ON QUOTIENT RESIDUATED LATTICES VIA FUZZY LIA-FILTERS

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Abstract. We construct the quotient residuated lattice induced by fuzzy filters of lattice implication algebras.

1. Introduction

In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, Xu [7] proposed the concept of lattice implication algebras, and discussed their some properties in [7] and [8]. Xu and Qin [9] introduced the notion of filter in a lattice implication algebra, and investigated their properties. In [11], Xu and Qin defined the fuzzy filter in a lattice implication algebra L, and they discussed their some properties. In [12], Xu et al. defined a congruence relation on lattice implication algebras induced by fuzzy filters and they proved the Fuzzy Homomorphism Fundamental Theorem. Pavelka introduced the notion of the residuated lattices in [6] and investigated their properties. In [2], Liu and Xu introduced the notion of new binary operation on lattice implication algebras and they lead to the residuated lattice by using the new operation of lattice implication algebras. In this paper, we construct the quotient residuated lattice induced by fuzzy filters of lattice implication algebras.

2. Preliminaries

We recall a few definitions and properties.

Definition 2.1 ([8]). By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\rightarrow$ " satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$. If $(L, \vee, \wedge, 0, 1)$ satisfies the conditions (I1) $\sim$ (I5), is called a *quasi lattice implication algebra*. A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

We can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \rightarrow y = 1$.

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In a lattice implication algebra L, the following hold([8]): for all $x, y, z \in L$,

(1) $0 \to x = 1, 1 \to x = x$ and $x \to 1 = 1$,
(2) $x \leq y$ implies $z \to x \leq z \to y$ and $x \to z \geq y \to z$,
(3) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(4) $x \to ((x \to y) \to y) = 1$.

Definition 2.2 ([10]). Let $(L, \vee, \wedge, \prime, \to)$ be a lattice implication algebra. A subset F of L is called a *filter* if it satisfies for all $x, y \in L$:

(i) $1 \in F$,
(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$.

The following proposition is clear.

Proposition 2.3. *Every filter F of L has the following property:*

$$x \leq y \text{ and } x \in F \text{ imply } y \in F.$$

Definition 2.4 ([6]). A *residuated lattice* is a triple $\mathbb{L} = (L, \circledast, \to)$ where

(R1) L is a bounded lattice with the least element 0 and the greatest element 1;
(R2) a couple $(\circledast, \to)$ of binary operations on L satisfies as follows:
 (i) $\circledast$ is isotone on $L \times L$;
 (ii) $\to$ is isotone in the first and antitone in the second variable;
 (iii) the adjointness condition

$$a \circledast b \leq c \text{ if and only if } a \leq b \to c$$

 holds for all $a, b, c \in Ł$;
(R3) $(L, \circledast, 1)$ is a commutative monoid.

Definition 2.5 ([2]). Let $(L, \vee, \wedge, ', \to, 0, 1)$ be a quasi lattice implication algebra and given elements a, b of L, we define

$$A(a, b) := \{x \in L | a \leq b \to x\}.$$

If for all $x, y \in L$, $A(x, y)$ has a least element, written $x \otimes y$, then the quasi lattice implication algebra is called to be *with property (P)*.

Lemma 2.6 ([2]). *Any lattice implication algebra is with property (P), in fact $a \otimes b = (a \to b')'$.*

Lemma 2.7 ([2]). *Let $(L, \vee, \wedge, ', \to, 0, 1)$ be a lattice implication algebra. Then the following hold: for all $a, b, c \in L$,*

(5) $a \otimes b \leq a \wedge b \leq a$,
(6) $a \leq b$ *if and only if* $a \otimes b' = 0$,
(7) $a \otimes b = b \otimes a$,
(8) $(a \to b) \otimes a \leq b$,
(9) $(a \otimes b) \to c = b \to (a \to c)$.

We now review some fuzzy logic concepts. Let X be a set. A function $\mu : X \to [0, 1]$ is called to a *fuzzy subset* on X.

