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A STRUCTURE OF AN INCREASING SEQUENCE OF CLOSED SETS WHOSE UNION IS A MULTIDIMENSIONAL INTERVAL

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Abstract. In order to study Fubini's theorem for multidimensional non-absolute integration, we introduced a lemma in the author's paper [1]. We found some errors in the proof of the lemma, but the lemma is still true. The purpose of this paper is to give a correction to the proof. In this paper, we will show a whole proof of the lemma to be easily understood.

There are some errors in the proof of Lemma 2 in [1] ([1, p.72]). The purpose of this paper is to give a correction to the proof of the lemma and to show that the lemma is true.

We denote the h-dimensional Euclidean space by E_h. For a set A in E_h, we denote the interior of A by A°, and the closure of A by $\overline{A}$. Given a system $a_1, b_1; a_2, b_2; \dots ; a_h, b_h$ of 2h real numbers such that $a_i < b_i$ for $i = 1, 2, \dots , h$, the set $\{(x_1, x_2, \dots , x_h) : a_i \le x_i \le b_i$ for $i = 1, 2, \dots , h\}$ in E_h is called an interval in E_h. A collection of intervals $I_i (i = 1, 2, \dots , n)$ such that $(I_i)^\circ \cap (I_j)^\circ = \emptyset$ for $i \ne j$ is said to be non-overlapping.

When we consider the space E_h as the product $E_h = E_{h_1} \times E_{h_2}$ of E_{h_1} and E_{h_2} such that $h = h_1 + h_2$, we denote the projection of a set $A \subset E_h$ on E_{h_1} by $\underset{E_{h_1}}{\mathrm{proj}}_x(A)$, and the projection of $A \subset E_h$ on E_{h_2} by $\underset{E_{h_2}}{\mathrm{proj}}_y(A)$. In particular, when $h = 2$ and $h_1 = h_2 = 1$, we denote $\underset{E_{h_1}}{\mathrm{proj}}_x(A)$ by $\mathrm{proj}_x(A)$ and $\underset{E_{h_2}}{\mathrm{proj}}_y(A)$ by $\mathrm{proj}_y(A)$ in short. For a set $A \subset E_h$, we denote for a point $q \in E_{h_2}$, the set $\{(p', q) : (p', q) \in A,\ p' \in E_{h_1}\}$ by A^q, similarly for a point $p \in E_{h_1}$, the set $\{(p, q') : (p, q') \in A,\ q' \in E_{h_2}\}$ by A^p.

We denote the Lebesgue measure of a set A in E_h which is measurable in the Lebesgue sense by $\mu_h(A)$. In particular, for an interval in E_1 we denote $\mu_1(I)$ by $|I|$.

For an interval in E_h, the least upper bound of the distances between a and b with $a, b \in I$ is called the diameter of I. For an interval $I = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_h, b_h]$, the point whose i-th coordinate is $(b_i - a_i)/2$ for $i = 1, 2, \dots , h$ is called the center of I. A point whose coordinates are rational numbers is called a rational point.

Let J be an interval in the 1-dimensional Euclidean space E_1, $M_k (k = 1, 2, \dots)$ a non-decreasing sequence of closed sets such that $\cup_{k=1}^{\infty} M_k \supset J$, and $\varepsilon_k (k = 1, 2, \dots)$ a sequence of numbers with $\varepsilon_k \downarrow 0$. Let F be a non-empty closed set in E_1 and n and m two positive integers with $n < m$. Then, we say that the *closed set F has the property (B_1) for $n < m$ in J associated with the sequences $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$* if the following condition is satisfied:

(B_1):(1) $F \subset J$ and $F \subset M_m$;

(2) Let $J_j (j = 1, 2, \dots)$ be the sequence of intervals contiguous to the set consisting of the set F and the end-points of the interval J. Then, the sequence can be classified

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into $m-n+1$ parts: $J_{kj}(j=1,2,\dots)$, possibly empty or finite, where $k=n,n+1,n+2,\dots,m$, in such a way that:

For $k=n,n+1,n+2,\dots,m$, we have

(i) $\sum_{j=1}^{\infty}|J_{kj}|<\varepsilon_k$;

(ii) $(J_{kj})^{\circ}\cap M_k=\emptyset$ for $j=1,2,\dots$; and

(iii) one at least of the end-points of the interval J_{kj} belongs to M_k for $j=1,2,\dots$.

Lemma 1. *Let J_0 be an interval in the 1-dimensional Euclidean space E_1. Let $M_k(k=1,2,\dots)$ be a non-decreasing sequence of closed sets such that $\cup_{k=1}^{\infty}M_k=J_0$. Then, for an arbitrary given sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ with $\varepsilon_k\downarrow 0$, any sub-interval J of J_0 has the following property (A_1) in J associated with the sequences $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$.*

(A_1): *There exist two increasing sequences of positive integers $n(i)$ and $m(i)(i=1,2,\dots)$ such that $n(i)<m(i)<n(i+1)$ and a non-decreasing sequence of non-empty closed sets $F_{n(i)m(i)}(i=1,2,\dots)$ in such a way that:*

(1) *each $F_{n(i)m(i)}$ has the property (B_1) for $n(i)<m(i)$ in J associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$; and*

(2) *$\cup_{i=1}^{\infty}F_{n(i)m(i)}=J$ holds.*

Lemma 2. *Let R_0 be an interval in the h-dimensional Euclidean space $E_h(h>1)$. Let $M_k(k=1,2,\dots)$ be a non-decreasing sequence of closed sets such that $\cup_{k=1}^{\infty}M_k=R_0$. Then, for an arbitrary given sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ with $\varepsilon_k\downarrow 0$, any sub-interval R of R_0 has the following property (A_h) in R assciated with the sequences $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$.*

(A_h): *There exist two increasing sequences of positive integers $n(i)$ and $m(i)(i=1,2,\dots)$ such that $n(i)<m(i)<n(i+1)$ and a non-decreasing sequence of non-empty closed sets $F_{n(i)m(i)}(i=1,2,\dots)$ such that*

(1) *$F_{n(i)m(i)}\subset R$ and $F_{n(i)m(i)}\subset M_{m(i)}$,*

in such a way that:

(2) *Put $Y=\cup_{i=1}^{\infty}\operatorname{proj}_{E_{h-1}}{}_y(F_{n(i)m(i)})$ and $Z=\operatorname{proj}_{E_{h-1}}{}_y(R)-Y$. Then, we have*

(i) *$\mu_{h-1}(Z)=0$;*

(ii) *for each $q\in Y$ and each $i=1,2,\dots$, if $(F_{n(i)m(i)})^q\neq\emptyset$, then the closed set $(F_{n(i)m(i)})^q$ has the property (B_1) for $n(i)<m(i)$ in R^q associated with $\{(M_k)^q\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$; and*

(iii) *$\cup_{i=1}^{\infty}(F_{n(i)m(i)})^q=R^q$ for each $q\in Y$.*

In the property (A_h), $F_{n(i)m(i)}$ is sometimes denoted by $F_{n(i)m(i)}(R)$, standing for that (A_h) is considered in R. Since $F_{n(i)m(i)}\uparrow$, (i) and (iii) imply that

(iv) $\mu_h(F_{n(i)m(i)})\uparrow\mu_h(R)$ holds as $i\to\infty$.

