A DUOPOLISTIC INVENTORY PROBLEM INCLUDING THE POSSIBILITY THAT THE CUSTOMERS GIVE UP PURCHASING THE MERCHANDISE

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ABSTRACT. This paper considers a competing inventory model with partial re-allocation over the unit interval. The model is described as follows: There are two retailers which handle the same kind of products. They open their stores at the both ends of the street with unit distance. Customers are uniformly distributed over the street. Though they are willing to purchase one of products at first, they may give up purchasing it on their way. Under this situation, each retailer is planning to minimize the sum of costs related with holding inventory, shortages and profits. The purpose of each retailer is to decide his order quantity. This model constructs a variation of the unit square games with pure strategies of continuous cardinary. We examine the optimal strategies for two players from the view point of non-zero sum game theory. We are interested in equilibrium analysis giving the optimal strategies.

1 Introduction We consider a duopolistic inventory problem including the possibility that the customers give up purchasing the merchandise. When many retailers are involved in a decision situation, each retailer needs to decide his strategy while giving careful consideration to the opposite strategy. We use the equilibrium concepts instead of the classical optimization concepts to analyze our inventory model.

Parlar [9] is the first paper which deals with an inventory problem using game theory. He proved the existence and uniqueness of the Nash solution for an inventory problem with two substitutable products having random demands. Hohjo [4] newly proposed a competitive inventory problem, which introduces the conception on time and distance, for two retailers selling the same kind of products, and then found the optimal strategies under equilibruim for such a competitive inventory problem. However, he was assumed that all customers are eager to purchase products even if the first visiting retailer stocks out. In fact, it is conceivable that some customers may give up purchasing the products with knowledge of stocking out. This paper suggests a concrete duopolistic model with partial re-allocation. We are interested in equilibrium analysis giving the optimal strategies. For researches on equilibrium, the author refers to Bryant [1], Kirman and Sobel [5], Levitan [6], Lippman and McCardle [7], and Topkis [10]. For works related to game theory, see Dresher [2], Nash [8] and Simaan and Cruz [11].

The papar is planed as follows. In Section 2, we introduce a duopolistic inventory model with partial re-allocation over the unit interval. Section 3 calculates the objective functions for two retailers. Section 4 is devote to investigate an equilibrium point. We analyze our model in a way to result in a bimatrix game and find an equilibrium point for two retailers. Finally, the paper is concluded with some remarks. We hope that a work in this paper will give one of criteria to decide the optimal order quantities in the competing cases.

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2 Model and Notation Consider a single period duopolistic inventory model with partial re-allocation. The model is described as follows: There are two retailers, Player 1 and 2, who open their stores at the both end points 0 and 1 on the unit interval $[0, 1]$, respectively. They simultaneously begin to sell some product to customers and share all customers. Their initial inventory levels are zero. Their orders for products are placed once at the beginning of period, and the products are delivered without lead time. When they order the products, the set-up costs are not charged but the ordering costs are charged in proportion to their order quantities. If they have some stock to sell, then they are charged the holding costs. On the other hand, if the all products have been already stocked out when the demand has occurred, then they will bear penalty costs for shortage. Holding costs, penalty costs and profits are incurred in proportion to their inventory quantities, shortage quantities and demand, respectively. In this model we can assume that the ordering costs are equal to 0. Players have a penalty in the sense that they will lose their confidence to customers if they have nothing to sell in inventory. Although excess demand for a player is not backlogged by himself, it is re-allocated to another player. Assume that two players mutually know values of the opposite unit holding cost, penalty cost and profit.

Customers are uniformly distributed over the range $[0, 1]$. Each customer goes to a shop located closer to him so as to purchase one unit product. As soon as he gets information such that in the first visiting shop the stock out occurs, he travels to another shop with a certain probability in order to satisfy his demand. The customers start from their positions at the same time and travel with the same speed. Then the arrival time taken from their positions to the shop is proportional to the travel distance. The customers do not know inventory quantities in the shops at any time. We treat a non-cooperative model for two players. The purpose of each player is to decide his order quantity which minimizes his total expected cost, taking account of the inventory holding, shortages and profit.