3. Main Results.

Definition 3.1 ([11]). A fuzzy subset μ of a lattice implication algebra $(L, \vee, \wedge, ', \rightarrow, 0, 1)$ is called a *fuzzy filter* if it satisfies

(F1) $\mu(1) \geq \mu(x)$ for all $x \in L$,

(F2) $\mu(y) \geq \min\{\mu(x \rightarrow y), \mu(x)\}$ for all $x, y \in L$.

Proposition 3.2 ([11]). *Let μ be a fuzzy filter of a lattice implication algebra $(L, \vee, \wedge, ', \rightarrow, 0, 1)$. Then for all $x, y \in L$, $x \leq y$ implies $\mu(x) \leq \mu(y)$.*

Remark 3.3. If $(L, \vee, \wedge, ', \rightarrow, 0, 1)$ is a lattice implication algebra, then by Lemma 2.6, we know that $a \otimes b \in L$ for all $a, b \in L$, and so we regarded $\otimes$ as a binary operation on L, *i.e.*,

$$\otimes : L \times L \rightarrow L, (a, b) \mapsto a \otimes b.$$

Theorem 3.4. *Let $(L, \vee, \wedge, ', \rightarrow, \otimes, 0, 1)$ be a lattice implication algebra. Then μ is a fuzzy filter of L if and only if for all $a, b \in L$,*

(i) $\mu(1) \geq \mu(a)$, *and*

(ii) *if $a \leq b$, then $\mu(a) \leq \mu(b)$, and*

(iii) $\mu(a \otimes b) \geq \min\{\mu(a), \mu(b)\}$.

Proof. Suppose that μ is a fuzzy filter of L. Then by the definition of fuzzy filters and Proposition 3.2, conditions (i) and (ii) are obvious. For any $a, b \in L$, we have

$$\mu(a \otimes b) \geq \min\{\mu(b \rightarrow (a \otimes b)), \mu(b)\} \geq \min\{\mu(a), \mu(b)\}.$$

Conversely, for all $a, b \in L$, we have

$$\mu(b) \geq \mu(a \wedge b) = \mu(a \otimes (a \rightarrow b)) \geq \min\{\mu(a), \mu(a \rightarrow b)\},$$

and so μ is a fuzzy filter of L. □

As is well known, the characteristic function of a set is a special fuzzy set. Assume that F is a subset of a lattice implication algebra L, denote by χ_F the characteristic function of F, *i.e.*,

$$\chi_F(x) := \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{otherwise.} \end{cases}$$

The following simple fact is sometimes useful.

Lemma 3.5 ([12]). *Let F be a subset of a lattice implication algebra $(L, \vee, \wedge, ', \rightarrow, 0, 1)$. Then χ_F is a fuzzy filter of L if and only if F is a filter of L.*

Proof. The proof is easy and is omitted. □

Now we construct the quotient residuated lattice induced by fuzzy filters. Let μ be a fuzzy filter of a lattice implication algebra $(L, \vee, \wedge, ', \rightarrow, \otimes, 0, 1)$. For any $a, b \in L$, define a binary relation $\underset{\mu}{\equiv}$ on L by

$$a \underset{\mu}{\equiv} b \text{ if and only if } \mu((a \rightarrow b) \otimes (b \rightarrow a)) = \mu(1).$$

Then we have the following Theorem.

