ON THE BCI-KG PART OF BCI-ALGEBRAS

Zhan Jianming and Tan Zhisong

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Abstract. In this paper, we introduce the concept of BCI-KG part BCI-algebras and study their properties.

1 Introduction We recall some definitions and elementary properties. The following identities hold for any BCI-algebra X:

(1) $x * 0 = x$;

(2) $(x * \gamma) * z = (x * z) * \gamma$

(3) $0 * (x * \gamma) = (0 * x) * (0 * \gamma)$

A nonempty subset A of a BCI-algebra X is called an ideal of X if $0 \in A$ and if $x*\gamma \in A$ and $\gamma \in A$ imply that $x \in A$. An ideal A of a BCI-algebra X is closed if $0 * x \in A$ for every $x \in A$.

For any elements x, γ in a BCI-algebra X, let us write $x * \gamma^n$ for $(\cdots(x * \gamma) * \gamma) * \cdots)$, where γ occurs n times.

Definition 1 For any BCI-algebra X and k a positive integer, the set $N_k(X) = \{x \in X \mid 0 * x^k = 0\}$. If $N_k(X) = 0$, then we say that X is a kp-semisimple BCI-algebra. In particular, $k = 1$, it's a p-semisimple BCI-algebra.

Definition 2 Let X be a BCI-algebra and K a positive integer. For any subset S of X, we difine

$$G_K(S) = \{x \in S \mid 0 * x^k = x\}.$$

In particular, if $S = X$, then we say that $G_K(X)$ is the BCI-KG part of X.

Definition 3 A nonempty subset I of a BCI-algebra X is called a k-ideal of X if

(1) $0 \in I$;

(2) $x * \gamma^k \in I$ and $\gamma \in I$ imply $x \in I$.

Lemma ([1]) *For any BCI-algebra X and any positive integer k,*

(1) $0 * (x * \gamma)^k = (0 * x^k) * (0 * \gamma^k)$

(2) $0 * (0 * x)^k = 0 * (0 * x^k)$

(3) $(x * \gamma^k) * z^k = (x * z^k) * \gamma^k$

(4) $(0 * x^m) * x^n = 0 * x^{m+n}$

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Proposition 1 $G_K(X) \cap N_K(X) = 0$

It's obvious.

Proposition 2 *If S is a subalgebra of a BCI-algebra X, then $G_K(S)$ is a subalgebra of X.*

Proof Let $x, \gamma \in G_K(S)$, then $0 * x^k = x$, $0 * \gamma^k = \gamma$ and $x, \gamma \in S$. Hence $0 * (x * \gamma)^k = (0 * x^k) * (0 * \gamma^k) = x * \gamma$ and $x * \gamma \in S$, because S is a subalgebra.

Therefore $x * \gamma \in G_K(S)$, which completes the proof.

Proposition 3 *Let X be a BCI-algebra. If $G_K(X) = X$, then X is Kp-semisimple.*

Proof Assume that $G_K(X) = X$, then by proposition 1, $0 = G_K(X) \cap N_K(X) = X \cap N_K(X) = N_K(X)$. Hence X is kp-semisimple.

2. Main Results

Theorem 1 *Let X be a BCI-algebra and K a positive integer, for any subalgebra S of X. If $G_K(S)$ is a k-ideal of X, then for any $x, \gamma \in N_K(X)$ and $a, b \in G_K(S)$,*

$$x * a^k = \gamma * b^k \text{ implies } x = \gamma \text{ and } a = b.$$

Proof Let $G_K(S)$ be a k-ideal of X. Assume that $x * a^k = \gamma * b^k$ for any $x, \gamma \in N_K(X)$ and $a, b \in G_K(S)$.

Then
$$\begin{aligned} a &= 0 * a^k = (0 * x^k) * (0 * a^k)^k = 0 * (x * a^k)^k \\ &= 0 * (\gamma * b^k)^k = (0 * \gamma^k) * (0 * b^k)^k \\ &= 0 * b^k = b \end{aligned}$$

Thus $x * a^k = \gamma * b^k$ implies

$$(x * \gamma) * a^k = (x * a^k) * \gamma = (\gamma * a^k) * \gamma = (\gamma * \gamma) * a^k = 0 * a^k = a \in G_K(S).$$

Since $G_K(S)$ is a k-ideal, it follows that $x * \gamma \in G_K(S)$.

As $x * \gamma \in N_K(X)$ and $N_K(X) \cap G_K(S) = 0$, we have $x * \gamma = 0$ *i.e.* $x \leq \gamma$.

Similarly, we get $\gamma \leq x$ and therefore $x = \gamma$.

This completes the proof.

For any subalgebra S of BCI-algebra X and any element a in S, we use $a_r^k(x)$ denotes the selfmap of S defined by $a_r^k(x) = x * a^k$ for any $x \in S$.

Theorem 2 Let S be a subalgebra of a BCI-algebra X. If $G_K(S)$ is a k-ideal of X, then a_r^k is bijective for any $a \in G_K(S)$.

