & $M \not\supseteq b$. Then it results $M^n = M^n + (M^n + a)(M^n + b) \supseteq M^n + (M^n + a)(M + b)^n = M^n + (M^n + a) = M^n + a$. □

By a multiplication AML we mean, of course, an AML satisfying condition (M).

Lemma 4.6. *Any multiplication AML $\mathfrak{A}$ has the archimedean property*

$$A^n \supseteq B \ \ (\forall\, n \in \mathbf{N}) \quad \Longrightarrow \quad AB = B. \tag{A}$$

Proof. Assume $A^n \supseteq B$ $(\forall\, n \in \mathbf{N})$. Then $\bigcap A^n$ $(n \in \mathbf{N}) \,|\, B$. Assume now $BA \subset B$. Then there exist some $b \subseteq B$ and some prime $P \supseteq A$ with $P^n \supseteq b \subseteq B$ $(\forall\, n \in \mathbf{N})$ & $bP \subset b$. So, if $Q := \bigcap P^n$ $(n \in \mathbf{N})$ is idempotent, we are through.

Otherwise we get $P \supseteq X \supseteq P^n \Longrightarrow X = P^k(P^k * X)$ $(\exists\, k \in \mathbf{N})$ with $P \not\supseteq P^k * X$. This implies $X = P^k(P^k * X + P^{n-k}) \supseteq P^k(P^k * X + P)^{n-k} = P^k$ and thereby $P^n \succ P^{n+1}$ whence Q is prime by 4. But this would imply $P \cdot b = P \cdot Q(Q*b) = Q(Q*b) = b$, a contradiction. □

We abbreviate "Archimedean and Prüfer" by AP. If $\mathfrak{A}$ is a Prüfer AML it is obviously an ideal monoid. AP-ideal-monoids are exactly those ideal monoids whose *localizations* have the Mori property (M). This is shown in [5]. Here we present only some special properties of AP-ideal-monoids.

Lemma 4.7. *In any AP-ideal-monoid prime elements P satisfy*

$$P \supseteq X \supseteq P^n \quad \Longrightarrow \quad X = P^k \ \ (\exists\, 0 \leq k \leq n).$$

Proof. If k is the maximal exponent with $P^k \supseteq X$ then by 4 it follows by the method above $X + P^n = X + (P^{k+1} + X)(P^{n-k-1}) = X + P^k \cdot (P^{n-k-1}) = X + P^{n-1}$ which implies $X = P^k$ inductively. □

Recall: Prüfer AMLs are always distributive, compare [5]. *Hence (RL) is already satisfied if only (HY) is satisfied, see above. Furthermore*

Lemma 4.8. *In a ringlike AP-ideal-monoid primes are idempotent or irreducible, or equivalently, prime elements P satisfy*

$$1 \neq A \supset P \quad \Longrightarrow \quad P = P^2\,.$$

Proof. $P \neq P^2$ would imply $P \supseteq c$ & $P^2 \not\supseteq c$ for some c and thereby also – by 4 and (I) –

$$P^2 \supseteq cA^* = 0 \quad \rightsquigarrow \quad P \supseteq A^* \rightsquigarrow A \supseteq P \supseteq A^*\,,$$

for at least one $A^* \perp A$, a contradiction! □

Lemma 4.9. *Ringlike AP-ideal-monoids satisfy*

$$1 \neq A \supset P, Q \ \ (P, Q \text{ prime}) \quad \Longrightarrow \quad P = Q\,.$$

Proof. By (I) and (A) we get $A \supset P \supseteq c \rightsquigarrow cA = c \rightsquigarrow cA^* = 0$ with some $A^* \perp A$,whence it follows $Q \supseteq c$, observe $A \not\supseteq A^*$. □

4.1. **Residue Classes.**

Let $\mathfrak{A} = (\mathcal{A}, \mathcal{A}_c, \Sigma, \cap, \cdot)$ be chosen as above.

In any AML the mapping $\phi_D : X \longmapsto D + X =: \overline{X}$ provides a $\Sigma-$respecting homomorphism with respect to $\overline{X} \circ \overline{Y} := D + XY$ whose algebraic image $\overline{\mathfrak{A}} =: \mathfrak{A}/D$ satisfies

$$X_i \supseteq D \ \ (\forall\, i \in I) \quad \Longrightarrow \quad \phi_D(\textstyle\bigcap_{i\in I} X_i) = \bigcap_{i \in I} \phi_D X_i\,,$$

as is shown by routine. Thus the residue classes of classical ideal theory are reflected. This in mind we get some interesting consequences:

Proposition 4.10. *Let $\mathfrak{A}$ be a ringlike Prüfer ideal monoid generated by $\mathcal{A}_c$ and let P be prime. Then $\mathfrak{A}/P$ is a Prüfer ideal monoid whose compact elements are* $0-$cancellable, *that is satisfy* $\overline{\mathrm{a}} \circ \overline{X} = \overline{\mathrm{a}} \circ \overline{Y} \neq \overline{0} = \overline{P} \Longrightarrow \overline{X} = \overline{Y}$.

