*Scientiae Mathematicae Japonicae Online,* **Vol. 5**, (2001), 241–244 

# SOME RESULTS ON ALMOST BIIDEALS

YUAN-HONG LIN

Received February 20, 2001; revised May 1, 2001

ABSTRACT. In this paper, we give some relations between biideals and almost biideals, and prove some properties on some special semigroups.

**Definition** 1[1] *A nonempty subset $T$ of a semigroup $S$ is a* **subsemigroup** *of $S$ if it is closed under the operation of $S$; i.e. if $a, b \in T$, then $ab \in T$.*

**Definition** 2[1],[2] *A nonempty subset $T$ of a semigroup $S$ is a* **two- sided ideal(or simply a biideal or an ideal)** *if $x, y \in S, t \in T$ imply $xt, ty \in T$.*

**Definition** 3[3] *A nonempty subset $T$ of a semigroup $S$ is a almost biideal if for any $s \in S$, there exists $x, y \in T$ such that $xsy \in T$.*
*It is clear that biideal are almost biideal.*

**Example 1.** t Let $S$ be a cyclic group generated by $a$ of order 4, $e$ is the identity of $S$,$\{e, a\}$ is a almost biideal of $S$, but $\{e, a\}$ is not a subsemigroup of $S$,$\{e, a^2\}$ is a subsemigroup of $S$, but $\{e, a^2\}$ is not a almost biideal of $S$.

**Definition** 4[1] *An element $a$ of a semigroup $S$ is* **regular** *if $a = axa$ for some $x \in S$. A semigroup $S$ is* **regular** *if every element of $S$ is regular. A nonempty subset $I$ of a semigroup $S$ is a* **regular set** *of $S$ if every element of $I$ is regular.*

**Definition** 5[2] *An element $a$ of a semigroup $S$ is* **quasi regular** *if $a^m = a^m x a^m$ for some positive integer $m$ and some $x \in S$. A semigroup $S$ is* **quasi regular** *if every element of $S$ is quasi regular. A nonempty subset $I$ of a semigroup $S$ is a* **quasi regular set** *of $S$ if every element of $I$ is quasi regular.*

**Definition** 6[1] *An element $A$ of a semigroup $S$ is* **idempotent** *if $a^2 = a$.*
*The set of all idempotent elements of semigroup $S$ is denote by $E$.*

**Theorem 1** *Let $S$ be a quasi regular semigroup and $E = \{e\}$, then we have*

1) *for any $y \in S$, there exists some positive integer $m$ and some $x \in S$ satisfy*

$$xy^m = y^m x = e, y^m e = ey^m = y^m, ye = ey;$$

2) *$Se = eS$ is a group.*

---

2000 *Mathematics Subject Classification.* 20M12.
*Key words and phrases.* Biideal, almost biideal, regular, minimal, group .

*Proof.* By the definition 5, for any $y \in S$, there exists some positive integer $m$ and some $x \in S$ satisfies $y^m = y^m x y^m$, further we have $(y^m x)^2 = y^m x, (xy^m)^2 = xy^m$. On the other hand, as $E = \{e\}$, so we have $xy^m = y^m x = e$, moreover $ey^m = y^m e = y^m; ey = xy^m y = xyy^m = xyy^m e = xyy^m y^m x = xy^{2m+1}x, ye = yy^m x = y^m yx = ey^m yx = xy^m y^m yx = xy^{2m+1}x$, therefore $ey = ye$, and $Se = eS$.

For any $u \in Se$, there exists $s \in S$, satisfy $u = se$. Therefore $eu = ese = see = se^2 = se = u, ue = see = se^2 = se = u$, then $e$ is the identity of $Se$. On the other hand, there exists some positive integer $n$ and some $t \in S$ satisfies $ts^n = s^n t = e, s^n e = es^n = s^n, se = es$. Let $v = s^{n-1}te$. Thus $uv = ses^{n-1}te = ess^{n-1}te = es^n te = eee = e^2 e = ee = e^2 = e$. Similarly, we have $vu = e$.So $Se = eS$ is a group. □

**Corollary 1.** *$S$ is a quasi regular semigroup and $E = \{e\}$ (Where $e$ is the identity of $S$) if and only if $S$ is a group.*

**Lemma 1.** *Let $B$ be both a almost biideal and a subsemigroup of a semigroup $S$, then for any $x, y \in S$, we have $xBy$ is a almost biideal of a semigroup $S$.*

*Proof* For any $x, y \in S, s \in S$, we have $ysx \in S$. Since $B$ is a almost biideal of a semigroup $S$,there exists $u, v \in B$ such that $uysxv \in B$. On the other hand, $B$ is a subsemigroup of $S$, so
$x(uysxv)y = (xuy)s(xvy) \in xBy$. Therefore $xBy$ is a almost biideal of a semigroup $S$. Thus, the proof is completed. □

**Remark 1.** Generally, for a semigroup $S$, there uncertain exists a almost biideal $M$ of $S$ such that for any almost biideal $B$ of $S$, have $x, y \in S$ satisfies $B = xMy$.