Remark. In (A_h) of Lemma 2, we can replace the conditions (ii) and (iii) by the following conditions (ii′) and (iii′):

(ii$'$) For almost all $q \in Y$ and all $i = 1, 2, \dots$, if $(F_{n(i)m(i)})^q \neq \emptyset$, then the closed set $(F_{n(i)m(i)})^q$ has the property (B_1) for $n(i) < m(i)$ in R^q associated with $\{(M_k)^q\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$;

(iii$'$) $\cup_{i=1}^\infty (F_{n(i)m(i)})^q = R^q$ for almost all $q \in Y$.

Proof. We prove only for the case when $h = 2$. By (ii$'$) and (iii$'$) there exists a set $K \subset Y$ with $\mu_1(K) = 0$ and such that for all $q \in Y - K$, the statements of (ii$'$) and (iii$'$) which correspond to q are true. Take a decreasing sequence of open sets $G(i)(i = 1, 2, \dots)$ such that $G(i) \supset K$ and so $\cap_{i=1}^\infty G(i) \supset K$, and $\mu_1(\cap_{i=1}^\infty G(i)) = 0$. Set $F^*_{n(i)m(i)} = F_{n(i)m(i)} \cap (\mathrm{proj}_x(R) \times (\mathrm{proj}_y(R) - G(i)))$. Then, the sequence $F^*_{n(i)m(i)}(i = 1, 2, \dots)$ satisfies the conditions (1) and (2) required in (A_2).

Next, we prove Lemma 2.

Proof of Lemma 2. The proof of the lemma requires four steps. We prove only for the case of $h = 2$.

The first step. *If an interval R has the property* (A_2) *in R associsted with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$, then any sub-interval R' of R has the property* (A_2) *in R' associated with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$.*

Proof. Let $n(i)$ and $m(i)(i = 1, 2, \dots)$ be the sequences of positive integers and $F_{n(i)m(i)}(R)$ $(i = 1, 2, \dots)$ the sequence of closed sets chosen to satisfy (A_2) in R associated with $\{M_k\}$ and $\{\varepsilon_k\}$. Let B$'$ and B$''$ be two sides of R$'$ which are parallel to the y-axis. Put $Y_k = \mathrm{proj}_y(B' \cap M_k) \cap \mathrm{proj}_y(B'' \cap M_k)$ and $Y^*_k = \mathrm{proj}_x(R') \times Y_k$. Then, $F^*_{n(i)m(i)}(R') = F_{n(i)m(i)}(R) \cap Y^*_{n(i)}(i = 1, 2, \dots)$ is a sequence of closed sets satisfying all of the conditions required in (A_2) in R' associated with $\{M_k\}$ and $\{\varepsilon_k\}$.

The second step. *Let R be an interval, F a closed set (empty or non-empty), and $Q(j)(j = 1, 2, \dots)$ a sequence of intervals. Suppose that*

(a) $R^\circ \subset (\cup_{j=1}^\infty Q(j) \cup F)$;

(b) $F \subset M_k$ *for some* k; *and*

(c) *each* $Q(j)(j = 1, 2, \dots)$ *has the property* (A_2) *in* $Q(j)$ *associated with* $\{M_k\}_{k=1}^\infty$ *and* $\{\varepsilon_k\}_{k=1}^\infty$.

Then, the interval R has the property (A_2) *in R associated with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$.*

As easily seen, by use of the first step to show that the second step is true, it is sufficient to prove the following statement.

The third step. *Let R be an interval in R_0. Suppose that there are a sequence $Q(j)(j = 1, 2, \dots)$ of non-overlapping intervals and a closed set F (empty or non-empty) such that*

(a) $F \subset R$, $F \cap Q(j) = \emptyset$ for $j = 1, 2, \dots$, and $R^\circ \subset (\cup_{j=1}^\infty Q(j) \cup F) \subset R$;

(b) $F \subset M_k$ *for some* k; *and*

(c) *each* $Q(j)(j = 1, 2, \dots)$ *has the property* (A_2) *in* $Q(j)$ *associated with* $\{M_k\}_{k=1}^\infty$ *and* $\{\varepsilon_k\}_{k=1}^\infty$.

Then, R has the property (A_2) *in R associated with* $\{M_k\}_{k=1}^{\infty}$ *and* $\{\varepsilon_k\}_{k=1}^{\infty}$.

Proof. We shall show that there exist two increasing sequences of positive integers $\{n(i)\}_{i=1}^{\infty}$ and $\{m(i)\}_{i=1}^{\infty}$, and a non-decreasing sequence of closed sets $\{F_{n(i)m(i)}\}_{i=1}^{\infty}$ which satisfy all of the conditions required in (A_2) in R associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$. By the assumption (c), for each $Q(j)$ there are two increasing sequences of positive integers $\{n(j,t)\}_{t=1}^{\infty}$ and $\{m(j,t)\}_{t=1}^{\infty}$ such that $n(j,t) < m(j,t) < n(j,t+1)$ for $t = 1,2,\dots$ and a sequence of closed sets $\{F_{n(j,t)m(j,t)}(Q(j))\}_{t=1}^{\infty}$ which satisfies the conditions (1) and (2) required in (A_2) in $Q(j)$ associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$. In particular, we have

$$\mathrm{proj}_y(F_{n(j,t)m(j,t)}(Q(j))) \subset \mathrm{proj}_y(Q(j)) \text{ for } j = 1,2,\dots \text{ and } t = 1,2,\dots. \qquad (1^\circ)$$

By (iii) of (2) we can suppose that for every $y \in \mathrm{proj}_y(F_{n(j,t)m(j,t)}(Q(j)))$ we have

$$(F_{n(j,t)m(j,t)}(Q(j)))^y \cap B'(j) \neq \emptyset \text{ and } (F_{n(j,t)m(j,t)}(Q(j)))^y \cap B''(j) \neq \emptyset$$
$$\text{for } j = 1,2,\dots \text{ and } t = 1,2,\dots, \qquad (2^\circ)$$

where $B'(j)$ and $B''(j)$ are the sides of $Q(j)$ which are parallel to the y-axis.

