COMPLEMENTED COPIES OF c_0 IN $L^{\infty}_{w^*}(\mu, Z)$

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Received June 1, 2001

Abstract. Let (Ω, Σ, μ) be a complete finite measure space, X a separable Banach space and Z a proper closed linear subspace of X^*. If the subspace of $L^{\infty}_{w^*}(\mu, X^*)$ (the Banach space of all [classes of] essentially bounded X^*-valued weak* measurable functions defined on Ω equipped with its usual norm) consisting of all those Z-valued functions contains a complemented copy of c_0, we show in this note that Z contains a copy of c_0.

1. Preliminaries

Throughout this paper (Ω, Σ, μ) will be a complete finite measure space and X a real or complex Banach space. Our notation is standard in this field [2]. We denote by $\mathcal{L}^{\infty}_{w^*}(\mu, X^*)$ the linear space over $\mathbb{K}$ of all weak* measurable functions $f : \Omega \to X^*$ for which there exists a scalar function $g \in \mathcal{L}_{\infty}(\mu)$ such that $\|f(\omega)\| \leq g(\omega)$ for μ-almost all $\omega \in \Omega$, whereas $L^{\infty}_{w^*}(\mu, X^*)$ stands for the quotient space of $\mathcal{L}^{\infty}_{w^*}(\mu, X^*)$ via the equivalence relation $\sim^*$ defined by $f_1 \sim^* f_2$ whenever $f_1(\ )x \sim f_2(\ )x$ for each $x \in X$ (where $\sim$ designs the usual equivalence relation in $\mathcal{L}_p(\mu)$). The space $L^{\infty}_{w^*}(\mu, X^*)$ is a Banach space when equipped with the norm $\left\|\widehat{f}\right\| = \inf \|g\|_{\mathcal{L}_{\infty}(\mu)}$, the infimum taken over all those functions $g \in \mathcal{L}_{\infty}(\mu)$ for which there is some $f \in \widehat{f}$ such that $\|f(\omega)\| \leq g(\omega)$ for μ-almost all $\omega \in \Omega$. It can be shown that there is always some $h \in \widehat{f}$ such that $\|h(\ )\| \in \mathcal{L}_{\infty}(\mu)$ and $\left\|\widehat{f}\right\| = \|\|h(\ )\|\|_{\mathcal{L}_{\infty}(\mu)}$. As it is well known, $L^{\infty}_{w^*}(\mu, X^*)$ identifies isometrically with $L_1(\mu, X)^*$ by means of the linear map $T : L^{\infty}_{w^*}(\mu, X^*) \to L_1(\mu, X)^*$ defined by $\left(T\widehat{f}\right)\widehat{g} = \int_{\Omega} \langle f(\omega), g(\omega)\rangle \, d\mu(\omega)$ for each $f \in \widehat{f}$ and each $g \in \widehat{g} \in L_1(\mu, X)$. A study of $L^{\infty}_{w^*}(\mu, X^*)$ can be found in [2, Section 1.5] and [6, Section 3]. When X is separable, $\mathcal{L}^{\infty}_{w^*}(\mu, X^*)$ coincides with the space of all weak* measurable functions $f : \Omega \to X^*$ such that $\|f(\ )\| \in \mathcal{L}_{\infty}(\mu)$. In this case $L^{\infty}_{w^*}(\mu, X^*)$ is the quotient of $\mathcal{L}^{\infty}_{w^*}(\mu, X^*)$ via the usual equivalence relation and $\left\|\widehat{f}\right\| = \|\|f(\ )\|\|_{\mathcal{L}_{\infty}(\mu)}$ for each $f \in \widehat{f}$. As usual, we represent by $L_{\infty}(\mu, X)$ the Banach space of all [classes of] essentially bounded μ-measurable functions equipped with the norm

$$\left\|\widehat{f}\right\|_{\text{ess}} = \|f\|_{\mathcal{L}_{\infty}(\mu)} = \inf\left\{\sup_{\omega \in \Omega - E} \|f(\omega)\| : E \in \Sigma, \mu(E) = 0\right\}$$

