DELTA OPERATORS ON SEQUENCE SPACES

J.Prada

Received March 18, 2001; revised June 12, 2001

Abstract. In this paper we study differentiation-invariant operators on sequence spaces showing that they can be considered as convolutions in a certain sense. When they are isomorphisms we prove that they have an exponential representation behaving as "translations".

1 Introduction It is known that polynomials of convolution type and certain kind of linear operators on polynomials are linked [1]. In fact, let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with $\deg q_n = n$, for all $n \in \mathbb{N}$; then $(q_n)_{n\in\mathbb{N}}$ is a basis for the vector space P of polynomials with coefficients in $\mathbb{R}$ or $\mathbb{C}$ (or any field K of characteristic zero). Hence there exists a unique linear operator Q on P with $Qq_n = q_{n-1}$, $n \geq 1$ and $Qq_0 = 0$, which turns out to be translation-invariant, where the translation operator E^a is defined by $(E^a p)(x) = p(x+a)$, $p \in P$. Translation-invariant operators Q on P such that $Qx =$ nonzero constant are called delta operators [1].

If D is the differentiation operator (example of a delta operator), then $q_n(x) = \frac{x^n}{n!}$ for all $n \in \mathbb{N}$ are the corresponding convolution polynomials. Algebraically, that is on P, any differentiation-invariant operator T is translation-invariant; besides T can be represented as a series, $T = \sum (Tq_n)(0)\ D^n$ [1]. Operators commuting with D (differentiation invariant) has been treated by several authors [1, 2, 5, 7, 9, 10, 11, 12] in different contexts.

Connected with the so-called "umbral calculus" Rota and his collaborators have studied certain sequences of special functions and related operators (Sheffer sequences and operators)

$$\lambda_{g,f} = \frac{1}{g(t)}\lambda_f = \lambda_f \frac{1}{g(\overline{f}(t))}$$

[11, formulas (3.4.5) and (3.5.1), p.42]. Then a Sheffer operator $\lambda_{g,f}$ is expressed as a composition of an umbral operator [11] and a multiplication operator (that is, an operator commuting with differentiation) and so in P these operators appear in the study of Sheffer operators and many of his properties are known [11, 12]. The question of extending them as continuous linear operators to Banach and Fréchet spaces (from this point of view) has been treated mainly by Grabiner [5]; using some expansion theorems she is able to extend some of the umbral methods, developed by Rota, Roman and some others to spaces of entire functions.

In section 2 we established the terminology used in this paper (following mainly [1], [5], [11]); we deal with weighted sequence spaces, $\ell^1(w_n)$ and $c_o(\frac{n!}{w_n})$ and state some basic results that will be used later on.

In section 3 we consider the problem of a differentiation-invariant operator being a translation- invariant operator and if so a convolution on a certain way.

In section 4 the spectrum of D is studied.

2000 *Mathematics Subject Classification.* 47B37, 46A45.

Key words and phrases. Deita Operators, Sequence Spaces.

Finally, section 5, treats the relationship between the continuity (and invertibility) of a differentiation-invariant operator and the spectrum of D; when $\ell^1(w_n) = \ell^1(r^n)$, these results are essentially known to Grabiner [5] without being explicitly mentioned. The question of an isomorphism being of the type $e^{f(D)}$, that is, behaving as a "translation" comes in a natural way (note that the translation operator $E^1 = e^D$). For certain Fréchet spaces (infinite power series spaces) results of this type can be found in [10]; for $\ell^1(w_n)$ being an algebra (that is, if and only if $w_{n+m} \leq C w_n w_m$, for all m, n) it turns out that all isomorphisms are of this type; in fact it is true for a much larger class of sequences (w_n) as it is proved in this paper.

2 Basic results and terminology

If (w_n) is a sequence of positive numbers, $\ell^1(w_n)$ is the Banach space of formal power series $f(t) = \sum c_n t^n$ for which the norm

$$\|f\| = \sum |c_n| \, w_n$$

is finite and $c_o(\frac{n!}{w_n})$ is the Banach space with the usual norm

$$\|f\| = \sup_n \left| \frac{c_n n!}{w_n} \right|$$

containing the formal series $f(x) = \sum c_n x^n$ such that

$$\lim \left| \frac{c_n n!}{w_n} \right| = 0$$

Note that the variables t and x are used in a formal sense; when we deal with $c_o(\frac{n!}{w_n})$ and its topological dual $\ell^1(w_n)$ (or P and P^*) the different variables are used to distinguish the elements of both spaces.