Now

1. Set $k(0) = 0, n(0) = 0$ and $m(0) = k'$, where k' is an integer such that $F \subset M_{k'} (k' \geq 1)$. There exsists such a k$'$ by (b).
2. Supposing $k(i-1), n(i-1)$ and $m(i-1)$ are defined for an $i \in \{1,2,\dots\}$, we shall define $k(i), n(i)$ and $m(i)$ as follows. Let us take an integer $k(i)$ so that

$$k(i-1) < k(i), \quad \text{and} \qquad (3^\circ)$$

$$\mu_2(R - (\cup_{j=1}^{k(i)} Q(j) \cup F)) < (\varepsilon_{m(i-1)+1}/2^i)^2. \qquad (4^\circ)$$

This is possible by (a). Next, choose two indices $n(j,t(j,i))$ and $m(j,t(j,i))$ for $j = 1,2,\dots,k(i)$ so that

$$m(i-1)+1 < n(1,t(1,i)) < m(1,t(1,i)) < n(2,t(2,i)) < m(2,t(2,i))$$
$$< \dots < n(k(i),t(k(i),i)) < m(k(i),t(k(i),i)); \text{ and} \qquad (5^\circ)$$

$$\mu_1(\mathrm{proj}_y(Q(j)) - \mathrm{proj}_y(F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))) < \varepsilon_{m(i-1)+1}/N(i)2^i, \qquad (6^\circ)$$

where $N(i) = \sum_{r=1}^{k(i)} r({}_{k(i)}C_r)$. This is possible by (iv). Put

$$n(i) = m(i-1)+1 \quad \text{and} \quad m(i) = m(k(i),t(k(i),i)). \qquad (7^\circ)$$

Since, for $i \geq 1$, $k(i) > k(0)$ and so $k(i) \geq 1$, we have $n(i) = m(i-1)+1 < m(1,t(1,i)) \leq m(k(i),t(k(i),i)) = m(i)$. Therefore

$$i < n(i) < m(i) < n(i+1) \quad \text{for} \quad i = 1,2,\dots. \qquad (8^\circ)$$

Repeating this process, we obtain $k(i), n(i)$ and $m(i) (i = 1,2,\dots)$. Since $n(j,t(j,i)) \leq m(k(i),t(k(i),i)) = m(i) < m(i)+1 (= n(i+1)) < n(j,t(j,i+1))$ for $1 \leq j \leq k(i)$, we have

$$t(j,i) < t(j,i+1) \quad \text{for} \quad i=1,2,\dots \quad \text{and} \quad j=1,2,\dots k(i). \tag{9°}$$

Fix an $i \in \{1,2,\dots\}$. Corresponding to $y \in \text{proj}_y(R)$ we consider a system consisting of all of the intervals $Q(j)$ belonging to $\{Q(1), Q(2), \dots, Q(k(i))\}$ and such that $((Q(j))^\circ)^y \neq \emptyset$. We denote the sysytem by $P(y)$(possible empty). We denote by $\triangle(i)$ the class of non-empty systems P consisiting of intervals chosen from $\{Q(1), Q(2), \dots, Q(k(i))\}$ for which there exists a $y \in \text{proj}_y(R)$ such that $P = P(y)$. We denote P belonging to $\triangle(i)$ by $P : Q(j(1,P)), Q(j(2,P)), \dots, Q(j(h(P),P))$. For $P \in \triangle(i)$, we have

$$\{y : P(y) = P\} = \cap_{h=1}^{h(P)} \text{proj}_y(Q(j(h,P)))^\circ - \cup_{Q(j)\notin P, 1\le j\le k(i)} (\text{proj}_y(Q(j)))^\circ.$$

Put

$$J(P) = \cap_{h=1}^{h(P)} \text{proj}_y(Q(j(h,P)))^\circ - \cup_{Q(j)\notin P, 1\le j\le k(i)} \text{proj}_y(Q(j)). \tag{10°}$$

Then, $J(P)$ consists of finite non-empty open intervals on y-axis. We have

$$J(P) \cap J(P') = \emptyset \quad \text{if} \quad P \neq P' \quad \text{for} \quad P, P' \in \triangle(i). \tag{11°}$$

Since $\cup_{P\in\triangle(i)}(\cap_{h=1}^{h(P)} \text{proj}_y(Q(j(h,P)))) \supset \cup_{P\in\triangle(i)} \overline{J(P)} \supset \cup_{j=1}^{k(i)} \text{proj}_y(Q(j))$, we have

$$\cup_{P\in\triangle(i)}(\cap_{h=1}^{h(P)} \text{proj}_y(Q(j(h,P)))) = \cup_{P\in\triangle(i)} \overline{J(P)} = \cup_{j=1}^{k(i)} \text{proj}_y(Q(j)). \tag{12°}$$

The number of systems belonging to $\triangle(i)$ is $\sum_{r=1}^{k(i)} {}_{k(i)}C_r$ at most.