where f is any member of the class $\widehat{f}$. According to [3], if $L_{\infty}(\mu, X)$ contains a complemented copy of c_0 then X contains a copy of c_0. Consequently, if Z is a proper closed linear subspace of X^* and $L^{\infty}_{w^*}(\mu, Z)$ stands for the (closed) linear subspace of $L^{\infty}_{w^*}(\mu, X^*)$ consisting of all those Z-valued functions, it is natural to ask whether or not Z contains a copy of c_0 whenever $L^{\infty}_{w^*}(\mu, Z)$ contains a complemented copy of c_0. In this note, we adapt the technique of [5,

2000 *Mathematics Subject Classification.* 46G10, 46E40.
Key words and phrases. Weak* measurable function, copy of c_0.
Supported by DGESIC PB97-0342 and Presidencia de la Generalitat Valenciana.

Section 2] to answer in the affirmative this question whenever X is a separable Banach space.

2. The main theorem

Let Z be a proper closed linear subspace of X^* and let us denote by $\ell_{w^*}^{\infty}(\Sigma, Z)$ the linear space over $\mathbb{K}$ of all those bounded functions $f : \Omega \to Z$ such that $f(\ )x$ is a scalar Σ-measurable function for each $x \in X$, provided with the supremum norm $\|f\|_{\infty} = \sup\{\|f(\omega)\| : \omega \in \Omega\}$.

Lemma 2.1. *If $\ell_{w^*}^{\infty}(\Sigma, Z)$ contains a complemented copy of c_0, then Z contains a copy of c_0.*

Proof. Let $\{f_n\}$ denote a basic sequence in $\ell_{w^*}^{\infty}(\Sigma, Z)$ that is equivalent to the unit vector basis of c_0 and let P be a bounded linear projection operator from $\ell_{w^*}^{\infty}(\Sigma, Z)$ onto $[f_n]$. Since the series $\sum_{n=1}^{\infty} f_n$ is wuC in $\ell_{w^*}^{\infty}(\Sigma, Z)$, there exists a constant $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} \xi_i f_i \right\|_{\infty} \leq C \|\xi\|_{\infty} \tag{1}$$

for each $\xi \in \ell_{\infty}$. On the other hand, given $x^{**} \in Z^*$ and $\omega \in \Omega$, the linear functional u on $\ell_{w^*}^{\infty}(\Sigma, Z)$ defined by $u(f) = x^{**} f(\omega)$ belongs to $\ell_{w^*}^{\infty}(\Sigma, Z)^*$, since $|u(f)| = |x^{**} f(\omega)| \leq \|x^{**}\| \|f\|_{\infty}$. Hence the series $\sum_{n=1}^{\infty} f_n(\omega)$ is wuC in Z for each $\omega \in \Omega$. Assume by contradiction that Z does not contain a copy of c_0. Then, according to the classical Bessaga and Pełczyński's criterion [1], the series $\sum_{n=1}^{\infty} f_n(\omega)$ is (BM)-convergent in Z for each $\omega \in \Omega$. This allows us to define a linear operator φ from ℓ_{∞} into $\ell_{w^*}^{\infty}(\Sigma, Z)$ by $(\varphi\xi)(\omega) = \sum_{i=1}^{\infty} \xi_i f_i(\omega)$ for each $\omega \in \Omega$. By virtue of (1) we have $\|\varphi\xi\|_{\infty} \leq C \|\xi\|_{\infty}$ for each $\xi \in \ell_{\infty}$, and clearly $(\varphi\xi)(\ )x$ is Σ-measurable since $(\varphi\xi)(\omega)x = \sum_{i=1}^{\infty} \xi_i f_i(\omega)x$ for each $\omega \in \Omega$ and $x \in X$. Hence φ is a bounded linear operator from ℓ_{∞} into $\ell_{w^*}^{\infty}(\Sigma, Z)$ such that $\varphi(e_n) = f_n$ for each $n \in \mathbb{N}$.