If moreover $\mathfrak{A}$ has even the multiplication property (M) then $\mathfrak{A}/P$ again has property (M) and is moreover cancellative with 0.

Proof. We repeat that any ringlike Prüfer AML has the Dilworth property. Next, by distributivity, divisors are sent to divisors. Hence all $\overline{a}_1 + \ldots + \overline{a}_n$ are divisors in $\mathfrak{A}/P$. Consequently $\mathfrak{A}/P$ has the Prüfer property (M-property), if $\mathfrak{A}$ has the Prüfer property (M-property).

Moreover, all generator images $\overline{a}$ are $0-$cancellable, recall 4. But this means that in the Prüfer case all $\overline{a}_1 + \ldots + \overline{a}_n$ are cancellable, since they are divisors of – for instance – $\overline{a}_1$, and by analogy in case of (M) we get that all elements $\overline{A}$ are $0-$cancellable divisors. □

Lemma 4.11. *Let $\mathfrak{A}$ be an AP-ideal-monoid. Then each prime element P satisfies the implication* $1 \neq M \supset P \Longrightarrow P = \bigcap M^n \ \ (n \in \mathbf{N})$.

Proof. The opposite would imply some $\mathfrak{A}/P = \overline{\mathfrak{A}}$ with $\bigcap \overline{M}^n \supseteq \overline{S} \supset \overline{0} = \overline{P}$ and thereby some $\overline{c} \neq \overline{0}$ with $\overline{c} \circ \overline{M} = \overline{c} \circ \overline{1} \rightsquigarrow \overline{M} = \overline{1}$, a contradiction! □

5. The Kernel

Throughout this section $\mathfrak{A}$ is assumed to be an AML with respect to some fixed submonoid of compact generators, not necessarily divisors.

We study the *kernel of an element,* introduced by Krull, see also [10]. To this end we need lemmata, basically due to Krull, whose proofs remain valid even in general since paper [15] is of purely multiplicative character.

We exhibit some equivalents of $\ker A = A$, valid in arbitrary AMLs of the above type, partly along the lines of Mori, [20], partly along the lines of Gilmer/Mott, [10].

For proofs of 5 through 4 the reader is referred to Krull, [15], or to [6]. First of all:

Krull's Separation Lemma 5.1. *Let $\mathfrak{A}$ be an AML, generated by a monoid of compact elements and let S be a multiplicatively closed system of compact elements not containing any $a \subseteq A$. Then there exists a prime element P with*

$$P \supseteq A \quad \& \quad \mathrm{s} \not\subseteq P \ \ (\forall\, \mathrm{s} \in S)\,.$$

Krull proved his separation lemma in [16] by means of well ordering. But, of course, it easily turns out that his separation lemma is equivalent to Zorn's lemma in its original version.

Definition 5.2. Let $\mathfrak{A}$ be an AML, generated by a monoid of compact elements and let P be a prime over A. By A_P we mean $\sum x_i \, (s \cdot x_i \subseteq A \ \ (\exists\, s \not\subseteq P))$.

Lemma 5.3. *Let M be maximal and let $P \subseteq M$ be minimal prime over A. Then A_M is P-primary.*

Proof. It holds $P \supseteq A_P \supseteq A_M$. So $x^n \subseteq A_M$ implies $x^n \subseteq A_P$ and thereby $x \subseteq P \subseteq M$.

Conversely in case of $p^n \not\subseteq A_M$ $(\forall n \in \mathbf{N})$ $S := \{s \cdot p^m \,\big|\, m \in \mathbf{N}^0 \ \& \ s \not\subseteq P\}$ is a multiplicatively closed set, containing all elements not contained in P. So, by $P \supseteq A_M$ and the separation lemma it would exist a prime element $Q \supseteq A_M$ not containing any element of S and thereby satisfying $P \supset Q \supseteq A$, a contradiction. □

Definition 5.4. Let $\mathfrak{A}$ be an AML and let P be minimal prime over A. Then A_P is called the isolated P-primary component of A.

As is easily seen A_P is equal to the intersection of all A containing P-primary elements over $\mathfrak{A}$.

Definition 5.5. Let $\mathfrak{A}$ be an AML and $A \in \mathcal{A}$. By the *kernel* $\ker A$ of A Krull defined the intersection of all *isolated primary components* A_P of A.

Krull's Kernel Lemma 5.6. *Let $\mathfrak{A}$ be an AML and suppose $a \subseteq A^* := \ker A \neq A$. Then each prime element $P \supseteq a * A$ properly contains at least one prime element minimal over A.*

Now we are in the position to prove some equivalents of $\ker A = A$.

Proposition 5.7. *Let $\mathfrak{A}$ be an AML, compactly generated by a submonoid. Then the following are equivalent:*

(*i*) $\ker A = A \ \ (\forall\, A \in \mathcal{A})$.
(*ii*) $X \supset P \supseteq p \Longrightarrow pX = p$.

Proof. Suppose that (i) is satisfied, and assume $X \supset P \supseteq p$, where P is prime and $p \supset pX$. Then by 2.1 there exists a prime element $Q \supseteq X$ satisfying $p \supset pQ$, where p and pQ have the same minimal prime supelements.

