SOME RESULTS IN HYPER BCK-ALGEBRAS

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Abstract. We introduce the concepts of a strong implicative hyper BCK-ideal and a strong positive implicative hyper BCK-ideal in hyper BCK-algebras, and investigates some related properties. Also we introduce the notions of a maximal hyper BCK-ideal and boundedness in hyper BCK-algebras, and investigates some related properties. First, we show that every strong (positive) implicative hyper BCK-ideal is a strong hyper BCK-ideal and (positive) implicative hyper BCK-ideal, but the converse is not true. Also we obtain the equivalent condition of strong implicative hyper BCK-ideals and strong hyper BCK-ideals, and we discuss the relations between strong positive implicative hyper BCK-ideals and strong implicative hyper BCK-ideals. Next, we show that if H is a bounded hyper BCK-algebra and $|H| \geq 2$, then H has at least one maximal strong hyper BCK-ideal, and if H is a bounded hyper BCK-algebra and I is a proper strong (positive) implicative hyper BCK-ideal of H, then there is a maximal strong (positive) implicative hyper BCK-ideal containing I.

1. Introduction

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras, In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras) is introduced in 1934 by F. Marty [9] at the 8th congress of Scandinavian Mathematiciens. In [8], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra, and investigated some related properties. They also introduced the notion of a hyper BCK-ideal and a weak hyper BCK-ideal, and gave relations between hyper BCK-ideals and weak hyper BCK-ideals. Y. B. Jun et al. [7] gave a condition for a hyper BCK-algebra to be a BCK-algebra, and introduced the notion of a strong hyper BCK-ideal, a weak hyper BCK-ideal and a reflexive hyper BCK-ideal. They showed that every strong hyper BCK-ideal is a hypersubalgebra, a weak hyper BCK-ideal and a hyper BCK-ideal; and every reflexive hyper BCK-ideal is a strong hyper BCK-ideal. In [5], Y. B. Jun and X. L. Xin introduced the notion of an implicative hyper BCK-ideal. They gave the relations among hyper BCK-ideals, implicative hyper BCK-ideals and positive implicative hyper BCK-ideals. They stated some characterizations of implicative hyper BCK-ideals. And they also introduced the notion of implicative hyper BCK-algebras and investigated the relation between implicative hyper BCK-ideals and implicative hyper BCK-algebras. In [6], Y. B. Jun and X. L. Xin introduced the notion of a positive implicative hyper BCK-ideal, and investigated some related properties. E. H. Roh et al. [11] gave the extension theorem for

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hyper BCK-algebras. In this paper we introduce the notions of strong implicative hyper BCK-ideals and strong positive implicative hyper BCK-ideals in hyper BCK-algebras, and also we introduce the concepts of a maximal hyper BCK-ideal and boundedness in hyper BCK-algebras, and investigates some related properties.

2. Preliminaries

Let H be a non-empty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

Definition 2.1 ([8])**.** By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfing the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,
for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

Example 2.2 ([8])**.** (1) Let $(H, *, 0)$ be a BCK-algebra and define a hyperoperation "$\circ$" on H by $x \circ y = \{x * y\}$ for all $x, y \in H$. Then $(H, \circ)$ is a hyper BCK-algebra.
(2) Define a hyperoperation "$\circ$" on $H := [0, \infty)$ by

$$x \circ y := \begin{cases} [0, x] & \text{if } x \leq y \\ (0, y] & \text{if } x > y \neq 0 \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circ)$ is a hyper BCK-algebra.
(3) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then $(H, \circ)$ is a hyper BCK-algebra.

Proposition 2.3 ([8])**.** *In a hyper BCK-algebra $(H, \circ)$, the condition* (HK3) *is equivalent to the condition:*
(i) $x \circ y \ll \{x\}$ *for all* $x, y \in H$.