As the first step in our analysis, we will calculate the total expected costs and find the optimal order quantities which attain their purposes. We describe the notation and assumptions used in this paper:

z_i	:	order quantity chosen by Player i, $z_i \geq 0$
q	:	probability with which customers give up purchasing a product, $0 < q \leq 1$
r_i	:	profit/unit for Player i, $r_i \geq 0$
h_i	:	holding cost/unit for Player i, $h_i \geq 0$
p_i	:	penalty cost/unit for Player i, $p_i \geq 0$
t	:	traveling time per unit distance.

Suppose that all values except for z_i are given. The planning period is $\frac{3}{2}t$ if they wait for the last possible customer until the time when he may come. We assume that the number of customers in a market is equal to 1 without loss of generality. In order to avoid an excessive inventory, he never place an order for products more than demand. Consequently, his strategy z_i can be restricted to the interval $[0, 1]$ if he has a positive holding cost h_i.

3 Objective Function There are six situations by taking account of the order quantity z_i and our model assumptions. We may consider just four situations because of symmetry on behavior of players. In order to explain these situations, we suppose that $C^i_j(z_1, z_2)$ denotes the total expected cost for Player $i (i = 1, 2)$, and the number $j (j = 1, 2, \dots, 6)$, corresponding to each situation described below. We give the inventory quantity $Q_i(T)$ for Player i at time T and calculate the total expected cost $C^i_j(z_1, z_2)$ as follows.

(i) Let $\frac{1}{2} \leq z_i \leq 1, i = 1, 2$. Then both players can supply products to all customers without stocking out by time $\frac{1}{2}t$. No demand occurs after that time.

For Player i the on-hand inventory $Q_i(T)$ is written as

$$Q_i(T) = \begin{cases} z_i - \frac{T}{t}, & 0 \le T < \frac{1}{2}t \\ z_i - \frac{1}{2}, & \frac{1}{2}t \le T \le \frac{3}{2}t. \end{cases} \tag{1}$$

Then the total expected cost $C_1^i(z_1, z_2)$ for Player i in this situation is given by

$$\begin{aligned} C_1^i(z_1, z_2) &= h_i \cdot I_1(z_1, z_2) + p_i \cdot I_2(z_1, z_2) - r_i \cdot \frac{1}{2} \\ &= h_i z_i - \left[\frac{5}{12}h_i + \frac{1}{2}r_i\right], \end{aligned} \tag{2}$$

where $I_1(z_1, z_2)$ and $I_2(z_1, z_2)$ denote the average inventory quantity and the average shortage quantity, respectively. We see that the total expected cost $C_1^i(z_1, z_2)$ is an increasing linear function of z_i. Therefore he chooses $z_i^* = \frac{1}{2}$ as his optimal strategy.

(ii) Let $0 \le z_1 < \frac{1}{2}$ and let $\frac{1}{2} + q\left(\frac{1}{2} - z_1\right) \le z_2 \le 1$. Then Player 1 stocks out by time $\frac{1}{2}t$ and Player 2 satisfies the demands not only for the first visiting customers but also for the re-allocated customers by Player 1. Hence they can supply all customers with products.

For Player 1, the on-hand inventory $Q_1(T)$ is given by Eq.(1). Then the total expected cost $C_2^1(z_1, z_2)$ is given by

$$\begin{aligned} C_2^1(z_1, z_2) &= h_1 \cdot I_1(z_1, z_2) + p_1 \cdot I_2(z_1, z_2) - r_1 \cdot z_1 \\ &= -[p_1 + r_1]z_1 + \frac{h_1 + p_1}{3}z_1^2 + \frac{5}{12}p_1. \end{aligned} \tag{3}$$

The total expected cost $C_2^1(z_1, z_2)$ is obviously a convex function of z_1. This fact leads to the following strategies:

- If $0 \le r_1 < \frac{h_1 - 2p_1}{3}$, the optimal strategy for Player 1 is $z_1^* = \frac{3(r_1 + p_1)}{2(h_1 + p_1)}$.
- If $r_1 \ge \frac{h_1 - 2p_1}{3}$, the optimal strategy for Player 1 is $z_1^* = \frac{1}{2}$.