We show that p and pQ have in addition the same isolated primary components.

To this end we start with a primary element $Q_1 \supseteq pQ$ whose radical be minimal prime over pQ. Q_1 contains p since Q is not minimal over pQ, observe $Q \supseteq X \supset P$. This leads to $pQ = p$, a contradiction.

Suppose now (ii) and assume $\ker A \supset A$. Then there exists for at least one $b \subseteq \ker A$ some prime element $Q \supseteq b * A$ with $b \neq bQ$. But by 4 for at least one P, minimal prime over A, this leads to

$$Q \supset P \supseteq A \quad \rightsquigarrow \quad Q \supset P \supseteq b \rightsquigarrow b = bQ\,,$$

a contradiction. Thus the proof is complete. □

We now turn to the ringlike case.

Proposition 5.8. *Let $\mathfrak{A}$ be an AML that is compactly generated by a submonoid and ringlike with respect to this submonoid. Then the following are equivalent:*

(i) $\ker A = A$ $(\forall\, A \in \mathcal{A})$.
(iii) *Non maximal primes are idempotent divisors.*

Proof. Obviously by 4 condition (iii) implies condition (i).

So, it remains to prove $(i) \Longrightarrow (iii)$. We suppose $1 \neq A \supset P \supseteq p$. Then (ii) implies $pA = p \rightsquigarrow pA^* = 0$ $(\exists\, A^* \perp A)$, that is $pP = p(P + A^*) = p$ by $P + A^* = P \Longrightarrow A \supseteq P \supseteq A^*$, whence non maximal primes are divisors. □

Lemma 5.9. *Let $\mathfrak{A}$ be an AML that is compactly generated by a submonoid and ringlike with respect to this submonoid and suppose $1 \neq A \supset P$. Then any P-primary element Q is equal to P.*

Proof. Assume $a \subseteq A$ & $a \not\subseteq P$ and $p \subseteq P$ & $p \not\subseteq Q$. Then it follows:

$$ab \subseteq Q + ap \;\&\; a \not\subseteq P \Longrightarrow p \subseteq Q + ap \Longrightarrow pa^* \subseteq Q \text{ with } a^* \perp a.$$

But it holds $a^* \not\subseteq P$, since otherwise it would follow $A \supseteq a + a^* = 1$. Hence no power of a^* is contained in P. So we get $Q \supseteq p$ and thereby in general $Q \supseteq P$, that is $Q = P$. □

Proposition 5.10. *Let $\mathfrak{A}$ be an AML that is compactly generated by a submonoid and ringlike with respect to this submonoid. Then the following are equivalent:*

(i) $\ker A = A$ $(\forall\, A \in \mathcal{A})$.
(iv) $\mathrm{Rad}A$ *prime* $\Longrightarrow A$ *is primary.*

Proof. Suppose (iv) and $M \supseteq P$ with minimal P and $P \supseteq p$. Then by 4 we get $0_M = P$. Hence there exists some $s \not\subseteq M$ with $ps = 0$ leading to $pM = p(M + s) = p$ since M is maximal. Thus it results (ii).

Let now $\mathrm{Rad}A = P$ be prime and $\ker A = A$. Then P is the only minimal prime over A whence A is primary. □

Proposition 5.11. *If $\mathfrak{A}$ is distributive then divisors of $\mathfrak{A}$ are sent to divisors of $\mathfrak{A}/D$.*

Proof. Let A be a divisor in $\mathfrak{A}$ and suppose $B \supseteq D$. Then by distributivity the homomorphism under consideration is even a $\cap$-homomorphism. Hence it results:

$$\begin{aligned}\overline{A} \supseteq \overline{B} \Longrightarrow \overline{B} &= D + B \\ &= (D+B) \cap (D+A) \\ &= D + (A \cap B) \\ &= D + (D+A)(D + A * B) \\ &= \overline{A} \circ (\overline{A * B})\end{aligned}$$

which is the divisor property of $\overline{A}$ in $\mathfrak{A}/D$ since $B \supseteq D$ is no restriction. □

Infact, the preceding proof works already in the modular case. In this paper, however, modularity is not employed.

Proposition 5.12. *Let $\mathfrak{A}$ satisfy (JP) ((RL)). Then $\mathfrak{A}/D$ satisfies (JP) ((RL)), too, with respect to $\overline{A} \star \overline{B} := (D \cap A) * (D \cap B)$*

Proof. Suppose that (JP) is satisfied. We calculate:

$$\begin{aligned}\overline{a} \circ \overline{X} \subseteq \overline{B} + \overline{a} \circ \overline{C} &\Longrightarrow aX \subseteq (D+B) + aC \\ &\Longrightarrow X \subseteq a * ((D+B) + C) \\ &\Longrightarrow D + X \subseteq a * ((D+B)+C) \\ &\Longrightarrow \overline{X} \subseteq \overline{a} \star \overline{B} + \overline{C}\,.\end{aligned}$$

