SOME PROPERTIES ON COMMUTATIVE BCK–ALGEBRAS

Yuan-Hong Lin

Received February 20, 2001; revised May 1, 2001

Abstract. In paper[1], the author had shown that commuta- tive BCK–algebras with condition (S) under natural partial order are distributive lattices. In this paper, we shall extend above mentioned conclusion, and obtain seven important properties on commutative BCK–algebra $(X, *, \leq; 0)$ if it is a lattice under natural partial order, and answer problem 1 in paper [1] if X is a finite set.etc.

The following Lemma 1,2 are well known:

Lemma 1 *Let $(X, *, \leq; 0)$ be a BCK–algebra, then for any $x, y, z \in X$ we have*

$$(x * y) * z = (x * z) * y; x * (x * (x * y)) = x * y$$

And $x \leq y \to x * z \leq y * z, z * y \leq z * x.$

Lemma 2 *Let $(X, *, \leq; 0)$ be a BCK-algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, then for any $x, y, z \in X$ we have*

$$(x * z) \vee (y * z) \leq (x \vee y) * z, (x \wedge y) * z \leq (x * z) \wedge (y * z)$$

Theorem 1 *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra, $x, y, z \in X$, and $x \leq z, y \leq z$, then $x \vee y = z * ((z * x) \wedge (z * y))$.*

Proof. Since $(X, *, \leq; 0)$ is a commutative BCK–algebra, then we have

$$x = x * (x * z) = z * (z * x) \leq z * ((z * x) \wedge (z * y))$$

$$y = y * (y * z) = z * (z * y) \leq z * ((z * x) \wedge (z * y))$$

Put $z_0 = z * ((z * x) \wedge (z * y))$, then $x \leq z_0, y \leq z_0$. Therefore we have

$$z_0 * x = (z * ((z * x) \wedge (z * y))) * x = (z * x) * ((z * x) * ((z * x) * (z * y)))$$

$$= (z * x) * (z * y) = (z * (z * y)) * x = (y * (y * z)) * x = y * x$$

2000 *Mathematics Subject Classification.* 06F35.
Key words and phrases. Commutative BCK-algebra, lattice, partially ordered.

Similarly, we have $z_0 * y = x * y$.
Therefore,

(1) $(z_0*x)\wedge(z_0*y) = (z_0*x)*((z_0*x)*(z_0*y)) = (y*x)*((z_0*(z_0*y))*x) = (y*x)*(y*x) = 0$

Let $x \leq u, y \leq u$, It follows from Lemma 1 that $z_0 * u \leq z_0 * x, z_0 * u \leq z_0 * y$, further $z_0 * u \leq (z_0 * x) \wedge (z_0 * y) = 0$,therefore $z_0 \leq u$, so $z_0 = x \vee y$. □

In fact,by the process of the proof of Theorem 1, we have the following remark:

Remark Let $(X, *, \leq; 0)$ be a commutative BCK–algebra.If $x \leq z_1, y \leq z_1; x \leq z_2, y \leq z_2$,Then $z_1 * ((z_1 * x) \wedge (z_1 * y)) = z_2 * ((z_2 * x) \wedge (z_2 * y))$.

Corollary 1. *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra with condition (S), then $(X, \leq)$ is a lattice under natural partial order $\leq$*[1].
(In fact, there $x \wedge y = x * (x * y), x \vee y = (x \circ y) * (((x \circ y) * x) \wedge ((x \circ y) * y))$)

Corollary 2. *Let $(X, *, \leq; 0)$ be a bounded commutative BCK–algebra, then $(X, \leq)$ is a lattice under natural partial order $\leq$*[2].

(In fact,there $x \wedge y = x * (x * y), x \vee y = 1 * ((1 * x) \wedge (1 * y))$, 1 is the greatest element of X)

Theorem 2. *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, then for any $x, y, z \in X$, we have*

$$(x \vee y) * z = (x * z) \vee (y * z), (x \wedge y) * z = (x * z) \wedge (y * z)$$

Proof For any $x, y, z \in X$, from $x * z \leq x \leq x \vee y, y * z \leq y \leq y \vee z$ and Theorem 1, we have

$$(x * z) \vee (y * z) = (x \vee y) * (((x \vee y) * (x * z)) \wedge ((x \vee y) * (y * z))).$$