For Player 2, the on-hand inventory $Q_2(T)$ is written as

$$Q_2(T) = \begin{cases} z_2 - \frac{T}{t}, & 0 \le T < \frac{1}{2}t \\ z_2 - \frac{1}{2}, & \frac{1}{2}t \le T < (1 + z_1)t \\ z_2 - \frac{1}{2} + q\left(z_1 + 1 - \frac{T}{t}\right), & (1 + z_1)t \le T \le \frac{3}{2}t. \end{cases} \tag{4}$$

Then the total expected cost $C_2^2(z_1, z_2)$ is given by

$$\begin{aligned} C_2^2(z_1, z_2) &= h_2 \cdot I_1(z_1, z_2) + p_2 \cdot I_2(z_1, z_2) - r_2 \cdot \left(\frac{1}{2} + q\left(\frac{1}{2} - z_1\right)\right) \\ &= h_2 z_2 - h_2\left(\frac{5}{12} + \frac{q}{3}\left(z_1 - \frac{1}{2}\right)^2\right) - r_2\left(\frac{1}{2} + q\left(\frac{1}{2} - z_1\right)\right). \end{aligned} \tag{5}$$

Given a value of z_1, the total expected cost $C_2^2(z_1, z_2)$ is an increasing linear function of z_2. Therefore he chooses $z_2^* = \frac{1}{2} + q\left(\frac{1}{2} - z_1^*\right)$ as his optimal strategy if Player 1 chooses his optimal strategy z_1^*.

(iii) Let $0 \le z_1 < \frac{1}{2}$ and let $\frac{1}{2} \le z_2 < \frac{1}{2} + q\left(\frac{1}{2} - z_1\right)$. Then Player 1 stocks out by time $\frac{1}{2}t$ and Player 2 satisfies the demands for the first visiting customers and a part of the demands for customers re-allocated from Player 1. Hence they can not supply all customers with products.

Player 1 has the same strategies as those of Case (ii).

For Player 2, the inventory quantity $Q_2(T)$ is given by Eq.(4). Given a fixed z_1, let t_1 denote the time T satisfying $z_2 - \frac{1}{2} + q\left(z_1 + 1 - \frac{T}{t}\right) = 0$ and $(1+z_1)t \le T < \frac{3}{2}t$ simultaneously. The expected total cost $C_3^2(z_1, z_2)$ is given by

$$\begin{aligned} C_3^2(z_1, z_2) &= h_2 \cdot I_1(z_1, z_2) + p_2 \cdot I_2(z_1, z_2) - r_2 \cdot z_2 \\ &= -r_2 z_2 + \frac{h_2 + p_2}{3}\frac{q t_1^2}{t^2} + h_2\left(\frac{1}{12} - \frac{q}{3}(1+z_1)^2\right) + p_2 q\left(\frac{3}{4} - \frac{t_1}{t}\right). \end{aligned} \tag{6}$$

For the fixed z_1, the total expected cost $C_3^2(z_1, z_2)$ is convex in z_2. We suppose that Player 1 always chooses his optimal strategy z_1^*. Then it tells us to the following strategies:

- If $0 \le r_2 < \frac{2(h_2+p_2)}{3}z_1^* + \frac{2h_2 - p_2}{3}$, the optimal strategy for Player 2 is $z_2^* = \frac{1}{2}$.
- If $\frac{2(h_2+p_2)}{3}z_1^* + \frac{2h_2-p_2}{3} \le r_2 < h_2$, the optimal strategy for Player 2 is $z_2^* = \frac{3(r_2+p_2)}{2(h_2+p_2)}q + \frac{1}{2} - q(z_1^* + 1)$.
- If $r_2 \ge h_2$, the optimal strategy for Player 2 is $z_2^* = \frac{1}{2} + q\left(\frac{1}{2} - z_1^*\right)$.