The proof for (RL) is done by putting $X = 1$. □

Proposition 5.13. *Let $\mathfrak{A}$ satisfy (JP) and let P be prime in $\mathfrak{A}$. Then any $\mathfrak{A}/P$ satisfies for all generators a of $\mathfrak{A}$ the implication $\overline{a} \circ \overline{X} = \overline{a} \circ \overline{Y} \neq \overline{0} = \overline{P} \Longrightarrow \overline{X} = \overline{Y}$.*

Proof. It holds $\overline{a} \circ \overline{X} = \overline{a} \circ \overline{Y} \neq \overline{0} \Longrightarrow P + aX = P + aY \neq P \Longrightarrow aX \subseteq P + aY \Longrightarrow X \subseteq a * P + Y = P + Y \Longrightarrow \overline{X} \subseteq \overline{Y}$. The rest follows by symmetry. □

6. Idempotency

We consider multiplication AMLs.

Recall, lower case roman letters denote compact elements. We start with a lemma, which was proven for rings by Mori, [20] and [22], respectively :

Lemma 6.1. *Let $\mathfrak{A}$ be a multiplication AML, not necessarily with compact identity. Put $N := \mathrm{Rad}\, 0$. Then to each compact c there exists a compact $\mathrm{u} \subseteq (\mathrm{c} * N) * N$ with $\mathrm{c} \subseteq N + \mathrm{cu}$.*

Proof. First by

$$y \subseteq \mathrm{c} * N \cap (\mathrm{c} * N) * N \quad \Longrightarrow \quad y^2 \subseteq N \Longrightarrow y \subseteq N \tag{28}$$

it follows

$$N \;=\; \mathrm{c} * N \cap (\mathrm{c} * N) * N\,. \tag{29}$$

Next by (M) there exists some divisor D with

$$(\mathrm{c} + \mathrm{c} * N)D \;=\; \mathrm{c} \;=\; cD + (c * N)D\,. \tag{30}$$

This implies in particular – recall (29) –

$$\begin{aligned}(\mathrm{c} * N)D &\subseteq \mathrm{c} \cap \mathrm{c} * N \\ &\subseteq (\mathrm{c} * N) * N \cap \mathrm{c} * N = N \\ \rightsquigarrow \quad D &\subseteq (\mathrm{c} * N) * N\,.\end{aligned}$$

Thus by (30) we get

$$\mathrm{c} \subseteq \mathrm{c} \cdot ((\mathrm{c} * N) * N) + N \tag{31}$$

which leads to some compact $\mathrm{u} \in (\mathrm{c} * N) * N$ with

$$\mathrm{c} \subseteq N + \mathrm{cu}\,. \tag{32}$$

This completes the proof. ☐

Applying (32) we get in particular:

Corollary 6.2. *Any ringlike multiplication AML satisfies:*

$$(\mathrm{N^c}) \qquad \mathrm{c} * N + (\mathrm{c} * N) * N = 1\,.$$

Proof. $\mathrm{c} \subseteq N + \mathrm{cu}$ leads to $\mathrm{c} * N \perp \mathrm{u}$ and hence to some $\mathrm{u}^* \perp \mathrm{u} \subseteq (c * N) * N$. ☐

Now we are in the position to show, which was done by GILMER/MOTT in [10] for rings:

Proposition 6.3. *Let $\mathfrak{A}$ be a ringlike multiplication ideal monoid. Then any idempotent element is a sum of idempotent compact elements.*

Proof. Let U be idempotent, and let A be the subelement that is generated by the set of all compact idempotents contained in U. This set is not empty because 0 is idempotent. We prove:

$$U = U^2 \supset A \implies \exists\, \mathrm{e} : U \supseteq \mathrm{e} = \mathrm{e}^2 \not\subseteq A\,.$$

To this end suppose $A \subset U$ and $\mathrm{c} \subseteq U$ but $\mathrm{c} \not\subseteq A$. By property (M) we get $U \cdot \mathrm{c} = \mathrm{c}$, whence there exists some $\mathrm{f} \subseteq U$ with $\mathrm{fc} = \mathrm{c}$ and thereby with $\mathrm{f}^n\mathrm{c} = \mathrm{c}$. So we may assume that already c satisfies $\mathrm{c}^n \not\subseteq A$, in particular that c is not contained in N. Then applying ($\mathrm{N^c}$) we get

$$\mathrm{c} = \mathrm{c} \cdot (\mathrm{c} * N) + \mathrm{c} \cdot U \cdot ((\mathrm{c} * N) * N)\,,$$

whence we find some $\mathrm{u} \subseteq U$ with

$$\mathrm{c} \subset N + \mathrm{cu} \;\; (\mathrm{u} \subseteq (\mathrm{c} * N) * N)\,.$$

This leads in $\mathfrak{A}$ to some $\mathrm{u}^* \subseteq \mathrm{c} * N$ with $\mathrm{u}^* \perp \mathrm{u}$ and $u\mathrm{u}^* \subseteq N$ and hence to some power $(\mathrm{uu}^*)^k = 0 \;\; (\exists\, k \in \mathbf{N})$. Therefore by (7) we get next

$$\begin{aligned}\mathrm{u}^k &= \mathrm{u}^k(\mathrm{u}^k + \mathrm{u}^{*k}) \\ &= (\mathrm{u}^k)^2 + (\mathrm{uu}^*)^k \\ &= (\mathrm{u}^k)^2\,.\end{aligned}$$

But by $\mathrm{c} \subseteq N + \mathrm{cu} \rightsquigarrow \mathrm{cu} \subseteq N\mathrm{u} + \mathrm{cu}^2$ we get $\mathrm{c} \subseteq N + \mathrm{cu}^k$. Hence, the element u of lemma 6.1 may be assumed to be idempotent.