On the other hand, by Lemma 2 and formula (1), we have

$$(((x \vee y) * (x * z)) \wedge ((x \vee y) * (y * z))) * z \leq (((x \vee y) * (x * z)) * z) \wedge (((x \vee y) * (y * z)) * z)$$
$$= (((x \vee y) * z) * (x * z)) \wedge (((x \vee y) * z) * (y * z)) \leq ((x \vee y) * x) \wedge ((x \vee y) * y) = 0.$$

Therefore $((x \vee y) * (x * z)) \wedge ((x \vee y) * (y * z)) \leq z$, moreover

$$(x \vee y) * z \leq (x \vee y) * (((x \vee y) * (x * z)) \wedge ((x \vee y) * (y * z))) = (x * z) \vee (y * z).$$

By Lemma 2, $(x * z) \vee (y * z) \leq (x \vee y) * z$, then $(x \vee y) * z = (x * z) \vee (y * z)$.

Secondly, for any $u \leq x * z$ and $u \leq y * z$, we have

$$u * ((x \wedge y) * z) \leq (x * z) * ((x \wedge y) * z); u * ((x \wedge y) * z) \leq (y * z) * ((x \wedge y) * z).$$

On the other hand,

$$\begin{aligned}(x * z) * ((x \wedge y) * z) &= (x * z) * ((x * (x * y)) * z) \\ &= (x * z) * ((x * z) * (x * y)) = (x * z) \wedge (x * y), \\ (y * z) * ((x \wedge y) * z) &= (y * z) * ((y * (y * x)) * z) \\ &= (y * z) * ((y * z) * (y * x)) = (y * z) \wedge (y * x),\end{aligned}$$

moreover, by the formula (1), we have

$$u * ((x \wedge y) * z) \leq ((x * z) \wedge (x * y)) \wedge ((y * z) \wedge (y * x)) =$$
$$((x * z) \wedge (y * z)) \wedge ((x * y) \wedge (y * x)) = ((x * z) \wedge (y * z)) \wedge 0 = 0.$$

Therefore $u \leq (x \wedge y) * z$. Since $(x \wedge y) * z \leq x * z$, $(x \wedge y) * z \leq y * z$, then $(x \wedge y) * z = (x * z) \wedge (y * z)$. □

Corollary *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, then for any $x, y \in X$, we have*

$$x * (x \wedge y) = (x * y) = (x \vee y) * y$$

By Lemma 1, we have the conclusion:

Theorem 3. *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra, then for any $x, y, z \in X$, $x * y = x * z$ if and only if $x \wedge y = x \wedge z$.*

Theorem 4. *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, then for any $x, y, z \in X$, $x * z = y * z$ if and only if $x \vee z = y \vee z$.*

Proof Sufficiency. For any $x, y, z \in X$, if $x \vee z = y \vee z$, from Corollary of theorem 2, we have $x * z = (x \vee z) * z = (y \vee z) * z = y * z$.

Necessity. For any $x, y, z \in X$, if $x * z = y * z$, by Theorem 1, we have

$$\begin{aligned}x \vee z &= (x \vee y \vee z) * (((x \vee y \vee z) * z) \wedge ((x \vee y \vee z) * x)) \\ &= (x \vee y \vee z) * (((x \vee y \vee z) * z) * (((x \vee y \vee z) * z) * ((x \vee y \vee z) * x))) \\ &= (x \vee y \vee z) * (((x \vee y \vee z) * z) * (((x \vee y \vee z) * ((x \vee y \vee z) * x)) * z)) \\ &= (x \vee y \vee z) * (((x \vee y \vee z) * z) * ((x * (x * (x \vee y \vee z))) * z)) \\ &= (x \vee y \vee z) * (((x \vee y \vee z) * z) * (x * z)).\end{aligned}$$

Similarly, we have $y \vee z = (x \vee y \vee z) * (((x \vee y \vee z) * z) * (y * z))$. Therefore, we have $x \vee y = x \vee z$. □

Corollary *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra with condition (S), then for any $x, y, z \in X, x * z = y * z$ if and only if $x \vee z = y \vee z$.*

Theorem 5. *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, then for any $x, y, z \in X$ and $z \leq x, z \leq y$, we have*

$$x * y = (x * z) * (y * z)$$

Proof Since $z \leq y$, then $x * y \leq x * z$, moreover, from Theorem 2 and its corollary, we have

$$\begin{aligned} x * y &= (x * z) * ((x * z) * (x * y)) = (x * z) * ((x * (x * y)) * z) \\ &= (x * z) * ((x \wedge y) * z) = (x * z) * ((x * z) \wedge (y * z)) \\ &= (x * z) * (y * z). \end{aligned}$$

Thus, the proof is completed. □

Let $(X, *, \leq; 0)$ be a BCK–algebra, for any $a, b \in X$, the set $\{x \mid x * a \leq b, x \in X\}$ is denote by $A(a, b)$.