(iv) Let $0 \le z_i < \frac{1}{2}, i = 1, 2$. Then both players stock out by time $\frac{1}{2}t$. They satisfy a part of the demands for the first visiting customers. Hence no players do not supply their demands for re-allocated customers.

For Player 1, the on-hand inventory $Q_1(T)$ is written as

$$Q_1(T) = \begin{cases} z_1 - \frac{T}{t}, & 0 \le T < \frac{1}{2}t \\ z_1 - \frac{1}{2}, & \frac{1}{2}t \le T < (1+z_2)t \\ z_1 - \frac{1}{2} + q\left(z_2 + 1 - \frac{T}{t}\right), & (1+z_2)t \le T \le \frac{3}{2}t. \end{cases} \tag{7}$$

Then the total expected cost $C_4^1(z_1, z_2)$ is given by

$$\begin{aligned} C_4^1 &= h_1 \cdot I_1(z_1, z_2) + p_1 \cdot I_2(z_1, z_2) - r_1 \cdot z_1 \\ &= -[p_1 + r_1]z_1 + \frac{h_1 + p_1}{3}z_1^2 + p_1\left(\frac{q}{3}\left(z_2 - \frac{1}{2}\right)^2 + \frac{5}{12}\right). \end{aligned} \tag{8}$$

Hence the total expected cost $C_4^1(z_1, z_2)$ is a convex function of z_1 for fixed z_2. Therefore it leads to the following strategies:

- If $0 \le r_1 < \frac{h_1 - 2p_1}{3}$, the optimal strategy for Player 1 is $z_1^* = \frac{3(r_1+p_1)}{2(h_1+p_1)}$.
- If $r_1 \ge \frac{h_1 - 2p_1}{3}$, the optimal strategy for Player 1 is $z_1^* = \frac{1}{2}$.

For Player 2 we obtain the similar results to those of Player 1.

It can be seen that the cases (v) and (vi) correspond to the situations in which are exchanged a role by Player 1 and 2 in the situations (ii) and (iii) respectively, although we omit to show it.

4 Equilibrium In the previous section, given the order quantity of a player, we can obtain the optimal order quantity z_i^* with conditions related to r_i so as to minimize the total expected cost for opposite player. Those conditions on r_i give some dominated pure strategies. We shall analyze our model in a way to give a reduced payoff bimatrix under these strategies and find an equilibrium point for both Player 1 and 2.

Definition 4.1 *Let X, Y denote sets of strategies for Player 1 and 2, respectively. A pair of strategies $x^* \in X$, $y^* \in Y$ is an equilibrium point for a non-zero-sum minimizing game if for any strategies $x \in X$, $y \in Y$ it holds the following equations:*

$$M_1(x, y^*) \geq M_1(x^*, y^*); \qquad M_2(x^*, y) \geq M_2(x^*, y^*),$$

where $M_1(\,\cdot\,,\,\cdot\,)$ denotes Player 1's loss and $M_2(\,\cdot\,,\,\cdot\,)$ denotes Player 2's loss.