We set

$$E(i) = \{y : y \in \text{proj}_y(R^\circ), \quad \mu_1((R)^y - (\cup_{j=1}^{k(i)} Q(j) \cup F)^y) < \varepsilon_{n(i)}/2^i\}. \tag{13°}$$

Take a closed set $H(i)$ so that

$$H(i) \subset E(i), \quad \mu_1(E(i) - H(i)) < \varepsilon_{n(i)}/2^{i+1}.$$

Take a closed set $S^*(i)$ so that

$$S^*(i) \subset H(i) - \cup_{j=1}^{k(i)} \text{proj}_y(Q(j)), \quad \text{and}$$

$$\mu_1((H(i) - \cup_{j=1}^{k(i)} \text{proj}_y(Q(j))) - S^*(i)) < \varepsilon_{n(i)}/2^{i+1}. \tag{14°}$$

Put

$$S(i) = (H(i) \cap \cup_{j=1}^{k(i)} \text{proj}_y(Q(j))) \cup S^*(i).$$

Since $H(i) = (H(i) \cap \cup_{j=1}^{k(i)} \text{proj}_y(Q(j))) \cup (H(i) - \cup_{j=1}^{k(i)} \text{proj}_y(Q(j)))$, $S(i)$ is a closed set contained in $E(i)$ such that

$$\mu_1(E(i)-S(i)) = \mu_1(E(i)-H(i)) + \mu_1((H(i) - \cup_{j=1}^{k(i)} \text{proj}_y(Q(j))) - S^*(i)) < \varepsilon_{n(i)}/2^i. \tag{15°}$$

We have

$$S(i) - S^*(i) = H(i) \cap \cup_{j=1}^{k(i)} \mathrm{proj}_y(Q(j)) \subset \cup_{j=1}^{k(i)} \mathrm{proj}_y(Q(j)). \tag{16°}$$

Put

$$T(i) = \cup_{P \in \triangle(i)} (\cap_{h=1}^{h(P)} \mathrm{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap \overline{J(P)}), \tag{17°}$$

$$U(i) = S(i) \cap (S^*(i) \cup T(i)) = S^*(i) \cup (S(i) \cap T(i)), \tag{18°}$$

$$V(i) = Y(i) \cap (\cap_{i'=i}^{\infty} U(i')),$$

where $Y(i) = \mathrm{proj}_y(B' \cap M_1) \cap \mathrm{proj}_y(B'' \cap M_1)$ and B' and B'' are the two sides of R which are parallel to y-axis.
We have

$$Y(i) \subset \mathrm{proj}_y(R),\ Y(i) \uparrow \quad \text{as} \quad i \to \infty, \quad \lim_{i \to \infty} \mu_1(Y(i)) = \mu_1(\mathrm{proj}_y(R)). \tag{19°}$$

Hence, $V(i) \uparrow$ as $i \to \infty$, and since $V(i) \subset E(i),\ V(i) \subset \mathrm{proj}_y(R^\circ)$ holds.

Put $V^*(i) = \mathrm{proj}_x(R) \times V(i)$, then $V^*(i)$ is a closed set such that

$$V^*(i) \uparrow \quad \text{as} \quad i \to \infty \quad \text{and} \quad V^*(i) \subset \mathrm{proj}_x(R) \times \mathrm{proj}_y(R^\circ) \subset R. \tag{20°}$$

We now define $F_{n(i)m(i)}(R)$ for $i = 1, 2, \ldots$ as follows:

$$F_{n(i)m(i)}(R) = V^*(i) \cap (\cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)) \cup M_{n(i)}). \tag{21°}$$

We next prove that this is a sequence of non-empty closed sets satisfying the conditions (1) and (2) required in (A$_2$) in R associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$. It is clear taht $F_{n(i)m(i)}(R)$ is a non-empty closed set. Now, put

$$M^*(i) = \cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)).$$

Since by (9°) $F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)) \subset F_{n(j,t(j,i+1))m(j,t(j,i+1))}(Q(j))$ for $j = 1, 2, \ldots, k(i)$ and $k(i) < k(i+1)$, we have $M^*(i) \uparrow$. Further, $V^*(i) \uparrow$ and $M_{n(i)} \uparrow$ by $n(i) < n(i+1)$. Hence

$$F_{n(i)m(i)}(R) \uparrow \quad \text{as} \quad i \to \infty.$$

Since $V^*(i) \subset R$ by (20°), we have

$$F_{n(i)m(i)}(R) \subset R. \tag{22°}$$

By (1) of the property (A$_2$) in $Q(j)$ associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$, (5°) and (7°), we have $F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)) \subset M_{m(j,t(j,i))} \subset M_{m(k(i),t(k(i),i))} = M_{m(i)}$ for $j = 1, 2, \ldots, k(i)$. Further $m(i) > n(i)$. Hence

$$F_{n(i)m(i)}(R) \subset M_{m(i)}.$$

Next, we prove that $\lim_{i\to\infty}\mu_2(F_{n(i)m(i)}(R)) = \mu_2(R)$. We first have $M_{n(i)} - M^*(i) \supset F$. Because, we have $M_{n(i)} \supset M_{m(0)} \supset F$ for $i \geq 1$, and since $Q(j) \cap F = \emptyset$ for $j = 1, 2, \ldots, k(i)$, we have $M^*(i) \cap F = \emptyset$. Therefore

$$\lim_{i\to\infty}\mu_2(F_{n(i)m(i)}(R)) = \lim_{i\to\infty}\mu_2(V^*(i) \cap (M_{n(i)} \cup M^*(i)))$$

$$= \lim_{i\to\infty}\mu_2(V^*(i) \cap (M_{n(i)} - M^*(i))) + \lim_{i\to\infty}\mu_2(V^*(i) \cap M^*(i))$$

$$\geq \lim_{i\to\infty}\mu_2(V^*(i) \cap F) + \lim_{i\to\infty}\mu_2(V^*(i) \cap M^*(i)). \tag{23°}$$

By (13°), if $y \in \mathrm{proj}_y(R^\circ) - E(i)$, then $\mu_1((R)^y - (\cup_{j=1}^{k(i)}Q(j) \cup F)^y) \geq \varepsilon_{n(i)}/2^i$. Hence

$$\begin{aligned}\mu_1(\mathrm{proj}_y(R) - E(i)) \cdot (\varepsilon_{n(i)}/2^i) &\leq \int_{\mathrm{proj}_y(R)-E(i)} \mu_1((R)^y - (\cup_{j=1}^{k(i)}Q(j) \cup F)^y)dy \\ &\leq \int_{\mathrm{proj}_y(R)} \mu_1((R)^y - (\cup_{j=1}^{k(i)}Q(j) \cup F)^y)dy \\ &\leq \mu_2(R - (\cup_{j=1}^{k(i)}Q(j) \cup F)) < (\varepsilon_{n(i)}/2^i)^2 \quad (\text{by}(4°) \text{ and } (7°)).\end{aligned}$$

Hence

$$\mu_1(\mathrm{proj}_y(R) - E(i)) < \varepsilon_{n(i)}/2^i. \tag{24°}$$

We have, by (10°) and (12°),

$$\cup_{j=1}^{k(i)}\mathrm{proj}_y(Q(j)) = \cup_{P\in\triangle(i)}(\cap_{h=1}^{h(P)} \mathrm{proj}_y(Q(j(h,P))) \cap \overline{J(P)}).$$

