EXTENSIONS OF FUZZY IDEALS IN BCK-ALGEBRAS

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Received July 31, 2001

Abstract. An extension of a fuzzy ideal in a BCK-algebra is established.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [6]. Since its inception, the theory of fuzzy sets has developed in many directions and is finding applications in a wide variety of fields. In [4], Rosenfeld used this concept to develop the theory of fuzzy groups. For the first time, Xi [5] applied this concept to BCK-algebras. Some elementary properties of fuzzy ideals in BCK-algebras were studied by Xi [5] and Jun [1, 2]. The purpose of this paper is to construct an extension of a fuzzy ideal in a BCK-algebra. Let S be a subalgebra of a BCK-algebra X. We give an extension of a fuzzy ideal μ of S to a fuzzy ideal μ^e of X such that μ and μ^e have the same image.

2. Preliminaries

A *BCK-algebra* is an algebra $(X; *, 0)$ of type (2, 0) satisfying the following axioms:
(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0$ and $y*x=0$ imply that $x=y$,
for all $x, y, z \in X$. A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x*y=0$.

A nonempty subset S of a BCK-algebra X is called a *subalgebra* of X if $x*y \in S$ whenever $x, y \in S$.

A nonempty subset I of a BCK-algebra X is called an *ideal* of X if
(i) $0 \in I$,
(ii) $x*y \in I$ and $y \in I$ imply that $x \in I$.

We note that the intersection of all ideals of a BCK-algebra X is also an ideal of X. Let I be a subset of a BCK-algebra X. The ideal generated by I is the intersection of all ideals of X which contain I. Let Λ be a totally ordered set and let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a family of ideals of a BCK-algebra X such that for all $\alpha, \beta \in \Lambda$, $\beta > \alpha$ if and only if $I_\beta \subset I_\alpha$. Then $\bigcup_{\alpha\in\Lambda} I_\alpha$ is an ideal of X.

By a *fuzzy set* μ in a nonempty set X, we mean a function μ from X into the closed interval $[0, 1]$. For $\alpha \in [0, 1]$, let

$$\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}.$$

Then μ_α is called a *level subset* of X.

2000 *Mathematics Subject Classification.* 06F35, 03G25, 94D05.
Key words and phrases. Fuzzy ideal, extension of fuzzy ideal.
The first author was supported by Korea Research Foundation Grant (KRF-99-005-D00003).

A fuzzy set μ in a set X has the *sup property* if for any subset T of X, there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t \in T} \mu(t)$.

Definition 2.1. [5] A fuzzy set μ in a BCK-algebra X is called a *fuzzy ideal* of X if it satisfies:
(i) $\mu(0) \geq \mu(x)$ for all $x \in X$,
(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Lemma 2.2. [5] *A fuzzy set μ in a BCK-algebra X is a fuzzy ideal of X if and only if for every $\alpha \in [0,1]$, μ_α is an ideal of a BCK-algebra X, when $\mu_\alpha \neq \emptyset$.*

3. Main Results

Theorem 3.1. *Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a collection of ideals of a BCK-algebra X such that*
(i) $X = \bigcup_{\alpha \in \Lambda} I_\alpha$,
(ii) $\beta > \alpha$ *if and only if* $I_\beta \subset I_\alpha$ *for all* $\alpha, \beta \in \Lambda$.
Define a fuzzy set μ in X by, for all $x \in X$,

$$\mu(x) = \sup\{\alpha \in \Lambda \mid x \in I_\alpha\}.$$

Then μ is a fuzzy ideal of X.

Proof. For any $\beta \in [0,1]$, we consider the following two cases:

$$(1)\ \beta = \sup\{\alpha \in \Lambda \mid \alpha < \beta\} \text{ and } (2)\ \beta \neq \sup\{\alpha \in \Lambda \mid \alpha < \beta\}.$$

For the case (1), we know that

$$x \in \mu_\beta \Leftrightarrow x \in I_\alpha \text{ for all } \alpha < \beta \Leftrightarrow x \in \bigcap_{\alpha < \beta} I_\alpha,$$

whence $\mu_\beta = \bigcap_{\alpha < \beta} I_\alpha$, which is an ideal of X. Case (2) implies that there exists $\varepsilon > 0$ such that $(\beta - \varepsilon, \beta) \cap \Lambda = \emptyset$. We claim that $\mu_\beta = \bigcup_{\alpha \geq \beta} I_\alpha$. If $x \in \bigcup_{\alpha \geq \beta} I_\alpha$, then $x \in I_\alpha$ for some $\alpha \geq \beta$. It follows that $\mu(x) \geq \alpha \geq \beta$, so that $x \in \mu_\beta$. Conversely if $x \notin \bigcup_{\alpha \geq \beta} I_\alpha$, then $x \notin I_\alpha$ for all $\alpha \geq \beta$, which implies that $x \notin I_\alpha$ for all $\alpha > \beta - \varepsilon$, that is, if $x \in I_\alpha$ then $\alpha \leq \beta - \varepsilon$. Thus $\mu(x) \leq \beta - \varepsilon$, and so $x \notin \mu_\beta$. Therefore $\mu_\beta = \bigcup_{\alpha \geq \beta} I_\alpha$, which is an ideal of X. Using Lemma 2.2, we know that μ is a fuzzy ideal of X. □

Definition 3.2. [3] Let S be a nonempty set. By an *extension* of fuzzy set μ in S to a fuzzy set ν in a set X containing S, we mean a fuzzy set ν in X such that $\nu = \mu$ in S.