It remains to show that $\mathrm{u}^k =: \mathrm{e}$ is not contained in A. But it holds $\mathrm{c} \subseteq N + \mathrm{ce}$ and hence $\mathrm{c} \subseteq \mathrm{n} + \mathrm{ce} \;\; (\exists\, \mathrm{n} \subseteq N)$, and this leads for some suitable $m \in \mathbf{N}$ to

$$\mathrm{c}^m = (\mathrm{n} + \mathrm{ce})^m = \mathrm{n}^m + c^m\mathrm{e} = c^m\mathrm{e}\,. \tag{34}$$

Therefore e cannot be contained in A, recall that c^m is not contained in A. ☐

As is easily checked, the proof that idempotent elements are sums of idempotent compact elements does not depend on compactness of the identity but merely on some u^* with $\mathrm{u}\cdot\mathrm{u}^* = 0$ and $\mathrm{u} = \mathrm{u}\cdot(\mathrm{u}+\mathrm{u}^*)$ for any compact u. Such elements u^* exist, for instance, in the ideal monoids of M-rings with *fixing elements,* that is elements e_a with $a \cdot e_a = a$.

Furthermore: commutative rings with fixing elements, monoids or d-monoids have ideal structures in which the product of any principal ideal with an arbitrary ideal is equal to the *complex product.* In those cases the proof above works even for principal ideals instead of compact, that is finitely generated ideals, as is easily verified by the reader.

In particular: property (M) guarantees fixing elements whence in analogy to the element $\langle c\rangle$ of (RL) one finds some $\langle c\rangle^*$ with $\langle c\rangle\cdot\langle c\rangle^* = \langle 0\rangle$ and $\langle c\rangle = \langle c\rangle\,(\langle c\rangle + \langle c\rangle^*)$. This means that M-rings and thereby also idempotent ideals of M-rings are generated idempotently, because $a^2x = a \Longrightarrow (ax)^2 = ax$.

7. Decomposition Theorems

Again we are concerned with multiplication AMLs only.

First a ringtheoretical result. Here we denote the *radical* N from above by $\mathfrak{n}$ and prime ideals by $\mathfrak{p}$.

Proposition 7.1. *Every M-ring with identity* 1 *has a subdirect decomposition into components which are cancellative with 0 or primary.*

Proof. Let $\mathfrak{R}$ be an M-ring with identity 1. Then the subdirect irreducible images of $\mathfrak{R}$ are again M-rings containing in particular no idempotents different from $\overline{0}$ and $\overline{1}$. But this means that the corresponding ideal monoids are *nilpotent* or otherwise that in 6 – because of $\langle e\rangle \subseteq (\,\langle c\rangle * \mathfrak{n}\,) * \mathfrak{n}$ – we get first $\langle e\rangle = \langle 1\rangle$ and thereby furthermore $(\,\langle c\rangle * \mathfrak{n}\,) * \mathfrak{n} = \langle 1\rangle$.

So, if a component is not nilpotent we get in this component $\langle c\rangle * \mathfrak{n} = \mathfrak{n}$, as is easily checked, and thereby $\langle c\rangle * \langle 0\rangle = \langle 0\rangle$, which results as follows:

Suppose below 6 $(\langle c\rangle * \mathfrak{n}) * \mathfrak{n} = \langle 1\rangle$. It follows $\langle c\rangle * \mathfrak{n} = \mathfrak{n} \rightsquigarrow \langle c\rangle^n * \mathfrak{n} = \mathfrak{n}$ $(\forall n \in \mathbf{N})$. So, if $c\cdot y = 0$, it follows $y \in \mathfrak{n}$ and thereby $y^n = 0$ for some $n \in \mathbf{N}$. Consequently every minimal prime element $\mathfrak{p}$ contains y.