We further have

$$Q(j(h,P)) \supset (F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P))) \quad \text{for} \quad h = 1, 2, \ldots, h(P).$$

Therefore, by (18°), (16°) and (17°) we have

$$\mu_1(S(i) - U(i)) = \mu_1((S(i) - S^*(i)) - T(i)) \leq \mu_1((\cup_{j=1}^{k(i)}\mathrm{proj}_y(Q(j))) - T(i))$$

$$= \mu_1(\cup_{P\in\triangle(i)}(\cap_{h=1}^{h(P)} \mathrm{proj}_y(Q(j(h,P))) \cap \overline{J(P)})$$

$$- \cup_{P\in\triangle(i)} (\cap_{h=1}^{h(P)}\mathrm{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap \overline{J(P)}))$$

$$= \sum_{P\in\triangle(i)} \mu_1(\cap_{h=1}^{h(P)} \mathrm{proj}_y(Q(j(h,P))) \cap J(P)$$

$$- \cap_{h=1}^{h(P)} \mathrm{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap J(P))$$

$$\leq \sum_{P\in\triangle(i)} \sum_{h=1}^{h(P)} (\mu_1(\cap_{h=1}^{h(P)} \mathrm{proj}_y(Q(j(h,P))) \cap J(P)$$

$$-\mathrm{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap J(P)))$$

$$\leq \sum_{P\in\triangle(i)} \sum_{h=1}^{h(P)} (\mu_1(\mathrm{proj}_y(Q(j(h,P))) \cap J(P)$$

$$-\mathrm{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap J(P)))$$

$$= \sum_{r=1}^{k(i)} \sum_{P\in\triangle(i),h(P)=r} \sum_{h=1}^{h(P)} (\mu_1(\mathrm{proj}_y(Q(j(h,P))) \cap J(P)$$

$$-\mathrm{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap J(P)))$$

$$< \sum_{r=1}^{k(i)} \left(\sum_{P\in\triangle(i),h(P)=r} \{r(\varepsilon_{n(i)}/N(i)2^i)\} \right) \quad (\text{by } (6^\circ)\text{and}(7^\circ))$$

$$\leq \sum_{r=1}^{k(i)} \left({}_{k(i)}C_r \left\{ r(\varepsilon_{n(i)}/\left(\sum_{r=1}^{k(i)} r({}_{k(i)}C_r)2^i \right) \right\} \right) = \varepsilon_{n(i)}/2^i. \tag{25°}$$

Hence, by (24°), (15°) and (25°),

$$\mu_1(\mathrm{proj}_y(R) - U(i))$$
$$\leq \mu_1(\mathrm{proj}_y(R) - E(i)) + \mu_1(E(i) - S(i)) + \mu_1(S(i) - U(i)) < 3(\varepsilon_{n(i)}/2^i).$$

Therefore, for $i = 1, 2, \ldots$ we have

$$\begin{aligned}
\mu_1(\mathrm{proj}_y(R) - V(i)) &= \mu_1(\mathrm{proj}_y(R) - (Y(i) \cap (\cap_{i'=i}^{\infty} U(i'))) \\
&\leq \mu_1(\mathrm{proj}_y(R) - Y(i)) + \sum_{i'=i}^{\infty} \mu_1(\mathrm{proj}_y(R) - U(i')) \\
&< \mu_1(\mathrm{proj}_y(R) - Y(i)) + \sum_{i'=i}^{\infty} 3(\varepsilon_{n(i')}/2^{i'}) \\
&< \mu_1(\mathrm{proj}_y(R) - Y(i)) + 3\varepsilon_{n(i)}.
\end{aligned}$$

Thus, by (19°) $\lim_{i\to\infty} \mu_1(V(i)) = \mu_1(\mathrm{proj}_y(R))$, and so

$$\lim_{i\to\infty} \mu_2(V^*(i)) = \mu_2(R). \tag{26°}$$

By (20°)

$$\lim_{i\to\infty} \mu_2(V^*(i)\cap F) = \mu_2(F). \tag{27°}$$

Next, for any positive integer j^*, take an i^* with $k(i^*) > j^*$. This is possible by (3°). Then

$$\begin{aligned}
\lim_{i\to\infty,\ i\geq i^*} \mu_2(V^*(i)\cap M^*(i)) &= \lim_{i\to\infty,\ i\geq i^*} \mu_2(V^*(i)\cap(\cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j))))\\
&\geq \lim_{i\to\infty,\ i\geq i^*} \sum_{j=1}^{j^*} \mu_2(V^*(i)\cap F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))\\
&= \sum_{j=1}^{j^*} \lim_{i\to\infty,\ i\geq i^*} \mu_2(V^*(i)\cap F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))\\
&= \sum_{j=1}^{j^*} \mu_2(Q(j))
\end{aligned}$$

by (20°), (26°), the fact that $t(j,i)\uparrow\infty$ as $i\to\infty$ for $j=1,2,\dots j^*$, and (iv) in $Q(j)$. Therefore, $\lim_{i\to\infty} \mu_2(V^*(i)\cap M^*(i)) = \lim_{i\to\infty,\ j\geq i^*} \mu_2(V^*(i)\cap M^*(i)) \geq \sum_{j=1}^{j^*} \mu_2(Q(j))$ and so

$$\lim_{i\to\infty} \mu_2(V^*(i)\cap M^*(i)) \geq \sum_{j=1}^{\infty} \mu_2(Q(j)). \tag{28°}$$

By (23°), (27°), (28°), (22°) and (a), we obtain $\lim_{i\to\infty} \mu_2(F_{n(i)m(i)}(R)) = \mu_2(R)$.

Consequently, when we put

$$Y = \cup_{i=1}^{\infty} \mathrm{proj}_y(F_{n(i)m(i)}(R)) \quad \text{and} \quad Z = \mathrm{proj}_y(R) - Y,$$

we have $\mu_1(Z) = 0$. We set

$$Y^* = Y - \cup_{i=1}^{\infty} \cup_{P\in\triangle(i)} (\overline{J(P)} - J(P)). \tag{29°}$$

Then. $Y^*\subset Y$ and $\mu_1(Y-Y^*) = 0$.

Next, we prove (ii′) and (iii′) of Remark. To prove them, we prepare the following statement(∗).