Suppose that T is a linear transformation on the space P of polynomials, and let T^* be its adjoint on P^*. The following lemma will be frequently used [5].

Lemma 2.1. *Suppose that T is a linear operator on the space P of polynomials with $T^*(t^k) = h_k(t)$ and that (w_n) and (μ_n) are sequences of positive numbers. Then the following are equivalent:*

(a) *T has a (necessarily unique) extension to a bounded operator from $c_o(\frac{n!}{w_n})$ to $c_o(\frac{n!}{\mu_n})$.*

(b) *T^* maps $\ell^1(\mu_n)$ to $\ell^1(w_n)$ (the restriction map is necessarily continuous).*

(c) *$h_k(t)$ belongs to $\ell^1(w_n)$ for all k, and $\|h_k(t)\|_{w_n} = O(\mu_k)$.*

Moreover, when the above conditions hold, the maps T and T^* have the same operator norm, which is $M = \sup_k \frac{\|h_k(t)\|}{\mu_k}$.

Considering the algebra of formal power series denoted by P^* (or $\mathbb{C}[[t]]$ for Grabiner) as the dual space of P ($\mathbb{C}[x]$ for Grabiner), that is, explicitly defining the duality by

$$\left\langle \sum \frac{a_n t^n}{n!} \;\; / \;\; \sum b_n x^n \right\rangle = \sum a_n b_n,$$

then multiplication by $f(t) = \sum c_n t^n$ on P^* is the adjoint of the operator $f(t)p(x) = \sum c_n p^{(n)}(x)$ on P [11]; this is a fundamental tool throughout our paper. With this duality $\ell^1(w_n)$ has predual $c_o(\frac{n!}{w_n})$.

In [5] multiplication for $f(t)$ means not only power-series multiplication on P^* but also the action on P of the operator mentioned above. In this paper we talk of differentiation-invariant (commuting with D) and multiplication operators, following [5].

3 Continuous operators commuting with D

Proposition 3.1. *A linear operator T commuting with D from $c_o(\frac{n!}{w_n})$ to $c_o(\frac{n!}{\mu_n})$ is continuous if and only if, for all n*

$$\sum_{k=0}^{k=\infty} \frac{|c_k|}{k!} w_{n+k} \leq C\mu_n$$

A necessary condition for T to be continuous is

$$\left\| \sum_{k=0}^{k=n} c_k \binom{n}{k} x^{n-k} \right\|_{c_o(\frac{n!}{\mu_n})} \leq C \parallel x^n \|_{c_o(\frac{n!}{w_n})}$$

or equivalently, for all n

$$\sup_{0 \leq k \leq n} \left(\frac{|c_k|}{k!} \frac{1}{\mu_{n-k}} \right) \leq \frac{C}{w_n}$$

Proof. As T commutes with D, $T = \sum_{k=0}^{k=\infty} \frac{c_k}{k!} D^k$; then $Tx^n = \sum_{k=0}^{k=n} c_k \binom{n}{k} x^{n-k}$ and the result follows from c of lemma 2.1. Writing $T = \sum c_k D^k$, the necessary and sufficient condition is $\sum_{k=0}^{k=\infty} |c_k| \, w_{n+k} \leq C\mu_n$, for all n while the necessary one is $\sup_{0 \leq k \leq n} (|c_k| \, \frac{1}{\mu_{n-k}}) \leq \frac{C}{w_n}$, for all n . ☐

Proposition 3.2. *Let $T = \sum c_k D^k$ a continuous linear operator commuting with D. If $\limsup (w_k)^{\frac{1}{k}} = a > 0$, then the function $f(z) = \sum c_k z^k$ is analytic on the open disc $\{z : |z| < a\}$.*

Proof. Continuity of T implies $|c_k| \leq C\frac{\mu_0}{w_k}$, for all k and so the result. ☐

Remark. Observe that the operator D is continuous if and only if $\sup \frac{w_k}{\mu_{k-1}} < \infty$.