Theorem 6. *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, if for any $a, b \in X$, there exists a maximum element $c \in A(a, b)$, then X is a BCK-algebra with condition (S).*

Proof For any $a, b \in X$, there exists a maximum element $c \in A(a, b)$, then for any $z \in A(a, b)$, we have $z * a \leq b, c * a \leq b$, moreover $(z * a) \vee (c * a) \leq b$. By Theorem 2, we have $(z \vee c) * a \leq b$, therefore $z \vee c \in A(a, b)$. Since c is a maximum element of $A(a, b)$, so $z \vee c = c$, moreover $z \leq z \vee c = c$. Therefore c is a greatest element of $A(a, b)$, thus, X is a BCK–algebra with condition (S). □

Corollary *Let $(X, *, \leq; 0)$ be a finite commutative BCK–algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, then X is a BCK–algebra with condition (S).*

Theorem 7. *Let $(X, *, \leq; 0)$ be a commutative BCK–algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, $a_1, a_2, a_3, \cdots, a_s \in X$, if $a_1 * a_2 = a_2 * a_3 = \cdots = a_{s-1} * a_s = a_s * a_1$, then $a_1 = a_2 = \cdots = a_s$.*

Proof Take $b = a_1 * a_2 = a_2 * a_3 = \cdots = a_{s-1} * a_s = a_s * a_1, a = a_1 \wedge a_2 \wedge \cdots \wedge a_s$, then $a \leq a_i (1 \leq i \leq s)$. By Theorem 5, we have $(a_1 * a) * (a_2 * a) = (a_2 * a) * (a_3 * a) = \cdots = (a_{s-1} * a) * (a_s * a) = (a_s * a) * (a_1 * a) = b$, so $b \leq a_i * a (1 \leq i \leq s)$, moreover we have $b \leq (a_1 * a) \wedge (a_2 * a) \wedge \cdots \wedge (a_s * a)$. From Theorem 2, we have $b \leq (a_1 \wedge a_2 \wedge \cdots \wedge a_s) * a = a * a = 0$, so $b = 0$, i.e. $a_1 * a_2 = a_2 * a_3 = \cdots = a_{s-1} * a_s = a_s * a_1 = 0$, moreover

$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{s-1} \leq a_s \leq a_1$, therefore $a_1 = a_2 = a_3 = \cdots = a_{s-1} = a_s$. Thus, the proof is completed. □

Lemma 3 *Let $(S, \wedge, \vee)$ be a distributive lattice, $a, b, c \in S$, if $a \wedge b = a \wedge c, a \vee b = a \vee c$, then $b = c$*[3].

Theorem 8 *Let $(X, *, \leq, 0)$ be a commutative BCK–algebra and $(X, \leq)$ is a lattice under natural partial order $\leq$, for any distinct $a_1, a_2, a_3 \in X$, if $a_1 * a_2 = a_1 * a_3$, then $a_2 * a_1 \neq a_3 * a_1$; if $a_2 * a_1 = a_3 * a_1$, then $a_1 * a_2 \neq a_1 * a_3$.*

Proof If $a_1 * a_2 = a_1 * a_3$ and $a_2 * a_1 = a_3 * a_1$, by Theorem 3 and Theorem 4, we have $a_1 \wedge a_2 = a_1 \wedge a_3$ and $a_1 \vee a_2 = a_1 \vee a_3$. On the other hand, by *the process of the remark 2 of theorem 4 in [1],* we have $(X, \leq)$ is a distributive lattice, therefore, from Lemma 3, we have $a_2 = a_3$. It contradicts to the assumption a_2, a_3 are distinct, thus, if $a_1 * a_2 = a_1 * a_3$, then $a_2 * a_1 \neq a_3 * a_1$; if $a_2 * a_1 = a_3 * a_1$, then $a_1 * a_2 \neq a_1 * a_3$. □

References

[1] Yuan-Hong Lin, *Some Results on BCK–algebras,* Math. Japon., 37(1992), 529–534

[2] K. Iseki and S. Tanaka, *An introduction to the theory of BCK–algebras,* Math. Japon., 23(1978), 1-26

[3] N. Jacobson, *Basic Algebra I,* 1974

Department of Basic Courses, Jimei University, Xiamen 361021,CHINA E-mail: jmlyh 2000 @ 21 cn. com