(∗): Let $y\in Y^*$, $y\in \mathrm{proj}_y(V^*(i))$ for some i, and $(Q(j_s))^y \neq \emptyset$ for $s=1,2,\dots,r$. Let

$$i^* = \min\{i : y\in \mathrm{proj}_y(V^*(i)) \ \text{ and } \ k(i)\geq j_s \ \ (s=1,2,\dots,r)\}.$$

Then, for each $i\geq i^*$ we have $y\in \mathrm{proj}_y(V^*(i))$ and there exists a $P\in\triangle(i)$ such that

(d) $Q(j_s)$ is $Q(j(h_s,P))$ for some h_s with $1\leq h_s\leq h(P)(s=1,2,\dots,r)$, and

(e) $y\in \cap_{h=1}^{h(P)} \mathrm{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P))))\cap J(P)$.

Indeed, let $i\geq i^*$. Then, we have $y\in \mathrm{proj}_y(V^*(i))$ by (20°). Since then $y\in V(i)$, $y\in U(i)$. Hence, by (18°)

$$y \in S^*(i) \quad \text{or} \quad y \in T(i). \tag{30°}$$

By the first part of (14°),

$$\text{if} \quad y \in S^*(i), \quad \text{then} \quad y \notin \cup_{j=1}^{k(i)} \text{proj}_y(Q(j)). \tag{31°}$$

Since $k(i) \geq j_s$ and $(Q(j_s))^y \neq \emptyset$ for $s = 1, 2, \ldots, r$, we have $y \in \cup_{j=1}^{k(i)} \text{proj}_y(Q(j))$. Hence, $y \in T(i)$ by (30°) and (31°). Since moreover $y \in Y^*$, by (17°) and (29°)

$$y \in \cup_{P \in \triangle(i)} (\cap_{h=1}^{h(P)} \text{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap J(P)).$$

Hence, there exists a $P \in \triangle(i)$ such that

$$y \in \cap_{h=1}^{h(P)} \text{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap J(P).$$

Further, this fact implies $y \in J(P)$. Hence, by (10°) $y \notin \cup \text{proj}_y(Q(j))$, where the union is over all j such that $Q(j) \notin P$ with $1 \leq j \leq k(i)$. Hence, $Q(j_s)$ must be $Q(j(h_s, P))$ for some h_s with $1 \leq h_s \leq h(P)$ for $s = 1, 2, \ldots, r$, The proof of $(*)$ is complete.

As a corollary of $(*)$, the following statement $(**)$ holds.

$(**)$: Let $y \in Y^*$ and $y \in \text{proj}_y(V^*(i)) \cap \text{proj}_y(\cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))$ for an i. Then, there exists a $P \in \triangle(i)$ such that

$$y \in \cap_{h=1}^{h(P)} \text{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap J(P).$$

Because, by the assumption of $(**)$ and (1°), $P \in \triangle(i)$ chosen in $(*)$ is the desired one.

Now, we prove (ii′) of Remark. Let $y \in \text{proj}_y(F_{n(i)m(i)}(R)) \cap Y^*$. Then, by (21°)

(i) $y \in \text{proj}_y(V^*(i) \cap \cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))$, or

(ii) $y \in \text{proj}_y(V^*(i) \cap M_{n(i)}) - \text{proj}_y(V^*(i) \cap \cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))$.

First of all, we remark that, since $y \in \text{proj}_y(V^*(i)) = V(i)$, we have $y \in E(i)$, and so by (13°)

$$\mu_1((R)^y - (\cup_{j=1}^{k(i)} Q(j) \cup F)^y) < \varepsilon_{n(i)}/2^i. \tag{32°}$$

For the case of (i): By $(**)$ there exists a $P \in \triangle(i)$ such that

$$y \in \cap_{h=1}^{h(P)} \text{proj}_y(F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \cap J(P).$$

In this case, we can suppose that $j(1,P) < j(2,P) < \ldots < j(h(P),P)$. Hence, by (5°) we have

$$\begin{aligned} n(i) &< n(j(1,P),t(j(1,P),i)) < m(j(1,P),t(j(1,P),i)) \\ &< n(j(2,P),t(j(2,P),i)) < m(j(2,P),t(j(2,P),i)) \\ &< \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &< n(j(h(P),P),t(j(h(P),P),i)) < m(j(h(P),P),t(j(h(P),P),i)) \\ &\leq m(k(i),t(k(i),i)) = m(i). \end{aligned}$$

And by the assumption (c) for $Q(j), (F_{n(j(h,P),t(j(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P))))^y$ has the property (B_1) in $(Q(j(h,P)))^y$ associated with $\{(M_k)^y\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$ for $h = 1, 2, \dots, h(P)$.

Further, since

$$\begin{gathered}(F_{n(j(h,P),t)m(j(h,P),t)}(Q(j(h,P))))^y \cap B'(j(h,P)) \neq \emptyset \quad \text{and} \\ (F_{n(j(h,P),t)m(j(h,P),t)}(Q(j(h,P))))^y \cap B''(j(h,P)) \neq \emptyset \\ \text{for} \quad h = 1, 2, \dots, h(P) \quad \text{and} \quad t = 1, 2, \dots \quad \text{by } (2^\circ);\end{gathered}$$

$$(M_i)^y \cap B' \neq \emptyset \quad \text{and} \quad (M_i)^y \cap B'' \neq \emptyset, \quad \text{because} \quad y \in V(i) \subset Y(i); \quad \text{and} \tag{33°}$$

$$M_i \subset M_{n(i)} \quad \text{by} \quad n(i) > i,$$

the set $(R)^y - (\cup_{h=1}^{h(P)} Q(j(h,P)) \cup M_{n(i)})^y$ consisits of a sequence of non-overlapping intervals such that one at least of the end-points of each interval belongs to $M_{n(i)}$. Furthermore, since $y \in J(P)$, we have, by (10°), $y \notin \cup \operatorname{proj}_y(Q(j))$, where the union is over all $Q(j) \notin P$ with $1 \leq j \leq k(i)$. Hence

$$(\cup_{j=1}^{k(i)} Q(j))^y = (\cup_{h=1}^{h(P)} Q(j(h,P)))^y.$$

Therefore, by (32°) and the fact that $F \subset M_{n(i)}$,

$$\mu_1((R)^y - (\cup_{h=1}^{h(P)} Q(j(h,P)) \cup M_{n(i)})^y) < \varepsilon_{n(i)}/2^i < \varepsilon_{n(i)}.$$

Thus, for the case (i) the property (ii′) of Remark holds.