Proposition 2.4 ([4, 8])**.** *Let $(H, \circ)$ be a hyper BCK-algebra. Then the following hold:*
(1) $x \circ 0 \ll \{x\}$, $0 \circ x \ll \{0\}$ *and* $0 \circ 0 \ll \{0\}$,
(2) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ *and* $0 \circ A \ll \{0\}$
(3) $0 \ll x$,
(4) $0 \circ 0 = \{0\}$,
(5) $x \ll x$,
(6) $A \ll A$,
(7) $A \subseteq B$ *implies* $A \ll B$,
(8) $0 \circ x = \{0\}$,
(9) $0 \circ A = \{0\}$,

(10) $A \ll \{0\}$ *implies* $A = \{0\}$,
(11) $A \circ B \ll A$,
(12) $x \circ 0 = \{x\}$,
(13) $A \circ 0 = A$,
(14) $x \circ 0 \ll \{y\}$ *implies* $x \ll y$,
(15) $y \ll z$ *implies* $x \circ z \ll x \circ y$,
(16) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$,
(17) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$,
for all $x, y, z \in H$ *and all non-empty subsetes* A, B *and* C *of* H.

Definition 2.5 ([8]). Let $(H, \circ)$ be a hyper BCK-algebra and let S be a subset of H containing 0. If S is a hyper BCK-algebra with respect to the hyperoperation "$\circ$" on H, we say that S is a *hypersubalgebra* of H.

Definition 2.6 ([5, 6, 7, 8]). Let I be a non-empty subset of a hyper BCK-algebra H. Then
I is said to be a *hyper BCK-ideal* of H if
(HI1) $0 \in I$,
(HI2) $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.
I is called a *weak hyper BCK-ideal* of H if
(HI1) $0 \in I$,
(WHI) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.
I is called a *strong hyper BCK-ideal* of H if
(HI1) $0 \in I$,
(SHI) $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.
I is called an *implicative hyper BCK-ideal* if it satisfies:
(HI1) $0 \in I$,
(IHI) $(x \circ z) \circ (y \circ x) \ll I$ and $z \in I$ imply $x \in I$
for all $x, y, z \in H$.
I is called a *positive implicative hyper BCK-ideal* if it satisfies:
(HI1) $0 \in I$,
(PIHI) $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$ for all $x, y, z \in H$.

Note from [4] that every hyper BCK-ideal of a hyper BCK-algebra H is a hypersubalgebra of H, but the converse may not be true. Note from [8] that every hyper BCK-ideal of a hyper BCK-algebra H is a weak hyper BCK-ideal of H, but the converse may not be true.

Lemma 2.7 ([7]). *Let I be a reflexive hyper BCK-ideal of a hyper BCK-algebra H. Then*

$$(x \circ y) \cap I \neq \emptyset \text{ implies } x \circ y \ll I \text{ for all } x, y \in H.$$

Proposition 2.8 ([4]). *Let A be a subset of a hyper BCK-algebra H. If I is a hyper BCK-ideal of H such that $A \ll I$, then $A \subseteq I$.*

Proposition 2.9 ([5]). *In a hyper BCK-algebra H, the following axiom holds:*

$$((x \circ z) \circ (y \circ z)) \circ u \ll (x \circ y) \circ u \text{ for all } x, y, z, u \in H.$$

3. Strong implicative hyper BCK-ideals

Definition 3.1. Let H be a hyper BCK-algebra. A non-empty subset I of H is called a *strong implicative hyper BCK-ideal* if it satisfies:

(HI1) $0 \in I$,

(SIHI) $((x \circ z) \circ (y \circ x)) \cap I \neq \emptyset$ and $z \in I$ imply $x \in I$ for all $x, y, z \in H$.

Example 3.2. Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Then $(H, \circ)$ is a hyper BCK-algebra([7]). Putting $I := \{0, 2\}$, we can see that I is a strong implicative hyper BCK-ideal of H. But $\{0, 1\}$ is not a strong implicative hyper BCK-ideal of H since $((2 \circ 0) \circ (2 \circ 2)) \cap \{0, 1\} \neq \emptyset$ and $0 \in \{0, 1\}$ but $2 \notin \{0, 1\}$.