We show that not only each minimal prime $\mathfrak{p}$ contains y but also all its powers $\mathfrak{p}^n$. From this, by ker $0 = 0$, it then results $y = 0$ (and thereby $\langle c\rangle * \langle 0\rangle = \langle 0\rangle$), for observe:

It holds $\mathfrak{p} \supseteq \mathfrak{n}$ and if $y \notin \mathfrak{p}^{m+1}$ we get $\mathfrak{p} \not\supseteq \mathfrak{p}^m * \langle y\rangle$ and thereby $c \notin \mathfrak{p}$, because $0 = c\cdot y \notin \mathfrak{p}^{m+1}$. But this leads to the contradiction

$$\mathfrak{p} \supseteq \mathfrak{n} = \langle c\rangle^m * \mathfrak{n} \supseteq \mathfrak{p}^m * \mathfrak{n} \supseteq \mathfrak{p}^m * \langle y\rangle \not\subseteq \mathfrak{p}\,.$$

Thus the proof is complete. □

As an immediate consequence we obtain:

Corollary 7.2. *Any ringlike multiplication AML admits a subdirect decomposition into factors satisfying* $\langle c\rangle * \langle 0\rangle = \langle 0\rangle$ *or* $M^n = 0$ *for all and thereby for exactly one maximal element.*

Next we present

A first decomposition theorem 7.3 *Any multiplication AML satisfies:*

$$\text{(DC)} \qquad A = \bigcap P_i^{\,e_i} \qquad (P \textit{ prime and } P_i^{\,e_i} \supseteq A)\,.$$

Proof. Put $B := \bigcap P_i^{\,e_i}$ $(P_i^{\,e_i} \supseteq A)$ and assume $1 \neq P \supseteq B * A \supseteq b$. Then, in case of $bP \neq b$, by (A) it would hold:

$$P^m \supseteq A \quad \& \quad P^{m+1} \not\supseteq A$$
$$\rightsquigarrow \quad P \not\supseteq P^m * A \quad \subseteq \quad B * A \subseteq P\,.$$

□

Obviously by 7 and 4 we obtain again 7, since in the M-case any $\mathfrak{A}/P$ is again a multiplication AML and thereby 0-cancellative.

A second decomposition theorem 7.4 *A ringlike M-ideal-monoid $\mathfrak{A}$ is a direct product in the sense of 7 if and only if for each family of idempotent elements B_i $(i \in I)$ the equation holds:*

$$(\mathrm{D}_\cap) \qquad A + \bigcap B_i \quad = \quad \bigcap (A + B_i\,) \quad (i \in I)\,.$$

Proof. $(\mathrm{D}_\cap)$ is obviously necessary. Suppose next that $\bigcap P_i{}^{e_i} = 0$ is the representation of 0 by minimal prime powers. Then each $P_i{}^{e_i} =: U_i$ is idempotent and each $A \in \mathcal{A}$ can be decomposed into

$$A = A + 0 = A + \bigcap U_i \quad = \quad \bigcap (A + U_i\,) \ (=: \bigcap A_i\,) \ \ (i \in I)\,.$$

Now, by 4.1 we get $P_i \neq P_j \Longrightarrow P_i \perp P_j$. But by $(\mathrm{D}_\cap)$ this leads to $U_i \perp \bigcap U_j \ \ (j \neq i)$. Hereby the proof is complete. □

The most natural question arises, when a ringlike multiplication AML has the Noether property. There is an abundance of necessary and sufficient conditions. In particular, since multiplication AMLs are AP-ideal monoids, the reader may consult [5]. Moreover, since components $\mathfrak{A}/P$ with idempotent minimal P may by Jaffard, [13], be considered as the ideal structure of some integral domain, and since any component of type $\mathfrak{A}/P^n$ with $P^{n-1} \neq P^n = P^{n+1}$ may be considered as ideal structure of some residue class ring $\mathbf{Z}/p^n$ we are arrived at Mott, [25].

For the sake of completeness only one characterization, which was not mentioned in [5].

A third decomposition theorem 7.5 *A ringlike M-ideal-monoid $\mathfrak{A}$ with compact identity 1 is a finite direct product in the sense of 7.1 if and only if:*

$$(\mathrm{D}^*) \qquad U = U^2 \quad \Longrightarrow \quad U + U * 0 = 1\,.$$

Proof. By assumption it follows immediately that idempotent elements are finitely generated since $U + U * 0 = 1 \Longrightarrow \mathrm{u} + \mathrm{u}^* = 1 \ \ (\exists\, \mathrm{u} \subseteq U, \mathrm{u}^* \subseteq U * 0)$. Observe $U \cdot \mathrm{u}^* = 0 \Longrightarrow \mathrm{u} \supseteq U$, apply (8).

Hence the set of idempotent elements satisfies the ascending chain condition. So, since all kernel components of 0 are idempotent, the set of kernel components of 0 is finite. □

Finally, a ringtheoretical result, due to D. D. Anderson, compare [2], proved in an alternate manner.