Theorem 3.6. *Let μ be a fuzzy filter of a lattice implication algebra $(L, \vee, \wedge, ', \rightarrow, \otimes, 0, 1)$ and $a, b \in L$. Then*

$$a \underset{\mu}{\equiv} b \text{ if and only if } \mu(a \rightarrow b) = \mu(1) \text{ and } \mu(b \rightarrow a) = \mu(1).$$

Proof. Let $a, b \in L$ be such that $a \underset{\mu}{\equiv} b$. Then, Since $(a \rightarrow b) \otimes (b \rightarrow a) = ((a \rightarrow b) \rightarrow (b \rightarrow a)')'$ and $(b \rightarrow a)' \leq (a \rightarrow b) \rightarrow (b \rightarrow a)'$, we have $(a \rightarrow b) \otimes (b \rightarrow a) \leq (b \rightarrow a)$. Thus by Proposition 3.2, we get

$$\mu(1) = \mu((a \rightarrow b) \otimes (b \rightarrow a)) \leq \mu(b \rightarrow a),$$

and so $\mu(b \rightarrow a) = \mu(1)$. Similarly, we obtain $\mu(a \rightarrow b) = \mu(1)$.

Conversely, let $a, b \in L$ be such that $\mu(a \rightarrow b) = \mu(1) = \mu(b \rightarrow a)$. Since

$$\begin{aligned} 1 &= (a \rightarrow b) \rightarrow ((a \rightarrow b) \vee (b \rightarrow a)') \\ &= (a \rightarrow b) \rightarrow (((a \rightarrow b) \rightarrow (b \rightarrow a)') \rightarrow (b \rightarrow a)') \\ &= (a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow ((a \rightarrow b) \rightarrow (b \rightarrow a)')') \\ &= (a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow ((a \rightarrow b) \otimes (b \rightarrow a))), \end{aligned}$$

we have $\mu((a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow ((a \rightarrow b) \otimes (b \rightarrow a)))) = \mu(1)$. By $\mu(a \rightarrow b) = \mu(1)$ and (F2), we get $\mu((b \rightarrow a) \rightarrow ((a \rightarrow b) \otimes (b \rightarrow a))) = \mu(1)$. By $\mu(b \rightarrow a) = \mu(1)$ and (F2), we get $\mu((a \rightarrow b) \otimes (b \rightarrow a)) = \mu(1)$, *i.e.*, $a \underset{\mu}{\equiv} b$. □

By Theorem 3.6, we obtain the following

Lemma 3.7 ([12])**.** $\underset{\mu}{\equiv}$ *is an equivalence relation on a lattice implication algebra L. Moreover, the relation is a congruence relation on L with respect to $\rightarrow$, i.e., $a \underset{\mu}{\equiv} b$ and $u \underset{\mu}{\equiv} v$ imply $a \rightarrow u \underset{\mu}{\equiv} b \rightarrow v$ for all $a, b, u, v \in L$.*

Theorem 3.8. *The relation $\underset{\mu}{\equiv}$ is a congruence relation on a lattice implication algebra L with respect to $\otimes$, i.e., $a \underset{\mu}{\equiv} b$ and $u \underset{\mu}{\equiv} v$ imply $a \otimes u \underset{\mu}{\equiv} b \otimes v$ for all $a, b, u, v \in L$.*

Proof. By Lemma 3.7, $\underset{\mu}{\equiv}$ is an equivalence relation on L. Now, we will prove the remainder part. Let $a, b, u, v \in L$ be such that $a \underset{\mu}{\equiv} b$ and $u \underset{\mu}{\equiv} v$. Then by Theorem 3.6, we have

$$\mu(a \rightarrow b) = \mu(b \rightarrow a) = \mu(u \rightarrow v) = \mu(v \rightarrow u) = \mu(1).$$

By (I3), we get $\mu(a' \rightarrow b') = \mu(b' \rightarrow a') = \mu(u' \rightarrow v') = \mu(v' \rightarrow u') = \mu(1)$. Thus by Theorem 3.6, we have

$$a' \underset{\mu}{\equiv} b' \text{ and } u' \underset{\mu}{\equiv} v'.$$

Hence by Lemma 3.7, we obtain $a \rightarrow u' \underset{\mu}{\equiv} b \rightarrow v'$ and $u \rightarrow a' \underset{\mu}{\equiv} v \rightarrow b'$. By Theorem 3.6, we have