Theorem 3.3. *If $\{I_\lambda | \lambda \in \Lambda\}$ is a family of strong implicative hyper BCK-ideals of a hyper BCK-algebra H, then so is $\bigcap_{\lambda \in \Lambda} I_\lambda$.*

Proof. For any $\lambda \in \Lambda$, let I_λ be a strong implicative hyper BCK-ideal of a hyper BCK-algebra H. Then clearly $0 \in \bigcap_{\lambda \in \Lambda} I_\lambda$. Let $x, y, z \in H$ be such that $((x \circ z) \circ (y \circ x)) \cap (\bigcap_{\lambda \in \Lambda} I_\lambda) \neq \emptyset$ and $z \in \bigcap_{\lambda \in \Lambda} I_\lambda$. Then $((x \circ z) \circ (y \circ x)) \cap I_\lambda \neq \emptyset$ and $z \in I_\lambda$ for all $\lambda \in \Lambda$. By using (SIHI), we have $x \in I_\lambda$ for all $\lambda \in \Lambda$, and hence $x \in \bigcap_{\lambda \in \Lambda} I_\lambda$. □

Theorem 3.4. *Every strong implicative hyper BCK-ideal of a hyper BCK-algebra H is a strong hyper BCK-ideal of H.*

Proof. Let I be a strong implicative hyper BCK-ideal of a hyper BCK-algebra H and let $x, y \in H$ be such that $(x \circ y) \cap I \neq \emptyset$ and $y \in I$. Note that $(x \circ y) \circ (0 \circ x) = (x \circ y) \circ 0 = x \circ y$. Thus we have $((x \circ y) \circ (0 \circ x)) \cap I \neq \emptyset$ and $y \in I$. Using (SIHI), we get $x \in I$ and so I is a strong hyper BCK-ideal of H. □

The converse of Theorem 3.4 may not be true since $\{0, 1\}$ is a strong hyper BCK-ideal of H in Example 3.2.

By Theorem 3.8 in [7] and Theorem 3.4, we have the following result.

Corollary 3.5. *Let I be a strong implicative hyper BCK-ideal of a hyper BCK-algebra H. Then*

(i) *I is a hypersubalgebra of H,*

(ii) *I is a weak hyper BCK-ideal of H,*

(iii) *I is a hyper BCK-ideal of H.*

Theorem 3.6. *Every strong implicative hyper BCK-ideal of a hyper BCK-algebra H is an implicative hyper BCK-ideal of H.*

Proof. Let I be a strong implicative hyper BCK-ideal of a hyper BCK-algebra H and let $x, y, z \in H$ be such that $(x \circ z) \circ (y \circ x) \ll I$ and $z \in I$. Then for each $a \in (x \circ z) \circ (y \circ x)$ there exists $b \in I$ such that $a \ll b$, i.e., $0 \in a \circ b$. It follows that $(a \circ b) \cap I \neq \emptyset$. By Theorem 3.4, we have $a \in I$. Thus $(x \circ z) \circ (y \circ x) \subseteq I$ and so $((x \circ z) \circ (y \circ x)) \cap I \neq \emptyset$, and using (SIHI) we get $x \in I$. Hence I is an implicative hyper BCK-ideal of H. □

In general, the converse of Theorem 3.6 may not be true.

Example 3.7. Let $H = \{0,1,2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then $(H,\circ)$ is a hyper BCK-algebra and $I := \{0,2\}$ is an implicative hyper BCK-ideal of H([5]). But I is not a strong implicative hyper BCK-ideal of H since $((2\circ 1)\circ(0\circ 2))\cap I \neq \emptyset$ and $1 \in I$ but $2 \notin I$.