On the other hand $\lim_k \sup \left\| D^k \right\|^{\frac{1}{k}} \geq \lim \ \sup (w_k)^{\frac{1}{k}}$; this relationship will be made more precise later on.

Proposition 3.3. *Assume that D is continuous from $c_o(\frac{n!}{w_n})$ to $c_o(\frac{n!}{w_n})$. Then, the continuity of D implies the continuity of the translation operator E^a, for all $a \in \mathbb{C}$.*

Proof. As $E^a(x^n) = \sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} x^k$, E^a (symbolically e^{aD}) is continuous if and only if

$$\sum_{k=0}^{k=\infty} \frac{|a|^k}{k!} w_{n+k} \leq C w_n, \; for\ all\ n$$

D continuous implies $\frac{w_k}{w_{k-1}} < A$, for all k and so $\frac{w_{n+k}}{w_n} < A^k$, for all k and n. Then

$$\sum_{k=0}^{k=\infty} \frac{|a|^k}{k!} w_{n+k} \leq e^{|a|A} w_n, \; for\ all\ n.$$

□

Remark. The continuity of D does not implies the continuity of E^a in all cases. It is enough to consider the sequences $w_n = \frac{1}{(n-1)!}$ and $\mu_n = \frac{1}{n!}$ and the previous result is false. Therefore in P differentiation-invariant operators are translation-invariant but in a more general setting the statement translation-invariant does not even "makes sense".

The results stated above are true if the weighted sequences spaces are substituted by a Banach space B with a Schauder basis but in the context of Fréchet spaces it is no longer so. On an infinite power series space, as in [10], continuity of both operators is not simultaneous. The study of conditions to ensure both things happening will be the object of a separate paper.

Theorem 3.1. *Assume that D is a continuous operator on $c_o(\frac{n!}{w_n})$ (and so E^a by the previous proposition). If T is a differentiation-invariant continuous linear operator on $c_o(\frac{n!}{w_n})$, then T can be written as a "convolution" in the following way*

$$Tx^n = (T_o * E^a)(x^n)$$

where T_o is the continuous linear functional such that

$$T_o x^n = (Tx^n)_{x=0}.$$

Proof. $(T_o * E^a)(x^n)$ means $(T(x+a)^n)_{a=0}$ considering a and x as two variables.

Therefore, as $T = \sum c_n D^n$, computing we have

$T(x+a)^n = T\left(\sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} x^k\right) = \sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} T(x^k) =$

$= \sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} \left(\sum_{p=0}^{p=k} c_p \times (k(k-1)...(k-p+1))x^{k-p}\right)$ and it is enough to take $a = 0$.

The continuity of all operators involved guaranteed the extension to all $c_o(\frac{n!}{w_n})$. □

Remark. The problem of characterizing translation-invariant operators is a classical one; for instance, Hörmander [7] studies translation-invariant operators on the spaces $L^p(\mathbb{R}^n)$ finding that they are differentiation-invariant and a convolution.

Note, on the other hand, that translations are isomorphisms and have a exponential representation; the result is true, in certain cases, for all isomorphisms as have been proved in [2, 9, 10]. Finally, let us stress the importance, in this context, of the value 0 as the previous theorem and the representation $T = \sum (Tq_n)(0)\ D^n[1]$ shows.

4 Spectrum of D Grabiner [5], Prada [10] have proved that there is a strong relationship between the spectrum of D and the isomorphisms commuting with D; this fact is, also, implicitly stated in [2, 9], where they deal with exponential functions as eigenvectors of the differential equation $Df(z) = \lambda f(z)$ on spaces of analytic functions. The formal solutions of the equation $D(\sum a_n x^n) = \lambda \sum a_n x^n$ are, in fact, the sequences $a_n = \frac{\lambda^n}{n!}$, for all $\lambda \in \mathbb{C}$ (exponential functions); then the first step to determine the spectrum of D is to consider the functions $e^{\lambda x} = \sum \frac{\lambda^n}{n!} x^n$, what we do stating the following obvious proposition.