$$\mu(1) = \mu((b \rightarrow v') \rightarrow (a \rightarrow u')) = \mu((a \rightarrow u')' \rightarrow (b \rightarrow v')') = \mu((a \otimes u) \rightarrow (b \otimes v)),$$

and

$$\mu(1) = \mu((a \rightarrow u') \rightarrow (b \rightarrow v')) = \mu((b \rightarrow v')' \rightarrow (a \rightarrow u')') = \mu((b \otimes v) \rightarrow (a \otimes u)),$$

and so we get $a \otimes u \underset{\mu}{\equiv} b \otimes v$. □

Combine the above fact, we lead to the quotient residuated lattice induced by fuzzy filter of lattice implication algebras.

Theorem 3.9. *Let μ be a fuzzy filter of a lattice implication algebra $(L, \vee, \wedge, ', \to, \otimes, 0, 1)$. We denote $\mu_a := \{b \in L | a \underset{\mu}{\equiv} b\}$ the equivalence class containing a and $L/\mu := \{\mu_a | a \in L\}$ the set of all equivalence classes of L. An operations $\sqcup, \sqcap, {}^N, \underset{\mu}{\to}, \bigotimes$ on L/μ are defined by*

$$\begin{aligned}
&\mu_a \sqcup \mu_b := \mu_{a\vee b},\\
&\mu_a \sqcap \mu_b := \mu_{a\wedge b},\\
&{\mu_a}^N := \mu_{a'},\\
&\mu_a \underset{\mu}{\to} \mu_b := \mu_{a\to b},\\
&\mu_a \bigotimes \mu_b := \mu_{a\otimes b},\\
&\mu_a \preceq \mu_b \Leftrightarrow \mu_a \underset{\mu}{\to} \mu_b = \mu_1.
\end{aligned}$$

Then $(L/\mu, \bigotimes, \underset{\mu}{\to})$ is a residuated lattice.

Proof. We know that L/μ is a bounded lattice with the least element μ_0 and the greatest element μ_1, and $\underset{\mu}{\to}$ is isotone in the first and antitone in the second variable([12]). For any $\mu_a, \mu_b, \mu_c, \mu_d \in L/\mu$, let $\mu_a \preceq \mu_c$ and $\mu_b \preceq \mu_d$. Then we have

$$\mu_a \underset{\mu}{\to} {\mu_b}^N \succeq \mu_c \underset{\mu}{\to} {\mu_b}^N \text{ and } {\mu_b}^N \succeq {\mu_d}^N.$$

Thus we obtain

$$\mu_c \underset{\mu}{\to} {\mu_b}^N \succeq \mu_c \underset{\mu}{\to} {\mu_d}^N.$$

Hence we get

$$(\mu_a \underset{\mu}{\to} {\mu_b}^N)^N \preceq (\mu_c \underset{\mu}{\to} {\mu_d}^N)^N,$$

and so $\mu_a \bigotimes \mu_b \preceq \mu_c \bigotimes \mu_d$, *i.e.*, $\bigotimes$ is isotone on $L/\mu \times L/\mu$.

Next, we will prove that the adjointness condition

$$\mu_a \bigotimes \mu_b \preceq \mu_c \Leftrightarrow \mu_a \preceq \mu_b \to \mu_c$$

for all $\mu_a, \mu_b, \mu_c \in L/\mu$. Since

$$\begin{aligned}
\mu_a \bigotimes \mu_b = \mu_{a\otimes b} = \mu_{(a\to b')'} &= (\mu_{a\to b'})^N\\
&= (\mu_a \underset{\mu}{\to} \mu'_b)^N\\
&= (\mu_a \underset{\mu}{\to} {\mu_b}^N)^N,
\end{aligned}$$

the adjointness condition is easily prove and so is omitted.