We need to analyze it in the following eight parametric ranges:
(a) $0 \leq r_i < \frac{h_i - 2p_i}{3}, i = 1, 2$;
(b) $0 \leq r_1 < \frac{h_1 - 2p_1}{3}, \frac{h_2 - 2p_2}{3} \leq r_2 < (h_2 + p_2)\frac{r_1 + p_1}{h_1 + p_1} + \frac{2h_2 - p_2}{3}$;
(c) $0 \leq r_1 < \frac{h_1 - 2p_1}{3}, (h_2 + p_2)\frac{r_1 + p_1}{h_1 + p_1} + \frac{2h_2 - p_2}{3} \leq r_2 < h_2$;
(d) $0 \leq r_1 < \frac{h_1 - 2p_1}{3}, r_2 \geq h_2$;
(e) $\frac{h_1 - 2p_1}{3} \leq r_1 < (h_1 + p_1)\frac{r_2 + p_2}{h_2 + p_2} + \frac{2h_1 - p_1}{3}, 0 \leq r_2 < \frac{h_2 - 2p_2}{3}$;
(f) $(h_1 + p_1)\frac{r_2 + p_2}{h_2 + p_2} + \frac{2h_1 - p_1}{3} \leq r_1 < h_1, 0 \leq r_2 < \frac{h_2 - 2p_2}{3}$;
(g) $r_1 \geq h_1, 0 \leq r_2 < \frac{h_2 - 2p_2}{3}$;
(h) $r_i \geq \frac{h_i - 2p_i}{3}, i = 1, 2$.

Now we shall examine an equilibrium point in the case of (a). Player 1 has three pure strategies having dominated in some situations: $\mathrm{I}_1 = \frac{3(r_1 + p_1)}{2(h_1 + p_1)}$, $\mathrm{I}_2 = \frac{1}{2}$ and $\mathrm{I}_3 = \frac{1}{2} + q\left(\frac{1}{2} - \frac{3(r_2 + p_2)}{2(h_2 + p_2)}\right)$. Also Player 2 has three dominated pure strategies: $\mathrm{II}_1 = \frac{3(r_2 + p_2)}{2(h_2 + p_2)}$, $\mathrm{II}_2 = \frac{1}{2}$ and $\mathrm{II}_3 = \frac{1}{2} + q\left(\frac{1}{2} - \frac{3(r_1 + p_1)}{2(h_1 + p_1)}\right)$. It is seen that $\mathrm{I}_1 < \mathrm{I}_2 < \mathrm{I}_3$ and $\mathrm{II}_1 < \mathrm{II}_2 < \mathrm{II}_3$ from conditions on r_i. Then we make the following reduced payoff bimatrix by arranging these strategies for Player 1 and 2. Then, the values of the total expected costs are given by

$$\begin{array}{c} \\ \mathrm{I}_1 \\ \mathrm{I}_2 \\ \mathrm{I}_3 \end{array}
\begin{array}{c} \begin{array}{ccc} \qquad\mathrm{II}_1\qquad & \qquad\mathrm{II}_2\qquad & \qquad\mathrm{II}_3\qquad \end{array} \\
\left(\begin{array}{ccc}
(C_4^1(\mathrm{I}_1, \mathrm{II}_1), C_4^2(\mathrm{I}_1, \mathrm{II}_1)) & (C_3^1(\mathrm{I}_1, \mathrm{II}_2), C_3^2(\mathrm{I}_1, \mathrm{II}_2)) & (C_2^1(\mathrm{I}_1, \mathrm{II}_3), C_2^2(\mathrm{I}_1, \mathrm{II}_3)) \\
(C_6^1(\mathrm{I}_2, \mathrm{II}_1), C_6^2(\mathrm{I}_2, \mathrm{II}_1)) & (C_1^1(\mathrm{I}_2, \mathrm{II}_2), C_1^2(\mathrm{I}_2, \mathrm{II}_2)) & (C_1^1(\mathrm{I}_2, \mathrm{II}_3), C_1^2(\mathrm{I}_2, \mathrm{II}_3)) \\
(C_5^1(\mathrm{I}_3, \mathrm{II}_1), C_5^2(\mathrm{I}_3, \mathrm{II}_1)) & (C_1^1(\mathrm{I}_3, \mathrm{II}_2), C_1^2(\mathrm{I}_3, \mathrm{II}_2)) & (C_1^1(\mathrm{I}_3, \mathrm{II}_3), C_1^2(\mathrm{I}_3, \mathrm{II}_3))
\end{array}\right) \end{array}.$$

The indexes denote the situation's number. For instance, if Player 1 takes Strategy I_1 and Player 2 takes Strategy II_1, the total expected costs are calculated according to Situation 4.