We know that the converse of Theorem 3.4 is not true. It is then natural to ask that given a strong hyper BCK-ideal, under what condition the converse of the Theorem 3.4 is also true? Now we solve this question.

Theorem 3.8. *Let I be a non-empty subset of a hyper BCK-algebra H. Then I is a strong implicative hyper BCK-ideal of H if and only if I is a strong hyper BCK-ideal of H and $(x\circ(y\circ x))\cap I \neq \emptyset$ implies $x \in I$ for all $x, y \in H$.*

Proof. Assume that I is a strong implicative hyper BCK-ideal of a hyper BCK-algebra H. By Theorem 3.4, I is a strong hyper BCK-ideal of H. Let $x, y \in H$ be such that $(x\circ(y\circ x))\cap I \neq \emptyset$. Note that $(x\circ 0)\circ(y\circ x) = x\circ(y\circ x)$. Thus we get $((x\circ 0)\circ(y\circ x))\cap I \neq \emptyset$. Since $0 \in I$, it follows from (SIHI) that $x \in I$

Conversely, suppose that I is a strong hyper BCK-ideal of a hyper BCK-algebra H and $(x\circ(y\circ x))\cap I \neq \emptyset$ implies $x \in I$ for all $x, y \in H$. Let $x, y, z \in H$ be such that $((x\circ z)\circ(y\circ x))\cap I \neq \emptyset$ and $z \in I$. Then $((x\circ(y\circ x))\circ z)\cap I \neq \emptyset$ and so $(u\circ z)\cap I \neq \emptyset$ for all $u \in x\circ(y\circ x)$. Since I is a strong hyper BCK-ideal and $z \in I$, we have $u \in I$ for all $u \in x\circ(y\circ x)$ and thus $x\circ(y\circ x) \subseteq I$ which implies $(x\circ(y\circ x))\cap I \neq \emptyset$. By hypothesis, we get $x \in I$, which shows that I is a strong implicative hyper BCK-ideal of H. □

4. Strong positive implicative hyper BCK-ideals

Definition 4.1. Let H be a hyper BCK-algebra. A non-empty subset I of H is called a *strong positive implicative hyper BCK-ideal* if it satisfies:

(HI1) $0 \in I$,

(SPIHI) $((x\circ y)\circ z)\cap I \neq \emptyset$ and $y\circ z \subseteq I$ imply $x\circ z \subseteq I$ for all $x, y, z \in H$.

Example 4.2. In Example 3.2, we can see that $\{0,2\}$ is a strong positive implicative hyper BCK-ideal of H.

Theorem 4.3. *If $\{I_\lambda | \lambda \in \Lambda\}$ is a family of strong positive implicative hyper BCK-ideals of a hyper BCK-algebra H, then so is $\bigcap_{\lambda\in\Lambda} I_\lambda$.*

Proof. For any $\lambda \in \Lambda$, let I_λ be a strong positive implicative hyper BCK-ideal of a hyper BCK-algebra H. Then clearly $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Let $x, y, z \in H$ be such that $((x\circ y)\circ z)\cap(\bigcap_{\lambda\in\Lambda} I_\lambda) \neq \emptyset$ and $y\circ z \subseteq \bigcap_{\lambda\in\Lambda} I_\lambda$. Then $((x\circ y)\circ z)\cap I_\lambda \neq \emptyset$ and $y\circ z \subseteq I_\lambda$ for all $\lambda \in \Lambda$. By using (SPIHI), we have $x\circ z \subseteq I_\lambda$ for all $\lambda \in \Lambda$, and hence $x\circ z \subseteq \bigcap_{\lambda\in\Lambda} I_\lambda$. □

Theorem 4.4. *Every strong positive implicative hyper BCK-ideal of a hyper BCK-algebra H is a strong hyper BCK-ideal of H.*

Proof. Let I be a strong positive implicative hyper BCK-ideal of a hyper BCK-algebra H and let $x, y \in H$ be such that $(x\circ y)\cap I \neq \emptyset$ and $y \in I$. Putting $z = 0$ in (SPIHI), we have