Proposition 7.6. *Let $\mathfrak{R}$ be a ring with identity 1. Then $\mathfrak{R}[\,x\,]$ is a multiplication ring if and only if $\mathfrak{R}$ is a direct product of fields.*

Proof. The one direction is clear. So let $\mathfrak{R}[\,x\,]$ be a multiplication ring. Then $\mathfrak{R}$ is (von Neumann) regular, since by distributivity of the ideal lattice every $a \in R$ satisfies

$$\langle a \rangle \supseteq \langle x - a \rangle + \langle x \rangle \quad \Longrightarrow \quad \big(\langle a \rangle \cap \langle x - a \rangle\big) + \big(\langle a \rangle \cap \langle x \rangle\,,$$

which implies

$$\begin{aligned} a \ &= \ (x-a)\cdot f(x) + x \cdot g(x) \\ & \quad\ \text{with } a \,|\, x \cdot g(x) \rightsquigarrow a \,|\, g(x) \\ &\rightsquigarrow \ a \,|\, (x-a)\cdot f(x) \\ &\rightsquigarrow \ a \,|\, xf(x) \rightsquigarrow a \,|\, f(x) \\ &\rightsquigarrow \ a^2 \,|\, af(x) = x(\,f(x) + g(x)\,) - a \\ &\rightsquigarrow \ a^2 \,|\, a\,. \end{aligned}$$

Thus the principal ideals form a boolean algebra, whence in particular all finitely generated ideals are principal ideals, recall $u = u^2 \ \&\ v = v^2 \Longrightarrow \langle u, v\rangle = \langle u - uv + v\rangle$.

We now show that $\mathfrak{R}$ has the Noether property. From this it will follow that $\mathfrak{R}$ is indeed a direct product of fields. To this end let $A = \langle a_i\rangle$ $(a_i^2 = a_i)$ be an ideal of $\mathfrak{R}$ and suppose $\langle A, x\rangle \cdot B = \langle x\rangle$ in $\mathfrak{R}[\,x\,]$. Then we obtain $\langle A, x\rangle \cdot B = \langle x\rangle = \langle a_1, \dots, a_n, x\rangle \cdot B$ $(\exists\, a_i \in A, 1 \le i \le n) \supseteq \langle x\rangle \rightsquigarrow \langle a_1, \dots, a_n, x\rangle \cdot B = \langle x\rangle$.

Now, according to $B \,\big|\, \langle x\rangle$, B is cancellable since $\langle x\rangle$ is cancellable. Hence we get: $\langle A, x\rangle = \langle a_1, \dots, a_n, x\rangle = \langle a, x\rangle$ with $\langle a\rangle = \langle a_1, \dots, a_n\rangle$ and thereby for all $a_i \in A$ the equation $a_i = a \cdot u(x) + x \cdot v(x)$ $(\exists\, u(x), v(x)) = a \cdot s$ $(s \in R)$ whence $A = \langle a\rangle$. Thus we are through. □

8. M-characterizations

Throughout this section we are concerned with ringlike AMLs

In the ringlike case we may hope for M-characterizations based on ringlike particularities sheding some special light.

Lemma 8.1. *In a ringlike ideal monoid $\mathfrak{A}$ satisfying*

(M1) $$a * B + B * a \;=\; 1$$

(RP) $$P \succeq P \quad (\forall\, P \textit{ prime})$$

any prime power satisfies $P^n \supseteq AB \,\&\, P \not\supseteq B \Longrightarrow P^n \supseteq A$. Evidently this means in particular that any prime power is primary.

Proof. First of all observe that $\mathfrak{A}$ has the Prüfer property. We suppose $P^n \supseteq AB \,\&\, P \not\supseteq B$. Then it follows $P^n \supseteq (P^n + A)(P + B)^n$. We put $P + B =: D$ and we will show in general $D \supset P \supseteq p \Longrightarrow Dp = p$.

So, suppose $p \subseteq P \ \&\ p \not\subseteq P^2$. Then $PX \subseteq P^2 \Longrightarrow P(P + X) = P^2 \Longrightarrow P + X = P$, that is – by (I) – $P \supseteq P * P^2$ and thereby $P = P * P^2$.

This leads next to $p * P^2 = (p + P^2) * P^2 = P * P^2 = P$ and $P^2 * p = (P^2 + p) * p = P * p$. Consequently it holds $P \perp P * p$, that is $P \,\big|\, p$ by (8). Hence we get $D \supset P \supseteq p \Longrightarrow D \cdot p = D \cdot P \cdot (P * p) = P \cdot (P * p) = p$. □

As a first characterization we present:

Proposition 8.2. *A ringlike ideal monoid $\mathfrak{A}$ has property* (*M*) *if and only if it satisfies*†

(M1) $$a * B + B * a \;=\; 1$$

(M2) $$U = U^2 \Longrightarrow U = \sum \mathrm{u_i} \ \ (\mathrm{u_i} = \mathrm{u_i}^2)$$

(M3) $$P \textit{ prime} \ \&\ P \supset X \supseteq P^2 \quad \Longrightarrow \quad X \textit{ is } P\textit{-primary}$$

Proof. Necessity: Condition (M1) follows by (HY) and $a(a * B + B * a) = a$, recall $a * B = a * B + a * 0$.

Next, condition (M2) was proven in the preceding section.

Finally let's turn to condition (M3). Obviously, under our assumption above – according to condition (I) – we get $X = PY = P(P + Y)$. So X must be equal to P^2. But P^2 is primary because $P^2 \supseteq ab \ \&\ P \not\supseteq a \Longrightarrow P^2 \supseteq (P + a)^2(P^2 + b) = P^2 + b$.

Sufficency: By (M1) the underlying ideal monoid has the Prüfer property.

So, any idempotent U is a divisor because it is a sum of compact idempotents.