The optimality of the total expected cost $C_3^2(z_1, z_2)$ and the continuity of the total expected cost for Player 2 lead to the following relationship

$$C_3^2(\mathrm{I}_1, \mathrm{II}_2) < C_3^2(\mathrm{I}_1, \mathrm{II}_3) = C_2^2(\mathrm{I}_1, \mathrm{II}_3). \tag{9}$$

From the optimality of the total expected cost $C_1^2(z_1, z_2)$, we obtain

$$C_1^2(\mathrm{I}_k, \mathrm{II}_2) < C_1^2(\mathrm{I}_k, \mathrm{II}_3), \ k = 2, 3. \tag{10}$$

Hence Strategy II_3 is dominated by Strategy II_2. On the bimatrix reduced by domination, the similar argument gives that Strategy II_2 is dominated by Strategy II_1. Furthermore the optimality of the total expected cost $C_6^1(z_1, z_2)$ and the continuity of the total expected cost for Player 1 yield the relationship

$$C_6^1(\mathrm{I}_2, \mathrm{II}_1) < C_6^1(\mathrm{I}_3, \mathrm{II}_1) = C_5^1(\mathrm{I}_3, \mathrm{II}_1). \tag{11}$$

Also from the optimality of the total expected cost $C_4^1(z_1, z_2)$ and the continuity, we obtain

$$C_4^1(\mathrm{I}_1, \mathrm{II}_1) < C_4^1(\mathrm{I}_2, \mathrm{II}_1) = C_6^1(\mathrm{I}_2, \mathrm{II}_1). \tag{12}$$

Hence Strategies I_2 and I_3 are dominated by Strategy I_1 on the reduced matrix. Therefore the equilibrium point is $\left(\frac{3(r_1+p_1)}{2(h_1+p_1)}, \frac{3(r_2+p_2)}{2(h_2+p_2)}\right)$.

By the similar argument we obtain the following theorem.

Theorem 4.2 *Let* $k_i = \frac{3(r_i+p_i)}{2(h_i+p_i)}, i = 1, 2$. *This model is interpreted as one of the unit square games with pure strategies of continuous cardinary. We then have eight results as an equilibrium point* (z_1^*, z_2^*) *on the conditions described above.*

(a) (k_1, k_2); (b) $\left(k_1, \frac{1}{2}\right)$; (c) $\left(k_1, \frac{1}{2} - q\,(k_1 + 1 - k_2)\right)$; (d) $\left(k_1, \frac{1}{2} + q\left(\frac{1}{2} - k_1\right)\right)$;
(e) $\left(\frac{1}{2}, k_2\right)$; (f) $\left(\frac{1}{2} - q\,(k_2 + 1 - k_1)\,, k_2\right)$; (g) $\left(\frac{1}{2} + q\left(\frac{1}{2} - k_2\right), k_2\right)$; (h) $\left(\frac{1}{2}, \frac{1}{2}\right)$.

In the remaining of this section, we make a comparison between results obtained in the case of taking no thought of the opposite and our results. If both players never consider influence from the opposit, they should behave the optimal policies based on the first visiting customers. If they take no thought of the opposite, we obtain the following solutions, from the similar arguments to (i) and (ii),in above parametric ranges: (k_1, k_2) for (a); $(k_1, \frac{1}{2})$ for (b),(c) and (d); $(\frac{1}{2}, k_2)$ for (e),(f) and (g); $(\frac{1}{2}, \frac{1}{2})$ for (h). Hence we have characteristic solutions on the conditions (c),(d),(f) and (g), respectively.