Proof Let $a \in G_K(S)$ and $a_r^k(x) = a_r^k(\gamma)$ for some $x, \gamma \in S$, then $x * a^k = \gamma * a^k$.

$$\begin{aligned} ((x * \gamma) * a^k) * a &= ((x * a^k) * \gamma) * a \\ &= ((\gamma * a^k) * \gamma) * a \\ &= ((\gamma * \gamma) * a^k) * a \\ &= (0 * a^k) * a \\ &= a * a = 0 \end{aligned}$$

Similarly, we have $a * ((x * \gamma) * a^k) = 0$, and so $(x * \gamma) * a^k = a$.

Since $G_K(S)$ is a k-ideal, it follows that $x*\gamma \in G_K(S)$. Now

$$\begin{aligned} a*(x*\gamma)^k &= (0*a^k)*(x*\gamma)^k = (0*(x*\gamma)^k)*(0*a^k)^k \\ &= (0*(x*\gamma)^k)*(0*a^k)^k = 0*((x*\gamma)*a^k)^k \\ &= 0*a^k = a \end{aligned}$$

In particular, $0*(x*\gamma)^k = 0$, because $0 \in G_K(S)$.

Hence $x*\gamma = 0*(x*\gamma)^k = 0$. Likewise, we obtain that $\gamma * x = 0$, and thus $x = \gamma$.

This shows that a_r^k is injective.

To prove a_r^k is surjective, note that

$$(x*a^k)*a \leq x \text{ for any } x \in S \quad \text{i.e. } ((x*a^k)*a)*x = 0.$$

On the other hand,

$$\begin{aligned} a_r^k a_r^k(x*((x*a^k)*a)) &= a_r^k((x*((x*a^k)*a))*a^k) \\ &= a_r^k((x*a^k)*((x*a^k)*a)) \\ &= ((x*a^k)*((x*a^k)*a))*a^k \\ &= ((x*a^k)*a^k)*((x*a^k)*a) \\ &= (((x*a^k)*a^{k-1})*((x*a^k)*a))*a \\ &= (((x*a^k)*a)*((x*a^k)*a))*a^{k-1} \\ &= 0*a^{k-1} = (a*a)*a^{k-1} = a*a^k = (0*a^k)*a^k \\ &= a_r^k a_r^k(0) \end{aligned}$$

Since a_r^k is injective, it follows that $x*((x*a^k)*a) = 0$

Hence $x = (x*a^k)*a = (x*a)*a^k = a_r^k(x*a)$. So a_r^k is subjective. The proof is completed.

Theorem 3 *Let X be a BCI-algebra which satisfies tthe identity $a*b^k = a*b$ for all $a, b \in G_K(S)$. If $G_K(S)$ is closed ideal of X, then $a_r^k a_r^k = (a*b)_r^k$.*

Proof For any $x \in S$, we have

$$\begin{aligned} a_r^k a_r^k((x*(a*b)^k)*((x*b^k)*a)) &= (((x*(a*b)^k)*((x*b^k)*a^k))*b^k)*a^k \\ &= (((x*(a*b)^k)*b^k)*((x*b^k)*a^k))*a^k \\ &= (((x*b^k)*(a*b)^k)*((x*b^k)*a^k))*a^k \\ &= (((x*b^k)*a^k)*(a*b)^k)*((x*b^k)*a^k) \\ &= (((x*b^k)*a^k)*((x*b^k)*a^k))*(a*b)^k \\ &= 0*(a*b)^k \\ &= a*b = a*b^k = (0*a^k)*b^k = (0*b^k)*a^k \\ &= a_r^k a_r^k(0) \end{aligned}$$

and

$$\begin{aligned} (a*b)_r^k(((x*b^k)*a^k)*(x*(a*b)^k)) &= (((x*b^k)*a^k)*(x*(a*b)^k))*(a*b)^k \\ &= (((x*a^k)*b^k)*(a*b)^k))*(x*(a*b)^k) \\ &= (((x*(a*b)^k)*a^k)*b^k)*(x*(a*b)^k) \\ &= (((x*(a*b)^k)*(x*(a*b)^k))*a^k)*b^k \\ &= (0*a^k)*b^k \\ &= a*b^k = a*b \\ &= 0*(a*b)^k \\ &= (a*b)_r^k(0) \end{aligned}$$

As $G_K(S)$ is a ideal; hence a subalgebra.

Therefore $a * b \in G_K(S)$ and $(a * b)_r^k$ is injective by Theorem 2. Also since $a_r^k a_r^k$ is injective, we have $(x * (a * b)^k) * ((x * b^k) * a^k) = 0$ and $((x * b^k) * a^k) * (x * (a * b)^k) = 0$

Hence $a_r^k a_r^k(x) = (a * b)_r^k(x)$ for any $x \in S$, which implies $a_r^k a_r^k = (a * b)_r^k$ for any $a, b \in G_K(S)$.

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Dept. of Math. Hubei Institute for Nationalities, Enshi, Hubei Province, 445000. P. R. China

E-mail : zhanjianming@hotmail.com