Proposition 4.1. *Consider the Banach space* $c_o(\frac{n!}{w_n})$. *Then,*

a) $\lambda = 0$ *is always an eigenvalue.*

b) *If* $\sup(w_n)^{\frac{1}{n}} = a > 0$, *the set of eigenvalues is the open disc* $D(0, a)$.

c) *If* $\sup(w_n)^{\frac{1}{n}} = \infty$, *the set of eigenvalues can be* $\lambda = 0$ *only, an open disc or* $\mathbb{C}$.

Proof. We give some simple examples to show the possibilities in c. Take $w_n = n^2, n$ *even* and $w_n = \frac{1}{n^2}, n$ *odd*; in this case $\lambda = 0$ is the only eigenvalue. If $w_n = 1, n$ *even* and $w_n = n^2, n$ *odd*, then the set of eigenvalues is the open disc $D(0, 1)$; finally taking $w_n = n^2, n$ *even* and $w_n = n^3$, n *odd*, all complex numbers are eigenvalues. □

Remark. Notice that in case c) the operator D is not continuous on $c_o(\frac{n!}{w_n})$; then this case is not to be considered in what follows because we assume D to be continuous. Compare the above proposition with theorem 2 in [10].

Theorem 4.1. *Suppose that* D *is continuous on* $c_o(\frac{n!}{w_n})$ *(equivalent to* $\sup \frac{w_{n+1}}{w_n} < \infty$). *Then,*

a) *The set of eigenvalues of* D *is an open disc.*

b) *The spectral radius of* $D = \lim_{n\to\infty} \|D^n\|^{\frac{1}{n}} = r$, *where* $r = \lim_{n\to\infty} \left(\sup_k \frac{w_{n+k}}{w_k}\right)^{\frac{1}{n}}$.

c) *SpectrumD* $=$ *SpectrumD** $=$ *a closed disc of center* 0 *and radius equal to* r.

Proof. (a) is obvious from the previous proposition and b comes from $\|D^n\| = \sup_k \frac{\|D^n e_k\|}{\|e_k\|}$ $= \sup_k \frac{w_{n+k}}{w_k}$.

To prove (c) observe, first, that $(\lambda I - D)^* = \lambda I - D^*$ and, second, that $(\lambda I - D)$ is continuous on $c_o(\frac{n!}{w_n})$ if and only if $(\lambda I - D^*)$ is continuous on $\ell^1(w_n)$ (lemma 2.1). Therefore, let us determine those λ that make $(\lambda I - D^*)$ invertible and continuous; as $(\lambda I - D^*)^{-1} = \sum_{k=0}^{k=\infty} \frac{1}{\lambda^{n+1}} D^n$, the mapping $(\lambda I - D^*)^{-1}$ is continuous if and only if

$$\sum_{k=0}^{k=\infty} \left|\frac{1}{\lambda^{n+1}}\right| w_{n+k} \leq C w_k, \ \textit{for all } k \text{ or} \sum_{k=0}^{k=\infty} \left|\frac{1}{\lambda^{n+1}}\right| \frac{w_{n+k}}{w_k} \leq C.$$

Assume that $|\lambda| > \lim_{n\to\infty} \left(\sup k \frac{w_{n+k}}{w_k}\right)^{\frac{1}{n}}$; then, $\lim_{n\to\infty} \left(\frac{1}{|\lambda|^{n+1}} \left(\sup_k \frac{w_{n+k}}{w_k}\right)\right)^{\frac{1}{n}} < 1$ and so $\lambda \notin$ *spectrum* D^* or *spectrum* $D^* \subset \overline{D(0, r)}$.

Finally, it is enough to see that $D(0,r) \subset$ *spectrum* D^*. Take $\lambda \notin$ *spectrum* D^* from what follows that $(\lambda I - D^*)$ is continuous and therefore analytic on the disc $D(0,r)$ (notice that $\|\lambda I - D^*\| \geq \sum_{k=0}^{k=\infty} \left|\frac{1}{\lambda^{n+1}}\right| \frac{w_{n+k}}{w_k} \geq \frac{1}{|\lambda^{n+1}|} \frac{w_{n+k}}{w_k}$, *for all* k); consequently $\lim_{n\to\infty} \left(\frac{1}{|\lambda^{n+1}|}\right)^{\frac{1}{n}} \leq \frac{1}{r}$ and $\lambda \notin D(0,r)$. □

Remark. In the previous theorem it is assumed that r is greater than zero; when $r = 0$, *spectrum* $D = \{0\}$. Besides $\lim_{n\to\infty} \left(\sup_k \frac{w_{n+k}}{w_k}\right)^{\frac{1}{n}}$ exists because $\left(\frac{w_{n+1}}{w_n}\right)$ is bounded (D continuous). If $\ell^1(w_n)$ is an algebra, that is, if and only if $w_{n+m} \leq C w_n w_m$, for all m and n, then $\sup_k \frac{w_{n+k}}{w_k} = w_n$ and the closed disc $D(0,r)$ is, precisely, the maximal ideal space of the Banach algebra $\ell^1(w_n)$.