Consider the situation on the condition (c). To compare our results with ordering policies determined without taking thought of the opposite, we calculate difference of two total expected costs.

$$\begin{aligned} & C_3^2(k_1, \frac{1}{2}) - C_3^2(k_1, \frac{1}{2} - q(k_1 + 1 - k_2)) \\ = & \left\{ r_2 + p_2 - \frac{h_2 + p_2}{3}(1 + k_1 + k_2) \right\} q(k_2 - k_1 - 1) \\ \ge & \ 0. \end{aligned} \tag{13}$$

The last inequality is obtained from the condition $r_2 \ge \frac{2h_2 - p_2}{3} + (h_2 + p_2)\frac{r_1 + p_1}{h_1 + p_1}$. The total expected cost for the strategy $\frac{1}{2}$ is more expensive than that for the strategy in our results, $\frac{1}{2} - q(k_1 + 1 - k_2)$. Therefore Player 2 will save the costs by $\{r_2 + p_2 - \frac{h_2 + p_2}{3}(1 + k_1 + k_2)\}q(k_2 - k_1 - 1)$.

For the condition (d) we have

$$\begin{aligned} & C_3^2(k_1, \frac{1}{2}) - C_3^2(k_1, \frac{1}{2} + q(\frac{1}{2} - k_1)) \\ = & \left\{ r_2 + p_2 - \frac{h_2 + p_2}{3}(\frac{5}{2} + k_1) \right\} q \left(\frac{1}{2} - k_1 \right) \\ \ge & \ 0, \end{aligned} \tag{14}$$

where the last inequality follows because $r_1 < \frac{h_1 - 2p_1}{3}$ and $r_2 \ge h_2$. Therefore, using our results, Player 2 will save the costs by $\{r_2 + p_2 - \frac{h_2 + p_2}{3}(\frac{5}{2} + k_1)\}q(\frac{1}{2} - k_1)$. We have the similar arguments for (f) and (g). Hence we verified that our results are better solutions.

5 Concluding Remarks

We have considered the duopolistic inventory model with reallocation over the unit interval, and have concretely examined the equilibrium point by resulting in one of bimatrix games. It has been confirmed from the result that there exists a unique equilibrium point. Then we have obtained the characteristic result in the duopolistic model. Also we found that the optimal order quantity is dependent on time at which a player stocks out. This time is given by a fixed value, whether customers give up purchasing a product with a certain probability or not. Hence the order quantity is decided by the optimal stocking out time.

The optimal strategies under equilibrium are summarized as follows: If a player has some stock at time $\frac{3}{2}t$, he places an order so that his inventory level at that time is equal to 0. If a player stocks out by time $\frac{3}{2}t$, he places an order so that his inventory level at time $\frac{3(r_i+p_i)}{2(h_i+p_i)}t$ is equal to 0.

For instance, suppose that customers purchase a kind of foods. They choose whether parchasing at another store or substituting the other kind of foods, if they go to purchase foods at a store and the store stocks out. The former means that customers are re-allocated and the latter means that they give up purchasing the foods. Let $q = 0.5, h_1 = 0.7, p_1 = 0.2, h_2 = 0.6, p_2 = 0.1, r_2 = 0.1$. Then, ordering almost 0.43% of total demands is the optimal strategy for P2. On the other hand, the optimal strategy for P1 is approximately to order 0.42% if $r_1 = 0.05$; 0.5% if $r_1 = 0.28$; 0.52% if $r_1 = 0.68$; and 0.54% if $r_1 = 1.0$.When P2 is greedy for his profits, we suppose that he set $r_2 = 1.0$. Then, a pair of the optimal strategies for P1 and P2 is $(0.42, 0.54)$ if $r_1 = 0.05$; $(0.5, 0.5)$ if $r_1 = 0.28, 0.68$, or 1.0. In a word, the strategy has to change in consideration of the opposite profit.

There are several competitive inventory models, i.e. with many retailers, locations, and different ordering time and planning period for retailers. These models should be compared with our simple model, though they are still remained research problems.

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