$((x \circ y) \circ 0) \cap I \neq \emptyset$ since $(x \circ y) \circ 0 = x \circ y$. Because of $y \circ 0 = \{y\} \subseteq I$, it follows from (SPIHI) that $\{x\} = x \circ 0 \subseteq I$. Thus I is a strong hyper BCK-ideal of H. □

The converse of Theorem 4.4 may not be true. In Example 3.2, $I := \{0, 1\}$ is a strong hyper BCK-ideal of H, but I is not a strong positive implicative hyper BCK-ideal of H since $((2 \circ 0) \circ 2) \cap I \neq \emptyset$ and $0 \circ 2 \subseteq I$ but $2 \circ 2 \nsubseteq I$.

By Theorem 3.8 in [7] and Theorem 4.4, we have the following result.

Corollary 4.5. *Let I be a strong positive implicative hyper BCK-ideal of a hyper BCK-algebra H. Then*

(i) *I is a hypersubalgebra of H,*
(ii) *I is a weak hyper BCK-ideal of H,*
(iii) *I is a hyper BCK-ideal of H.*

Theorem 4.6. *Every strong positive implicative hyper BCK-ideal of a hyper BCK-algebra H is a positive implicative hyper BCK-ideal of H.*

Proof. Let I be a strong positive implicative hyper BCK-ideal of a hyper BCK-algebra H and let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$. Then for each $a \in (x \circ y) \circ z$ there exists $b \in I$ such that $a \ll b$, i.e., $0 \in a \circ b$. It follows that $(a \circ b) \cap I \neq \emptyset$. By Theorem 4.4, we have $a \in I$. Thus $(x \circ y) \circ z \subseteq I$ and so $((x \circ y) \circ z) \cap I \neq \emptyset$, and using (SPIHI) we get $x \circ z \subseteq I$. Hence I is a positive implicative hyper BCK-ideal of H. □

In general, the converse of Theorem 4.6 may not be true. In Example 3.2, we can see that $\{0, 1\}$ is a positive implicative hyper BCK-ideal of H, but $\{0, 1\}$ is not a strong positive implicative hyper BCK-ideal of H.

In the above section, we introduced the notion of strong (positive) implicative hyper BCK-ideals in hyper BCK-algebras. Now, we discuss relations between strong positive implicative hyper BCK-ideals and strong implicative hyper BCK-ideals.

Theorem 4.7. *Each reflexive strong implicative hyper BCK-ideal of a hyper BCK-algebra H is a strong positive implicative hyper BCK-ideal of H.*

Proof. Assume that I is a reflexive strong implicative hyper BCK-ideal of a hyper BCK-algebra H. Let $x, y, z \in H$ be such that $((x \circ y) \circ z) \cap I \neq \emptyset$ and $y \circ z \subseteq I$. First, we prove that $(x \circ z) \circ z \subseteq I$. Assume $t \in x \circ z$. Note that

$$(t \circ z) \circ (y \circ z) \ll t \circ y \subseteq (x \circ y) \circ z.$$

Since $((x \circ y) \circ z) \cap I \neq \emptyset$, we have $(x \circ y) \circ z \ll I$ by Lemma 2.7. By Proposition 2.8, we get $(x \circ y) \circ z \subseteq I$ and so $(t \circ z) \circ (y \circ z) \ll I$. Using Proposition 2.8 again, we have $(t \circ z) \circ (y \circ z) \subseteq I$ and hence $((t \circ z) \circ (y \circ z)) \cap I \neq \emptyset$. Since I is a strong hyper BCK-ideal, it follows from $y \circ z \subseteq I$ that $t \circ z \subseteq I$ for each $t \in x \circ z$. Hence $(x \circ z) \circ z \subseteq I$. Next, we prove that $x \circ z \subseteq I$. Let $t \in x \circ z$. Then we have