Lemma 3.3. [3] *Let S be a nonempty subset of a set X and let μ be a fuzzy set in S such that μ has the sup property. If $\mathcal{B} = \{B_\alpha \mid \alpha \in \mathrm{Im}(\mu)\}$ is a collection of subsets of X such that*
(i) $\bigcup_{\alpha \in \mathrm{Im}(\mu)} B_\alpha = X$;
(ii) $\beta > \alpha$ *if and only if* $B_\beta \subset B_\alpha$ *for all* $\alpha, \beta \in \mathrm{Im}(\mu)$;
(iii) $\mu_\alpha \cap B_\beta = \mu_\beta$ *for all* $\alpha, \beta \in \mathrm{Im}(\mu)$, $\beta \geq \alpha$;
then μ has a unique extension to a fuzzy set μ^e in X such that $(\mu^e)_\alpha = B_\alpha$ for all $\alpha \in \mathrm{Im}(\mu)$ and $\mathrm{Im}(\mu^e) = \mathrm{Im}(\mu)$.

Let I be a subset of a BCK-algebra X. The ideal *generated* by I, written I^e, is defined to be the intersection of all ideals of X which contain I. Note that

$$I^e = \{x \in X \mid (\cdots((x * a_1) * a_2) * \cdots) * a_n = 0 \text{ for some } a_1, a_2, \cdots, a_n \in I\}.$$

Note also that if I is an ideal of X, then $I^e = I$.

Proposition 3.4. *Let S be a subalgebra of a BCK-algebra X. If I is an ideal of S, then $S \cap I^e = I$.*

Proof. Clearly $I \subseteq S \cap I^e$. Let $x \in S \cap I^e$. Then there exist $a_1, a_2, \cdots, a_n \in I$ such that

$$(\cdots((x * a_1) * a_2) * \cdots) * a_n = 0.$$

Note that $a_1, a_2, \cdots a_n \in S$. Since I is an ideal of S, it follows that $x \in I$ so that $S \cap I^e \subseteq I$, ending the proof. □

Proposition 3.5. *Let μ be a fuzzy set in a BCK-algebra X and let $\alpha, \beta \in \text{Im}(\mu)$ be such that $\alpha < \beta$. Then $\mu_\beta \subsetneq \mu_\alpha$. Moreover if μ is a fuzzy ideal of X then $(\mu_\beta)^e \subsetneq (\mu_\alpha)^e$.*

Proof. Clearly $\mu_\beta \subseteq \mu_\alpha$ whenever $\alpha < \beta$. Let $x, y \in X$ be such that $\mu(x) = \alpha$ and $\mu(y) = \beta$, respectively. Since $\mu(x) = \alpha < \beta = \mu(y)$, it follows that $x \in \mu_\alpha$ but $x \notin \mu_\beta$. Therefore $\mu_\beta \subsetneq \mu_\alpha$. If μ is a fuzzy ideal of X, then μ_α and μ_β are ideals of X (see Lemma 2.2). Hence $(\mu_\beta)^e = \mu_\beta \subsetneq \mu_\alpha = (\mu_\alpha)^e$, ending the proof. □

Theorem 3.6. *Let S be a subalgebra of a BCK-algebra X and let μ be a fuzzy ideal of S such that μ has the sup property. If $\bigcup_{\alpha \in \text{Im}(\mu)} (\mu_\alpha)^e = X$, then μ has a unique extension to a fuzzy ideal μ^e of X such that $(\mu^e)_\alpha = (\mu_\alpha)^e$ for all $\alpha \in \text{Im}(\mu)$ and $\text{Im}(\mu^e) = \text{Im}(\mu)$.*

Proof. Since $\beta > \alpha$ if and only if $\mu_\beta \subset \mu_\alpha$ for all $\alpha, \beta \in \text{Im}(\mu)$, it follows that $\beta > \alpha$ if and only if $(\mu_\beta)^e \subset (\mu_\alpha)^e$. If we let $B_\alpha = (\mu_\alpha)^e$, then by Lemma 3.3 we know that μ has a unique extension to a fuzzy set μ^e in X such that $(\mu^e)_\alpha = (\mu_\alpha)^e$ for all $\alpha \in \text{Im}(\mu)$ and $\text{Im}(\mu^e) = \text{Im}(\mu)$. Noticing that $(\mu^e)_\alpha = (\mu_\alpha)^e$ is an ideal of X, and using Lemma 2.2, we conclude that μ^e is a fuzzy ideal of X. This completes the proof. □

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