If J is an isomorphism from $[f_n]$ onto c_0 such that $Jf_n = e_n$ for each $n \in \mathbb{N}$, the mapping $S = J \circ P \circ \varphi$ is a bounded linear operator from ℓ_{∞} onto c_0 such that $Se_n = e_n$ for each $n \in \mathbb{N}$. Thus S is a bounded projection from ℓ_{∞} onto c_0, a contradiction. □

Theorem 2.2. *Assume that X is a separable Banach space. If $L_{w^*}^{\infty}(\mu, Z)$ contains a complemented copy of c_0, then Z contains a copy of c_0.*

Proof. Since X is separable, $L_{w^*}^{\infty}(\mu, Z)$ is linearly isometric to the quotient of $\ell_{w^*}^{\infty}(\Sigma, Z)$ via the usual equivalence relation ' $\sim$ ' that identifies functions which differ in a μ-null set. In fact, given $\widehat{f} \in L_{w^*}^{\infty}(\mu, Z)$ and choosing any $g \in \widehat{f}$, there exists a μ-null set $N_g \in \Sigma$ such that

$$\sup\{\|g(\omega)\| : \omega \in \Omega - N_g\} = \left\|\widehat{f}\right\|.$$

Hence, if $f : \Omega \to Z$ verifies that $f(\omega) = g(\omega)$ for each $\omega \in \Omega - N_g$ and $f(\omega) = 0$ for each $\omega \in N_g$, then $f \in \widehat{f} \cap \ell_{w^*}^{\infty}(\Sigma, Z)$ and $\left\|\widehat{f}\right\| = \|f\|_{\infty}$. Consequently, if $\widetilde{f}$ denotes the class of all those $h \in \ell_{w^*}^{\infty}(\Sigma, Z)$ such that $h \sim f$, the linear map T from $L_{w^*}^{\infty}(\mu, Z)$ onto $\ell_{w^*}^{\infty}(\Sigma, Z)/\sim$ defined by $T\widehat{f} = \widetilde{f}$ satisfies that

$$\left\|T\widehat{f}\right\| = \left\|\widetilde{f}\right\| = \inf\{\|h\|_{\infty} : h \in \ell_{w^*}^{\infty}(\Sigma, Z), h \sim f\} \leq \|f\|_{\infty} = \left\|\widehat{f}\right\|.$$

On the other hand, if $h \in \ell_{w^*}^{\infty}(\Sigma, Z)$ is such that $h \sim f$, then $\|h\|_{\infty} \geq \|\|h(\ )\|\|_{\mathcal{L}_{\infty}(\mu)} = \left\|\widehat{f}\right\|$ and hence $\left\|\widehat{f}\right\| \leq \left\|\widetilde{f}\right\|$. Therefore $\left\|T\widehat{f}\right\| = \left\|\widehat{f}\right\|$.

Let $\left\{\widehat{h}_n\right\}$ be a normalized basic sequence in $L^\infty_{w^*}(\mu, Z)$ equivalent to the unit vector basis of c_0 such that $\left[\widehat{h}_n\right]$ is a complemented subspace of $L^\infty_{w^*}(\mu, Z)$. Since $\sum_{n=1}^\infty \widehat{h}_n$ is wuC in $L^\infty_{w^*}(\mu, Z)$, denoting by h_n a particular function in $\ell^\infty_{w^*}(\Sigma, Z)$ belonging to the class $\widehat{h}_n$, there is a $C > 0$ such that

$$\left\|\sum_{i=1}^n \varepsilon_i \widehat{h}_i\right\| = \left\|\left\|\sum_{i=1}^n \varepsilon_i h_i\,(\ )\right\|\right\|_{\mathcal{L}_\infty(\mu)} = \inf_{E\in\Sigma,\mu(E)=0}\ \sup_{\omega\in\Omega-E}\left\|\sum_{i=1}^n \varepsilon_i h_i\,(\omega)\right\| < C$$