$$\begin{aligned} ((x \circ z) \circ (x \circ t)) \circ (t \circ z) &= ((x \circ z) \circ (t \circ z)) \circ (x \circ t) \\ &= \bigcup_{u \in x \circ t} (((x \circ z) \circ (t \circ z)) \circ u) \\ &\ll \bigcup_{u \in x \circ t} ((x \circ t) \circ u) \quad \text{[by Proposition 2.9]} \\ &= (x \circ t) \circ (x \circ t) \\ &\ll x \circ x \subseteq I. \end{aligned}$$

By Proposition 2.8, we have that $((x \circ z) \circ (x \circ t)) \circ (t \circ z) \subseteq I$. Since $t \circ z \subseteq (x \circ z) \circ z \subseteq I$, we get $(x \circ z) \circ (x \circ t) \subseteq I$. It follows that $(x \circ z) \circ (x \circ (x \circ z)) \subseteq I$. Since $b \circ (x \circ b) \subseteq (x \circ z) \circ (x \circ (x \circ z))$ for all $b \in x \circ z$, we have $b \circ (x \circ b) \subseteq I$ and hence $(b \circ 0) \circ (x \circ b) \subseteq I$. Since I is a strong implicative hyper BCK-ideal and $0 \in I$, it follows that $b \in I$ for all $b \in x \circ z$. Therefore we obtain $x \circ z \subseteq I$, which show that I is a strong positive implicative hyper BCK-ideal of H. □

5. Bounded hyper BCK-algebras

Definition 5.1. Given a hyper BCK-algebra H, a hyper BCK-ideal I of H is said to be *maximal* if $I \neq H$ and for every hyper BCK-ideal J such that $I \subset J \subset H$, either $J = I$ or $J = H$.

Example 5.2 ([11])**.** Let $(H, \circ')$ be a hyper BCK-algebra and $u \notin H$. We define a hyper operation $\circ$ on $H_1 := H \cup \{u\}$ as follows: for all $x, y \in H_1$,

$$x \circ y := \begin{cases} x \circ' y & \text{if } x, y \in H, \\ \{0\} & \text{if } x \in H, y = u \text{ or } x = y = u, \\ \{u\} & \text{if } x = u, y \in H. \end{cases}$$

Then $(H_1, \circ)$ is a hyper BCK-algebra.

Theorem 5.3. *In Example 5.2, H is a maximal strong hyper BCK-ideal of a hyper BCK-algebra H_1.*

Proof. Clearly, $0 \in H$. Let $x, y \in H_1$ be such that $(x \circ y) \cap H \neq \emptyset$ and $y \in H$. If $x = u$, then we get $(x \circ y) \cap H = \{u\} \cap H = \emptyset$, it is a contradiction. Thus H is a maximal strong hyper BCK-ideal of a hyper BCK-algebra H_1. □

Definition 5.4. Let H be a hyper BCK-algebra. If there exists an element $u \in H$ such that $x \ll u$ for all $x \in H$, then H is said to be *bounded* and such u is called the *unit* of H.

Note that if H is a bounded hyper BCK-algebra, then the unit of H is unique by (HK4).

Example 5.5. (1) In Example 5.2, H_1 is a bounded hyper BCK-algebra and the unit is u.

(2) In Example 2.2(3), H is a bounded hyper BCK-algebra and the unit is 2.

(3) In Example 3.2, $(H, \circ)$ is not bounded since there does not exist $u \in H$ such that $x \ll u$ for all $x \in H$.