For all $\mu_a, \mu_b, \mu_c \in L/\mu$, we have

$$\mu_a \bigotimes \mu_b = \mu_{a\otimes b} = \mu_{b\otimes a} = \mu_b \bigotimes \mu_a, \text{ and}$$

$$\begin{aligned}
(\mu_a \bigotimes \mu_b) \bigotimes \mu_c &= ((\mu_a \underset{\mu}{\to} {\mu_b}^N)^N \underset{\mu}{\to} {\mu_c}^N)^N\\
&= (\mu_a \underset{\mu}{\to} (\mu_b \bigotimes \mu_c)^N)^N\\
&= \mu_a \bigotimes (\mu_b \bigotimes \mu_c), \text{ and}
\end{aligned}$$

$$\mu_a \bigotimes \mu_1 = \mu_{a\otimes 1} = \mu_{(a\to 1')'} = \mu_a.$$

Hence $(L/\mu, \bigotimes, \underset{\mu}{\to})$ is a commutative monoid. Therefore $(L/\mu, \bigotimes, \underset{\mu}{\to})$ is a residuated lattice. $\square$

In [2], Liu and Xu gave an equivalence relation on a lattice implication algebra $(L, \vee, \wedge, ', \to, \otimes, 0, 1)$ by using the filter, *i.e.*, let F be a filter of a lattice implication algebra $(L, \vee, \wedge, ', \to, \otimes, 0, 1)$. For all $a, b \in L$, we say that a is equivalent to b with respect to F denoted by

$$a \underset{F}{\equiv} b \text{ if and only if } (a \to b) \otimes (b \to a) \in F.$$

Theorem 3.10. *Let F be a filter of a lattice implication algebra $(L, \vee, \wedge, ', \to, \otimes, 0, 1)$ and χ_F be the characteristic funstion of F. Then for any $a, b \in L$, we have*

$$a \underset{F}{\equiv} b \textit{ if and only if } a \underset{\chi_F}{\equiv} b.$$

Proof. By definitions $a \underset{F}{\equiv} b$ and $a \underset{\chi_F}{\equiv} b$, we obtain

$$\begin{aligned} a \underset{F}{\equiv} b &\Leftrightarrow (a \to b) \otimes (b \to a) \in F \\ &\Leftrightarrow \chi_F((a \to b) \otimes (b \to a)) = 1 \\ &\Leftrightarrow \chi_F((a \to b) \otimes (b \to a)) = \chi_F(1) \\ &\Leftrightarrow a \underset{\chi_F}{\equiv} b. \quad \square \end{aligned}$$

By means of Theorem 3.8 and Theorem 3.10, we know that for any filter F of a lattice implication algebra $(L, \vee, \wedge, ', \to, 0, 1)$, $a \underset{F}{\equiv} b$ is a congruence relation on L. Denote by $F_a := \{b | a \underset{F}{\equiv} b\}$ the equivalence class containing a and L/F the set of all the equivalence classes of L via F. Obviously, $F_a = (\chi_F)_a$ and $L/F = L/\chi_F$. Thus we have the quotient residuated lattice induced by filter of lattice implication algebras.

Corollary 3.11. *Let F be a fuzzy filter of a lattice implication algebra $(L, \vee, \wedge, ', \to, \otimes, 0, 1)$, where $a \otimes b := (a \to b')'$ for all $a, b \in L$. An operations $\sqcup, \sqcap, ^N, \underset{F}{\to}, \bigotimes$ on L/F are defined by*

$$\begin{aligned} &F_a \sqcup F_b := F_{a\vee b}, \\ &F_a \sqcap F_b := F_{a\wedge b}, \\ &{F_a}^N := F_{a'}, \\ &F_a \underset{F}{\to} F_b := F_{a\to b}, \\ &F_a \bigotimes F_b := F_{a\otimes b}, \\ &F_a \preceq F_b \Leftrightarrow F_a \underset{F}{\to} F_b = F_1. \end{aligned}$$

Then $(L/F, \bigotimes, \underset{F}{\to})$ is a residuated lattice.

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