D^* is always one-to-one and so it is $(\lambda I - D^*)$ (D^* has not eigenvalues); then $\lambda \in$ *spectrum* D^* if and only if the space generated by $\left[(\lambda I - D^*)\,\ell^1(w_n)\right]$ is not $\ell^1(w_n)$.

If T is a continuous linear operator commuting with D (given by the function $f(t)$) and λ is an eigenvalue of D, from $T(e^{\lambda x}) = f(\lambda)e^{\lambda x}$ it follows that $f(\lambda)$ is an eigenvalue of T; in fact, it is easy to see that the set $\{f(\lambda), \lambda \in D(0,r)\} \subset$ *spectrum* T.

5 Invertible operators Linear translation-invariant operators on P are well -known; a family $(T_t)_{t>0}$ of linear translation-invariant operators is a semigroup if $T_{s+t} = T_s T_t$, *for all* s, $t > 0$. If $(T_t)_{t>0}$ is a semigroup on P , then T_t is invertible for all $t > 0$ and hence, $(T_t)_{t>0}$ can be extended to a group $(T_t)_{t\in R}$[1]. Besides as T_t can be expanded into powers of D (linear translation-invariant operators on P coincide with linear differentiation-invariant ones) the coefficients of these expansion are studied and the infinitesimal generator of the semigroup is determined. We mention *theorem* 2.5.4. of [1] for the sake of completeness:

Theorem 2.5.4. *Let* $(T_t)_{t>0}$ *be a semigroup of linear translation-invariant operators on* P *and let the functions* a_n $(n \in \mathbb{N})$ *be defined by* $T_t = \sum_{k=0}^{k=\infty} a_n(t) D^n$ *for all* $t > 0$ *and all* $n \in \mathbb{N}$. *Then:*

a) *The sequence* $(a_n)_{n\in\mathbb{N}}$ *is a sequence of functions of convolution type.*

b) *If* $(a_n)_{n\in\mathbb{N}}$ *is a sequence of measurable functions, then there exists a linear translation-invariant operator* T *on* P *such that* $T_t = e^{tT}$ *for all* $t > 0$ *and* $(T_t)_{t>0}$ *can be extended to a group* $(T_t)_{t\in R}$.

Therefore, the question of all invertible linear differentiation-invariant operators being exponential ones in a "certain sense" seems to be natural. In [10] it was proved that on $H(C)$ (space of entire functions) all differentiation-invariant isomorphism (translation-invariant too) are of the type e^{aD+b}, $a, b \in \mathbb{C}$ (in fact, this result had been found previously by Delsarte and Lions [2] and Nagnibida [9]; in [10] more general results are obtained including as a particular case the one mentioned); note that in this case $e^{aD+b} = e^b \sum_{k=0}^{k=\infty} \frac{a^n}{n!} D^n$ and the functions (a_n) are $(e^b \frac{a^n}{n!})$, for all $a, b \in \mathbb{C}$ (continuous which is not surprising considering *theorem*2.5.5. of [1]).

We prove here that the conjecture is true for a large class of weighted spaces that includes Grabiner results for $\ell^1(r^n)$; for $\ell^1(w_n)$ being an algebra they come straightforwardly from *theorem* (3.3) of [5] as the next theorem shows.