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†Evidently condition (M3) results from $\ker A = A$ as well as from $\mathrm{Rad}A$ prime $\Longrightarrow A$ primary.

Furthermore by condition (M3) prime elements are maximal or idempotent, recall 4. This means in particular (M3) & (M2) $\Longrightarrow \ker A = A$.

Next we get $(x+y)*B+B*(x+y)=1$ since by normality and 3 it results condition (K) and thereby

$$\begin{aligned}(x+y)*B+B*(x+y) &= (x*B)\cap(y*B)+B*(x+y)\\ &= ((x*B)+B*(x+y))\cap((y*B)+B*(x+y)) \supseteq 1\cap 1\,.\end{aligned}$$

Consider now some $x\subseteq M$ with $x\not\subseteq M^2$. Then by $M\succ M^2$ for each $m\subseteq M$, defining $\mathrm{b}:=x+m$ it follows $M^2*\mathrm{b}+\mathrm{b}*M^2=(M^2+\mathrm{b})*\mathrm{b}+(M^2+\mathrm{b})*M^2=M*\mathrm{b}+M=1$. But by (8) this leads to $M\cdot(M*\mathrm{b})=M\cap(M*\mathrm{b})\supseteq\mathrm{b}$, that is $M\big|\mathrm{b}$.

So, given some $b\subseteq M$ we find some $x\subseteq M$ with $x\not\subseteq M^2$ implying $M\,\big|\,b+x\,\big|\,b$ and thereby $M\supseteq B\Longrightarrow M\big|B$.

Summarizing: By (M1),(M2),(M3) any prime element is even a prime divisor.

Suppose now $A\supseteq b$ and $b\supset A(A*b)=b\cdot(b*A(A*b))$. Then by 2.1 there would exist a prime element P with $b\neq bP$ and containing $b*A(A*b)\supseteq A\cdot(A*b)$ and thereby containing – in any case – also b. So, P cannot be idempotent and must hence be maximal.

We assume $bM\neq b$ & $M\,\big|\,b$. By 4 this means $M^n\succ M^{n+1}$ $(\forall n\in\mathbf{N})$. But then – by assumption and 4 – it results that $Q:=\bigcap_{n\in N}P^n$ is a prime divisor of b, implying the contradiction $P\cdot b=P\cdot Q(Q*b)=Q(Q*b)=b$. □

Proposition 8.3. *A ringlike ideal monoid $\mathfrak{A}$ is a multiplication AML iff it satisfies*

(M1) $$a*B+B*a \;=\; 1$$

(DC) $$A=\bigcap {P_i}^{e_i}\qquad (P\ \textit{prime and}\ {P_i}^{e_i}\supseteq A)\,.$$

Proof. By the results above it suffices to verify

SUFFICIENCY: By (DC) $\mathfrak{A}$ satisfies condition (S) whence according to 8.1 any prime power is primary. Conversely by (DC) any primary element is a prime power.

Assume now $P^n\supseteq B$ $(\forall n\in\mathbf{N})$ but $B\neq PB$. Then any prime power Q^m with $Q^m\supseteq PB$ either satisfies $Q\supseteq P\Longrightarrow Q^m\supseteq B$ or we get $Q\not\supseteq P\Longrightarrow Q^m\supseteq B$, since Q^m is primary. Hence $\mathfrak{A}$ has the archimedean property, leading to $U^2=U\supseteq B\rightsquigarrow UB=B$. Therefore the rest is done along the proof lines of 8 by applying the Prüfer property. □

In [3] it is shown:

Proposition 8.4. *In an arbitrary AML the following are equivalent:*

(*i*) $\mathfrak{A}$ *is a multiplication AML.*
(*ii*) $\mathfrak{A}$ *is a weak multiplication AML, that is* $\mathfrak{A}$ *satisfies* $P\supseteq B\Longrightarrow P\,\big|\,B$.
(*iii*) $\mathfrak{A}$ *satisfies:*
(a) *Every element is equal to its kernel.*
(b) *Every primary element is a power of its radical.*
(c) *If P is minimal prime over A, if n is the least positive integer such that P^n is the isolated P-primary component of A and if $P^n\neq P^{n+1}$, then P does not contain the meet of the remaining isolated primary components.*

For the ideal structure of commutative rings with identity this result is due to MOTT, [24], for ringlike AMLs it is due to ALARCON/ANDERSON/JAYARAM, [1]. Here we add:

Proposition 8.5. *A ringlike ideal monoid $\mathfrak{A}$ is a multiplication AML iff it satisfies*

(M1) $$a*B+B*a \;=\; 1$$

(RP) $$\mathrm{Rad}A=P\ \textit{prime}\ \Longrightarrow\ A=P^n\ (\exists n\in\mathbf{N}).$$

Proof. Let Rad$A$ be prime. Then by (RP) A is equal to some P^n. Hence by (DC) of 8 it suffices to show $\ker A = A$, which by 4 is equivalent to Rad$B$ is prime $\Longrightarrow B$ is primary. But, by (RP) it holds $P \succeq P^2$ whence B is primary by 8.1. □

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