for each $\varepsilon_i \in \{-1, 1\}$ with $1 \le i \le n$ and each $n \in \mathbb{N}$ [4, Chapter 5, Thm. 6]. For each fixed positive integer n choose $E(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma$, with $\mu(E(\varepsilon_1, \dots, \varepsilon_n)) = 0$, such that $\|\sum_{i=1}^n \varepsilon_i h_i(\omega)\| \le C$ for each $\omega \in \Omega - E(\varepsilon_1, \dots, \varepsilon_n)$, set

$$E := \bigcup_{n=1}^\infty \bigcup \left\{E(\varepsilon_1, \dots, \varepsilon_n) : \varepsilon_i \in \{-1, 1\}, 1 \le i \le n\right\}$$

and note that $\mu(E) = 0$. For each $n \in \mathbb{N}$ define $f_n \in \ell^\infty_{w^*}(\Sigma, Z)$ such that $f_n(\omega) = h_n(\omega)$ if $\omega \in \Omega - E$ and $f_n(\omega) = 0$ otherwise. Since $\|f_n(\omega)\| = \|h_n(\omega)\| \le 2C$ for each $\omega \in \Omega - E$, then f_n is bounded and $f_n \sim h_n$ for each $n \in \mathbb{N}$. On the other hand, since $f_n(\omega)x = h_n(\omega)x$ or $f_n(\omega)x = 0$ depending on $\omega \in \Omega - E$ or $\omega \in E$, respectively, then $f_n(\ )x$ is μ-measurable and, actually, $f_n(\ )x \in \mathcal{L}_\infty(\mu)$ for each $x \in X$. Besides, given that $\|\sum_{i=1}^n \varepsilon_i f_i(\omega)\| \le C$ for $\varepsilon_i \in \{-1, 1\}$, $1 \le i \le n$, $\omega \in \Omega$ and $n \in \mathbb{N}$, the series $\sum_{n=1}^\infty f_n$ is wuC in $\ell^\infty_{w^*}(\Sigma, Z)$.

Considering that $C \ge \|f_n\|_\infty \ge \left\|\widehat{h}_n\right\| = 1$ for each $n \in \mathbb{N}$, an application of the Bessaga-Pełczyński selection principle guarantees that $\{f_n\}$ contains a c_0-subsequence $\{f_{n_i}\}$ [4, Chapter 5, Corollary 7]. Since the closed linear span of any subsequence of the unit vector basis of c_0 is a copy of c_0 complemented in c_0, it follows that $\left[\widehat{h}_{n_i}\right]$ embeds complementably into $L^\infty_{w^*}(\mu, Z)$. Hence, there is a bounded linear projection operator P from $L^\infty_{w^*}(\mu, Z)$ onto $\left[\widehat{h}_{n_i}\right]$. Let Q be the quotient map from $\ell^\infty_{w^*}(\Sigma, Z)$ onto $\ell^\infty_{w^*}(\Sigma, Z)/\sim$, let J denote an isomorphism from c_0 onto $[f_{n_i}]$ such that $Je_i = f_{n_i}$ and let K be an isomorphism from $\left[\widehat{h}_{n_i}\right]$ onto c_0 with $K\widehat{h}_{n_i} = e_i$ for each $i \in \mathbb{N}$. Since $Qf_{n_i} = T\widehat{h}_{n_i}$ for each $i \in \mathbb{N}$, the linear operator $S : \ell^\infty_{w^*}(\Sigma, Z) \to [f_{n_i}]$ defined by $S = J \circ K \circ P \circ T^{-1} \circ Q$ is bounded and verifies $Sf_{n_i} = f_{n_i}$ for each $i \in \mathbb{N}$. Consequently, $[f_{n_i}]$ is a complemented copy of c_0 in $\ell^\infty_{w^*}(\Sigma, Z)$ and the previous lemma applies. □

Open problem. Assuming that X is a nonseparable Banach space and Z a proper closed linear subspace of X^*, is it true that c_0 embeds into Z whenever $L^\infty_{w^*}(\mu, Z)$ contains a complemented copy of c_0?

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