For the case of (ii): Since $V^*(i) = \operatorname{proj}_x(R) \times V(i)$, we have

$$\begin{aligned}&\operatorname{proj}_y(V^*(i)) \cap \cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j))) \\ &\quad = \operatorname{proj}_y(V^*(i)) \cap \operatorname{proj}_y(\cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j))).\end{aligned}$$

Hence, in the case (ii) $y \in \operatorname{proj}_y(V^*(i))$, but $y \notin \operatorname{proj}_y(\cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))$. Hence, if $(Q(j))^y \neq \emptyset$ for some j with $1 \leq j \leq k(j)$, by $(*)$ we must have

$$y \in \cap_{h=1}^{h(P)} \operatorname{proj}_y(F_{n(j(h,P),t(j,(h,P),i))m(j(h,P),t(j(h,P),i))}(Q(j(h,P)))) \quad \text{for some} \quad P \in \triangle(i)$$

and so

$$y \in \operatorname{proj}_y(\cup_{j=1}^{k(i)} F_{n(j,t(j,i))m(j,t(j,i))}(Q(j))).$$

Hence, $(Q(j))^y = \emptyset$ for $j = 1, 2, \dots, k(i)$. Therefore, by (32°)

$$\mu_1((R)^y - (F)^y) = \mu_1((R)^y - (\cup_{j=1}^{k(i)} Q(j) \cup F)^y) < \varepsilon_{n(i)}/2^i.$$

Hence, by $F \subset M_{n(i)}$, we have $\mu_1((R)^y - (M_{n(i)})^y) < \varepsilon_{n(i)}$. By (33°), the set $(R)^y - (M_{n(i)})^y$ consists of a sequence of non-overlapping intervals such that one at least of the end-points of each interval belongs to $M_{n(i)}$. Hence, for the case (ii) the property (ii′) of Remark is true.

Next, we prove (iii′) of Remark in the following form:

$$R^y = \cup_{i=1}^{\infty}(F_{n(i)m(i)}(R))^y \quad \text{for every} \quad y \in Y^*.$$

Let $y \in Y^*$. By (a) we first have

$$R^y = \cup_{j=1}^{\infty}(Q(j))^y \cup F^y \cup (B')^y \cup (B'')^y.$$

Since then $y \in Y$, there exists an i' such that $y \in \text{proj}_y(F_{n(i')m(i')}(R))$. Then, $y \in \text{proj}_y(V^*(i'))$.

Now, suppose that $(Q(j))^y \neq \emptyset$ for some j. Then, by $(*)$ there exists an i^* with $i^* \geq i'$ and $k(i^*) \geq j$ such that

$$y \in \text{proj}_y(F_{n(j,t(j,i))m(j,t(j,i))}(Q(j))) \quad \text{for all} \quad i \geq i^*.$$

Hence, since $t(j,i) \uparrow \infty$ as $i \to \infty$ by (9°), we have, by (iii) of (2) in the property (A_2) for $Q(j)$,

$$(F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))^y \uparrow (Q(j))^y \quad \text{as} \quad i \to \infty,$$

and so

$$(Q(j))^y = \cup_{i=1}^{\infty}(F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))^y. \tag{34°}$$

On the other hand, by (21°) for all i

$$(F_{n(i)m(i)}(R))^y \supset (V^*(i) \cap (F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))^y.$$

Further, since $y \in \text{proj}_y(V^*(i))$ for $i \geq i^*$ and $V^*(i) = \text{proj}_x(R) \times V(i)$,

$$(F_{n(i)m(i)}(R))^y \supset (F_{n(j,t(j,i))m(j,t(j,i))}(Q(j)))^y \quad \text{for all} \quad i \geq i^*. \tag{35°}$$

Thus, by (34°) and (35°)

$$\cup_{i=1}^{\infty}(F_{n(i)m(i)}(R))^y \supset (Q(j))^y.$$

Thus

$$\cup_{i=1}^{\infty}(F_{n(i)m(i)}(R))^y \supset \cup_{j=1}^{\infty}(Q(j))^y. \tag{36°}$$

Similarly, by (21°), $(F_{n(i)m(i)}(R))^y \supset (V^*(i) \cap M_{n(i)})^y \supset F^y$ for all $i \geq i'$, and so

$$\cup_{i=1}^{\infty}(F_{n(i)m(i)}(R))^y \supset F^y. \tag{37°}$$

Further, since $y \in \text{proj}_y(V^*(i)) \subset Y(i)$ for all $i \geq i'$,

$$(B' \cap M_{n(i)})^y \neq \emptyset \quad \text{and} \quad (B'' \cap M_{n(i)})^y \neq \emptyset.$$

Therefore

$$\cup_{i=1}^{\infty}(F_{n(i)m(i)}(R))^y \supset \cup_{i=1}^{\infty}(V^*(i) \cap M_{n(i)})^y$$

$$\supset \cup_{i=1}^{\infty}(M_{n(i)})^y \supset (B')^y \cup (B'')^y. \qquad (38^\circ)$$

Consequently, by (36°), (37°) and (38°) we have $\cup_{i=1}^{\infty}(F_{n(i)m(i)}(R))^y = R^y$.

The fourth step. *Under the assumption of Lemma 2, R_0 has the property (A_2) in R_0 associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$.*

Proof. We prove the statement above by use of the method of transfinite induction. We denote the smallest ordinal number of the third class by Ω.

1. For the case of $\nu = 1$: Since $R_0 = \cup_{k=1}^{\infty} M_k$, by Baire's theorem (see [2, p.54] for example) there exists a k' and a square R_1 such that the center is a rational point and the diameter is a rational number, in such a way that

   $$R_1 \subset R_0, \quad \text{and} \quad R_1 \cap M_{k'} = R_1 \quad \text{and so} \quad M_{k'} \supset R_1.$$

   Putting, for $i = 1, 2, \dots$,

   $$n(i) = k' + 2i \quad \text{and} \quad m(i) = k' + 2i + 1; \quad \text{and} \quad F_{n(i)m(i)}(R_1) = R_1,$$

   R_1 has the property (A_2) in R_1 associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$.