Theorem 5.6. *If H is a bounded hyper BCK-algebra and $|H| \geq 2$, then H has at least one maximal strong hyper BCK-ideal.*

Proof. First, we prove that a strong hyper BCK-ideal I of H is proper if and only if $u \notin I$, where u is the unit of H. In fact, if $u \notin I$, then $I \neq H$, and so I is a proper strong hyper BCK-ideal. Conversely, assume that I is a proper strong hyper BCK-ideal of H and let $x \in H$. If $u \in I$, then since $x \ll u$, we have $(x \circ u) \cap I \neq \emptyset$. Since I is a strong hyper BCK-ideal, we get $x \in I$. This means that $I = X$, which contradicts that I is proper. Therefore $u \notin I$. We now prove that every strong hyper BCK-ideal A of H is contained in a maximal strong hyper BCK-ideal. The set of all proper strong hyper BCK-ideals containing A is denoted by $\mathcal{S}$. Obviously, $(\mathcal{S}, \subseteq)$ is a partially ordered set and $\mathcal{S} \neq \emptyset$. Let $\mathcal{S}_0$ be a chain of $\mathcal{S}$ and let $B := \cup\{I | I \in \mathcal{S}_0\}$. Noticing that A is the least element of $(\mathcal{S}, \subseteq)$, we have $A \subseteq B$. Hence $0 \in B$. Let $x, y \in H$ be such that $(x \circ y) \cap B \neq \emptyset$ and $y \in B$. Then there are $I_1, I_2 \in \mathcal{S}_0$ such that $(x \circ y) \cap I_1 \neq \emptyset$ and $y \in I_2$. We may assume $I_2 \subseteq I_1$, without loss of generality. Thus $(x \circ u) \cap I_1 \neq \emptyset, y \in I_1$ and so $x \in I_1$. It follows that $x \in B$. This means that B is a strong hyper BCK-ideal of H. Since every strong hyper BCK-ideal of $\mathcal{S}_0$ does not contain the element u, we have $u \notin B$, and so B is a proper strong hyper BCK-ideal. Hence $B \in \mathcal{S}$. This proves that every chain of $\mathcal{S}$ has an upper bound in $\mathcal{S}$. By Zorn's Lemma, $\mathcal{S}$ have a maximal element M. Clearly, $A \subseteq M$. Therefore M is indeed a maximal strong hyper BCK-ideal. □

As an immediate consequence of the above theorem, we have

Corollary 5.7. *Suppose that H is a bounded hyper BCK-algebra and let I be a proper strong (positive) implicative hyper BCK-ideal of H. Then there is a maximal strong (positive) implicative hyper BCK-ideal containing I.*

References

[1] P. Corsini, *Prolegomena of hypergroup theory*, Aviani Editore, 1993.

[2] K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japon. **21** (1976), 351-366.

[3] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon. **23(1)** (1978), 1-26.

[4] Y. B. Jun and X. L. Xin, *Scalar elements and hyperatoms of hyper BCK-algebras*, Scientiae Mathematicae **2(3)** (1999), 303-309.

[5] Y. B. Jun and X. L. Xin, *Implicative hyper BCK-ideals of hyper BCK-algebras*, Math. Japonica **52(3)** (2000), 435-443.

[6] Y. B. Jun and X. L. Xin, *Positive implicative hyper BCK-ideals of hyper BCK-algebras*, To appear in Math. Japonica.

[7] Y. B. Jun, X. L. Xin, M. M. Zahedi and E. H. Roh, *Strong hyper BCK-ideals of hyper BCK-algebras*, Math. Japonica **51(3)** (2000), 493-498.

[8] Y. B. Jun, M. M. Zahedi, X. L. Xin and R. A. Borzoei, *On hyper BCK-algebras*, Italian Journal of Pure and Applied Mathematics **8** (2000), 127-136.

[9] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm (1934), 45-49.

[10] J. Meng and Y. B. Jun, *BCK-algebras*, Kyungmoonsa, Seoul, Korea, 1994.

[11] E. H. Roh, Y. B. Jun and W. H. Shim, *On extension of hyper BCK-algebras*, Submitted.

[12] M. M. Zahedi and A. Hasankhani, *F-polygroups (I)*, J. Fuzzy Math. **3** (1996), 533-548.

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