**Example 2** Let $S$ be a cyclic group group generated by $a$ of order 4,$e$ is the identity of $S$, there not exists a almost biideal $M$ of $S$ such that for any almost biideal $B$ of $S$, have $x, y \in S$
satisfies $B = xMy$.

**Example 3.** There a almost biideal $B$ of a quasi regular semigroup $S$ and $E = \{e\}$, but $B$ is not a biideal of $S$.

*Proof* Let $S$ be a infinitely cyclic group generated by $a, e$ is the identity of $S$, it is clear that $S$ is also a quasi regular semi- group and $E = \{e\}$. Take $B = \{e, a, a^2, a^3, \cdots\}$, then $B$ is a almost biideal of $S$. Because for any $s \in S$, there exists integer $m$ such that $s = a^m$, so there exists positive integer $n =\mid m \mid +1$,
$a^n s a^n \in B$, therefore $B$ is a almost biideal of $S$.
On the other hand, $a^{-1} = ea^{-1}e \notin B$.In fact,if exists integer $n \geq 0$ satisfies $a^{-1} = a^n$, then $a^{n+1} = e$.It contradicts to the assumption $S$ is a infinitely cyclic group. Thus, $B$ is not a biideal of $S$. Hence there exists a proper almost biideal $B$ of a quasi regular semigroup $S$ and $E = \{e\}$. □

But we have the conclusions:

**Theorem 2.** *Let $B$ be a subgroup of semigroup $S$, and also is a almost biideal of semigroup $S$,then $B = S$.*

*Proof* As $B$ is a almost biideal of semigroup $S$,then for any $s \in S$, exists $u, v \in B$ satisfies $usv \in B$.On the other hand, $B$ is a subgroup of semigroup $S$, so $u^{-1}, v^{-1} \in B$,therefore $s = u^{-1}(usv)v^{-1}$
$= s \in B$, moreover, $S \subseteq B$.Since $B \subseteq S$, then $B = S$. □

**Corollary 1.** *For any a semigroup $S, B$ is a nonempty proper subset of $S$. If $B$ is a subgroup of $S$, then $B$ is not a almost biideal of $S$.*

**Corollary 2.** *For any a semigroup $S, B$ is a nonempty proper subset of $S$. If $B$ is a almost biideal of $S$, then $B$ is not a subgroup of $S$.*
*On the other hand, by the process of the proof of the theorem 2, we have the following conclusions:*

**Theorem 3.** *There exists a regular set $B$ of a semigroup $S$, and $B$ is a subsemigroup of $S$, but $B$ is not a regular semigroup.*
*Similarly, there exists a quasi regular set $B$ of a semigroup $S$, and $B$ is a subsemigroup of $S$, but $B$ is not a quasi regular*
*semigroup.*

**Theorem 4.** *There exists a proper almost biideal $B$ of a group $S$.*
But we have the following conclusion:

**Theorem 5.** *Let $S$ be a monoid,If $S$ have not exists proper almost biideal, then $S$ is a group.*

*Proof.* Assume contrary. Take $e$ be the identity of $S, S$ is not a group.
Order $B = S - \{e\}$,then esists $x \in S$, $x$ is not inverse,i.e. for any $y \in S, xy \neq e$.Therefore for any $s \in S$,let $y = sx$, we have $xy = xsx \neq e$, i.e. $x \in B, xsx \in B$. So $B$ is a proper almost biideal of $S$, it contradicts to the assumption $S$ have not exists proper almost biideal and we have the conclusion. □

**Lemma** 2[3]. *Let $S$ be a semigroup, then $S$ is a group if and only if for any $a \in S$ satisfies $aSa = S$.*

**Example 4.** There exists a regular semigroup $S$ and $u, v \in S$ such that $uSv \neq S$.
Take $S = \{0, 1\}$ with the multiplication operation,then $S$ is a regular semigroup, and $0S0 = \{0\} \neq S$. □ We have the following conclusion:

**Theorem 6** *Let $B$ be both a minimal almost biideal and a subsemigroup of $S$,then $B$ is a subgroup of $S$, moreover $B = S$.*

*Proof* For any $x \in B$, by the lemma 1, we have $xBx$ is a almost biideal of $S$. Since $xBx \subseteq B, B$ is a minimal almost biideal of $S$, so $xBx = B$. On the other hand, $B$ is also a subsemigroup of $S$,by the lemma 2, then $B$ is a group and $B$ is a subgroup of $S$.Therefore,from the theorem 2, we have $B = S$. □

Finally, I mention the following unsolved problem:

*Let $B$ be a minimal almost biideal of a semigroup $S$, for any $x, y \in S$, is $xBy$ a minimal almost biideal of $S$?*

## References

[1] Mario Petrich, *Introduction to Semigroups,* Chares E. Merrill Publishing Company, A Bell & Howell Company,Columbus,Ohio, 1973

[2] Bogdanovic S. *Semigroups with a System of Subsemigroups,* NoviSad, 1985, 1-196

[3] J.M.Howie, *An Introduction to Semigroup Theory,* Academic Press London New York San Francisco, 1975

Department of Basic Courses, Jimei University, Xiamen 361021,CHINA E-mail: jmlyh 2000 @ 21 cn. com