Theorem 5.1. *Let* $T = \sum_{k=0}^{k=\infty} a_n D^n$ *be a differentiation-invariant operator on* $c_o(\frac{n!}{w_n})$. *Then* T *is an isomorphism if and only if the following equivalent conditions are true:*

a) $f(t) = \sum_{k=0}^{k=\infty} a_n t^n$ *belongs to inv* $\ell^1(w_n) = \exp \ell^1(w_n)$, *that is,* $f(t) = e^{g(t)}$, $g(t) \in \ell^1(w_n)$.

b) $f(t) = \sum_{k=0}^{k=\infty} a_n t^n$ *belongs to* $\ell^1(w_n)$ *and* $\sum_{k=0}^{k=\infty} a_n z^n \neq 0$, *for all* $|z| \leq \rho$, $\rho = \lim(w_n)^{\frac{1}{n}}$.

c) $f(t) = \sum_{k=0}^{k=\infty} a_n t^n$ *belongs to* $\ell^1(w_n)$ *and does not vanish on the spectrum of* D.

Proof. The maximal ideal space of the Banach algebra $\ell^1(w_n)$ is the closed disc $\overline{D(0,\rho)}$ ([3], Section 19, pp. 116-120). As T is a multiplication operator on $\ell^1(w_n)$, then $f(t)$ and $\frac{1}{f(t)}$ are elements of $l^1(w_n)$ and the result follows ([1], Chapter 4, Section 4.2, p.88). □

When $\ell^1(w_n)$ is not an algebra the result is still true, at least, for a large class of weighted spaces; it is our conjecture that the result is valid for a larger class still. We assume (w_n) to satisfy a condition that includes all sequences (w_n) such that $\frac{w_{n+1}}{w_n}$ is increasing (as $\frac{w_{n+1}}{w_n}$ is bounded, then $\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = A$). Explicitly

Theorem 5.2. *Suppose that the sequence* (w_n) *satisfies the condition:*

$$\forall n, \forall \varepsilon, \exists k_n(\varepsilon) \text{ such that } \frac{w_{1+k_n}}{w_{k_n}}, \frac{w_{2+k_n}}{w_{1+k_n}}, \ldots. \frac{w_{n+k_n}}{w_{n-1+k_n}} \geq A - \varepsilon$$

where $A = \sup_n \frac{w_{n+1}}{w_n}$. *Then* $T = \sum_{k=0}^{k=\infty} a_n D^n$ *is a differentiation-invariant isomorphism on* $c_o(\frac{n!}{w_n})$ *if and only if the following equivalent conditions are true:*

a) $f(t) = \sum_{k=0}^{k=\infty} a_n t^n$ *belongs to inv* $\ell^1(A^n) = \exp \ell^1(A^n)$, *that is,* $f(t) = e^{g(t)}, g(t) \in \ell^1(A^n)$.

b) $f(t) = \sum_{k=0}^{k=\infty} a_n t^n$ *belongs to* $\ell^1(A^n)$ *and* $\sum_{k=0}^{k=\infty} a_n z^n \neq 0$ *for all* $|z| \leq A$.

c) $f(t) = \sum_{k=0}^{k=\infty} a_n t^n$ *belongs to* $\ell^1(A^n)$ *and does not vanish on the spectrum of* D.

Proof. To prove (a) we will see that T is continuous if and only if $f(t) = \sum_{k=0}^{k=\infty} a_n t^n$ belongs to $\ell^1(A^n)$. Therefore if T is an isomorphism $f(t)$ and $\frac{1}{f(t)}$ are elements of $\ell^1(A^n)$ and $a)$ follows. As $\ell^1(A^n)$ is an algebra, $a)$ and $b)$ are equivalent. Noting that $\sup_k \frac{w_{n+k}}{w_k} = A^n$ we have $c)$.

Assume that T is continuous; so

$$\sum_{k=0}^{k=\infty} |a_n| \frac{w_{n+k}}{w_k} \leq C, \; for\ all\ k$$

Taking ε and $k_1(\varepsilon)$

$$|a_1|\,(A-\varepsilon) \leq |a_1| \frac{w_{1+k_1}}{w_{k_1}} \leq C$$

For the same ε and $k_2(\varepsilon)$

$$(A-\varepsilon)\,|a_1| + (A-\varepsilon)^2\,|a_2| \leq |a_1| \frac{w_{1+k_2}}{w_{k_2}} + |a_2| \frac{w_{2+k_2}}{w_{k_2}} \leq C$$