2. For the case of $\nu < \Omega$: Suppose that an interval R_μ such that the center is a rational point and the diameter is a rational number is defined for every $0 < \mu < \nu$ in such a way that:

   (a) $R_\mu \subset R_0$ for every $0 < \mu < \nu$;

   (b) if $\mu \neq \mu'$, $0 < \mu < \nu$ and $0 < \mu' < \nu$, then $R_\mu \neq R_{\mu'}$; and

   (c) each $R_\mu (0 < \mu < \nu)$ has the property (A_2) in R_μ associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$.

   Now suppose that $R_0 - \cup_{0<\mu<\nu} R_\mu \neq \emptyset$. Put $A = R_0 - \cup_{0<\mu<\nu} R_\mu$. Since then A is a non-empty G_δ-set and $A \subset \cup_{k=1}^{\infty} M_k$, by Baire's theorem there exist a point $z \in A$ and a square R of center z such that for some k'' the set $A \cap M_{k''}$ is everywhere dense in $A \cap R^\circ$ and so we have

   (d) $\overline{A \cap M_{k''}} \supset A \cap R^\circ$.

   Then, taking a rational point $r \in R_0$ which is sufficiently near to the point z, we can find an interval R' so that:

   (e) $z \in R'$;

   (f) $R' \subset R^\circ$ and $R' \subset R_0$; and

   (g) the center of R' is r and the diameter of R' is a rational number.

   We remark that (e) contains that

   (h) $R' \cap A \neq \emptyset$.

   Next, we show that the interval R' has the property (A_2) in R' associated with $\{M_k\}_{k=1}^{\infty}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$. Put

$$A^* = \overline{A} \cap R'.$$

Then, A^* is a non-empty closed set by (h). By (d), we have $M_{k''} = \overline{M_{k''}} \supset \overline{A \cap M_{k''}} \supset \overline{A \cap R^\circ}$. Further, we have $\overline{A \cap R^\circ} \supset \overline{A} \cap R^\circ$. Because, let $x \in \overline{A} \cap R^\circ$, then there exists a sequence $\{x_i\}$ such that $x_i \in A$ and $\lim_{i\to\infty} x_i = x$. Since $x \in R^\circ$, there exists an i' such that $x_i \in R^\circ$ for all $i \geq i'$. Hence, $x_i \in A \cap R^\circ$ for all $i \geq i'$.Therefore, $x \in \overline{A \cap R^\circ}$. By (f) $\overline{A} \cap R^\circ \supset \overline{A} \cap R' = A^*$. Hence, we have $A^* \subset M_{k''}$.

Put $G = (R')^\circ - A^*$. Since then G is an open set, there is a sequence of non-overlapping intervals $J_i (i = 1, 2, \dots)$ whose union is G, and for $i = 1, 2, \dots$ we have

$$J_i \subset (R')^\circ - A^* \subset R' - A^* = R' - \overline{A} \subset R' - A = R' \cap (R_0 - A)$$
$$= R' \cap (\cup_{0<\mu<\nu} R_\mu) = \cup_{0<\mu<\nu}(R_\mu \cap R').$$

Hence, by (c) and the first and second steps each J_i has the property (A_2) in J_i associated with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$, and we have $(R')^\circ = A^* \cup (\cup_{i=1}^\infty J_i)$. Consequently, by the second step the interval R' has the property (A_2) in R' associated with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$.

Now put $R' = R_\nu$. Then, $\{R_\mu (0 < \mu \leq \nu)\}$ has the following three properties.

(i) $R_\mu \subset R_0$ for $0 < \mu \leq \nu$ by (a) and (f).

(j) If $\mu \neq \mu'$, $0 < \mu \leq \nu$ and $0 < \mu' \leq \nu$, then $R_\mu \neq R_{\mu'}$. This follows from (b) and the fact that we have $R_\nu \neq R_\mu$ for $0 < \mu < \nu$, because $R_\nu \cap A \neq \emptyset$ by (h).

(k) Each R_μ $(0 < \mu \leq \nu)$ has the property (A_2) in R_μ associated with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$. This follows from (c) and the result for R' mentioned in the above.

We should thus obtain a transfinite sequence of type Ω of distinct intervals such that each interval has the center that is a rational point and its diameter is a rational number. This is impossible. Consequently, there exists a $\kappa < \Omega$ such that $\cup_{0<\mu<\kappa} R_\mu = R_0$. Thus, by the second step R_0 has the property (A_2) in R_0 associated with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$.

By the first and fourth steps, the proof of Lemma 2 is complete.

Lemma 1 is obtained as a corollary of Lemma 2 as follows.

Proof of Lemma 1. Let I_0 be an interval on y-axis. Put $R_0 = J_0 \times I_0$, and $M_k^* = M_k \times I_0$ for $k = 1, 2, \dots$. Then, $\{M_k^*\}_{k=1}^\infty$ is a non-decreasing sequence of closed sets such that $\cup_{k=1}^\infty M_k^* = R_0$. Let J be any sub-interval of J_0. Put $R = J \times I_0$. By Lemma 2 there exist two increasing sequences of positive integers $n(i)$ and $m(i)$ $(i = 1, 2, \dots)$ such that $n(i) < m(i) < n(i+1)$ and a non-decreasing sequence of non-empty closed sets $F_{n(i)m(i)}$ $(i = 1, 2, \dots)$ satisfying the conditions (1) and (2) of (A_2) in R associated with $\{M_k^*\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$. Hence, there exists a $y \in I_0$ such that $(F_{n(i)m(i)})^y \subset R^y$ and $(F_{n(i)m(i)})^y \subset (M_k^*)^y$ and such that for $i = 1, 2, \dots$, the closed set $(F_{n(i)m(i)})^y$ has the property (B_1) for $n(i) < m(i)$ in R^y associated with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$, and $\cup_{i=1}^\infty (F_{n(i)m(i)})^y = R^y$.

Put $F^*_{n(i)m(i)} = \mathrm{proj}_x((F_{n(i)m(i)})^y)$. Then $F^*_{n(i)m(i)}(i = 1, 2, \dots)$ is a non-decreasing sequence of closed sets in J such that: (1) for $i = 1, 2, \dots$, the closed set $F^*_{n(i)m(i)}$ has the property (B_1) for $n(i) < m(i)$ in J associated with $\{M_k\}_{k=1}^\infty$ and $\{\varepsilon_k\}_{k=1}^\infty$; and (2) $\cup_{i=1}^\infty F^*_{n(i)m(i)} = J$ holds. Thus, the proof of Lemma 1 is complete.

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