and so for the same ε and $k_n(\varepsilon)$

$$|a_1|\,(A-\varepsilon) + |a_2|\,(A-\varepsilon)^2 + \cdots + |a_n|\,(A-\varepsilon)^n \leq \\ |a_1| \frac{w_{1+k_n}}{w_{k_n}} + |a_2| \frac{w_{2+k_n}}{w_{k_n}} + \cdots + |a_n| \frac{w_{n+k_n}}{w_{k_n}} \leq C$$

which implies

$$\sum_{k=0}^{k=n} (A-\varepsilon)^k\,|a_k| \leq C, \; for\ all\ n\ and\ all\ \varepsilon$$

and therefore

$$\sum_{k=0}^{k=\infty} A^n\,|a_n| \leq C$$

Conversely, if $(a_n) \in \ell^1(A^n)$ it is obvious that T is a continuous operator. □

Remark. Note that $\ell^1(A^n) \subset \ell^1(w_n)$; besides, calling $\alpha_n = \sup_k \frac{w_{n+k}}{w_k}$, it is clear that $\ell^1(\alpha_n) \subset \ell^1(w_n)$ and that all elements of $\ell^1(\alpha_n)$ give differentiation-invariant continuous operators. The question is to know if $\ell^1(\alpha_n)$ is, precisely, the algebra that solves the problem in all cases as it seems.

Note, too, that all elements of $\ell^1(w_n)$ (when it is an algebra) are generators of a semi-group of differentiation-invariant isomorphisms. When it is not an algebra but (w_n) satisfies the above condition the roll is assumed by all elements of $\ell^1(A^n)$.

Finally we give some examples of sequences (w_n) that satisfy the hypothesis of *theorem* 5, 2; as it was said before all (w_n) such that $(\frac{w_{n+1}}{w_n})$ is increasing are included. Take, for instance, $w_n = \frac{1}{n}$, $\frac{w_{n+1}}{w_n} = \frac{n}{n+1}$; then $\ell^1(A^n) = \ell^1$.

The following sequence (w_n) is such that $\frac{w_{n+1}}{w_n}$ is not increasing but the above condition is true;

$$\frac{w_1}{w_0} = 1, \frac{w_2}{w_1} = 2, \frac{w_3}{w_2} = 1, \frac{w_4}{w_3} = 2, \frac{w_5}{w_4} = 2, \\ \frac{w_6}{w_5} = 1, \frac{w_7}{w_6} = 2, \frac{w_8}{w_7} = 2, \frac{w_9}{w_8} = 2, \frac{w_{10}}{w_9} = 1 \ldots$$

In this case $\ell^1(A^n) = \ell^1(2^n)$.

References

[1] A. Di Bucchianico, *Probabilistic and analytical aspects of the umbral calculus*, CW/TRACT, 119, 1996.

[2] J.Delsarte et J.L.Lions, *Transmutations d'opérateurs différentiels*, Comment.Math.Helv.32 (1957), 113-128.

[3] I. Gelfand, D.Raikov, and G. Shilov, *Commutative Normed Rings*, Chelsea, New York, (1964).

[4] S.Grabiner, *Dense subspaces of entire functions*, Michigan Math.J. 33 (1986), 417-422..

[5] S. Grabiner, *Convergent expansions and bounded operators in the Umbral Calculus*, Advances in Mathematics. 72, (1988), 132-167.

[6] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Providence, RI, 1957.

[7] L.Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Mathematica. 104, (1960), 93-140.

[8] I. I. Ibragimov and N. I. Nagnibida, *The matrix method and quasi-power bases in the space of analytic functions in a disc*, Russian Math. Surveys 30:6 (1975), 107-154.

[9] N. I. Nagnibida, *Isomorphisms of analytic spaces that commute with differentiation*, Math. Sbornik. Tom 72 (114) (1967). No. 2.

[10] J. Prada, *Operators commuting with differentiation*, Math. Japonica. 38, No.3 (1993), 461-467.

[11] S. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.

[12] S.Roman and G.-R. Rota, *The Umbral Calculus*, Advances in Mathematics. 27 (1978), 95-188.

[13] A. Wilansky, *Summability through Functional Analysis*, North-Holland, Amsterdam, 1984.

Sección de Matemáticas
Universidad de Salamanca.
37008, Salamanca.
Spain
E-mail: prada@gugu.usal.es