ON THE EVALUATION OF SOME BESSEL INTEGRALS

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ABSTRACT. Known integrals of Bessel functions currently avialable in handbooks, and computations of these integrals has no limit. The underlying result is an investigation of a class of some Bessel integrals where we prove that all of the Bessel integrals in this class vanish under certain conditions. The paper concludes by indicating the wide range of results that can be obtained.

1. INTRODUCTION

Integrals involving Bessel functions occur quite frequently in both mathematical and physical analysis. There exists a considerable body of information on the subject of integrals involving Bessel functions. Of special significance are those of Watson's monumental treatise of Bessel functions [3] which provide a variety of methods for evaluating such integrals and the excellent book by Luke [1] which provides a thorough summary of results known prior to 1962.

The aim of this work is to derive a number of infinite integrals involving Bessel functions which appear to be new. In fact, we investigate the class of integrals $\int_0^\infty t^{\nu+1} J_\nu(at) F(t) dt$ and $\int_0^\infty t^{\nu+1} Y_\nu(at) F(t) dt$, where $J_\nu(\cdot)$ is the Bessel function of the first kind, $Y_\nu(\cdot)$ is the Neumann's function, and F is a function that satisfies some certain conditions. We summarize our main results in two theorems and give their proofs. Particular examples of our main results are given. Throughout this article, $\not\!C$ denotes the set of all complex numbers.

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2. MAIN RESULTS

We start this section by the following definition:

Definition 1. *For $\lambda > 0$, let $\mathcal{F}_\lambda$ be the family of all functions F from $\not\!C$ into $\not\!C$ that satisfies*

(i) F has no singularties in the right upper half plane;

(ii) $F(iy) = F(-iy)$ for all $y \in \mathbf{R}$;

(iii) F has at most exponential growth in the sense that $\lim\limits_{|z| \to \infty} \dfrac{|F(z)|}{|e^{i\lambda z}|} = 0$.

It can be easily verified that the functions, for example, $J_\mu(\alpha z) J_{-\mu}(\beta z)$ for $\alpha, \beta > 0$, and $F(z) = \sum\limits_{k=1}^{\infty} \dfrac{1}{k^k \, (z^2 + 2i)^k}$ belong to the family $\mathcal{F}_\lambda$ in addition to the "$\cos z$" type elements..

Theorem 1. If $F \in \mathcal{F}_\lambda$, then *for* $a > \lambda$, the following equalities hold:

$$(2.1) \qquad \int_0^\infty t^{\nu+1} J_\nu(at) F(t) dt = 0$$

and

$$(2.2) \qquad \int_0^\infty t^{\nu+1} Y_\nu(at) F(t) dt = 0.$$

Proof. First we investigate the integral (2.1). If we write $J_\nu(at)$ in terms of the Hankel functions, we have

$$(2.3) \qquad J_\nu(at) = \frac{1}{2} \left[H_\nu^{(1)}(at) + H_\nu^{(2)}(at) \right].$$

Therefore,

$$I = \int_0^\infty t^{\nu+1} J_\nu(at) F(t) dt = \frac{1}{2} \left[\int_0^\infty t^{\nu+1} H_\nu^{(1)}(at) F(t) dt + \int_0^\infty t^{\nu+1} H_\nu^{(2)}(at) F(t) dt \right]$$

$$(2.4) \qquad = \frac{1}{2} \left[I_1 + I_2 \right].$$

Now if we use the contour $C = \gamma_\epsilon + C_1 + C_2 + C_3$ in the right half plane, we have

$$\int_{\gamma_\epsilon} z^{\nu+1} H_\nu^{(1)}(az) F(z) dz + \int_{C_1} t^{\nu+1} H_\nu^{(1)}(at) F(t) dt + \int_{C_2} z^{\nu+1} H_\nu^{(1)}(az) F(z) dz$$

$$(2.5) \qquad = - \int_{C_3} z^{\nu+1} H_\nu^{(1)}(az) F(z) dz,$$

where $\gamma_\epsilon = \left\{ z \in \mathcal{C} : z = \epsilon e^{i\theta}, 0 \leq \theta \leq \frac{\pi}{2} \right\}, C_1 = \left\{ z \in \mathcal{C} : z = t, \epsilon < t \leq R \right\}$,
$C_2 = \left\{ z \in \mathcal{C} : z = R e^{i\theta}, 0 \leq \theta \leq \frac{\pi}{2} \right\}$, and $C_3 = \left\{ z \in \mathcal{C} : z = it, \epsilon < t < R \right\}$ for some $0 < \epsilon < R < \infty$.

One can show easily that

$$(2.6) \qquad \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} z^{\nu+1} H_\nu^{(1)}(az) F(z) dz = 0.$$

Now from the asymptotic expansion of $H_\nu^{(1)}(z)$, we have

$$(2.7) \qquad \left| H_\nu^{(1)}(az) \right| \leq \sqrt{\frac{2}{\pi}} \left| \frac{e^{iaz}}{\sqrt{az}} \right|$$

which implies that

$$(2.8) \qquad \left| H_\nu^{(1)}(R e^{i\theta}) \right| \leq \sqrt{\frac{2}{\pi R}} e^{-R \sin \theta} \qquad \text{for } 0 < \theta < \frac{\pi}{2}.$$

Thus (2.7) and the inequality $\dfrac{2\theta}{\pi} < \sin\theta < \theta$ for $0 < \theta < \frac{\pi}{2}$, gives

$$\lim_{R\to\infty} \left| \int_{C_2} z^{\nu+1} H_\nu^{(1)}(az) F(z)dz \right| \leq \lim_{\epsilon\to 0}\lim_{R\to\infty} \int_\epsilon^{\frac{\pi}{2}} R^{\nu+2} e^{-aR\sin\theta} e^{-\lambda R \sin\theta} d\theta$$

$$\leq \lim_{\epsilon\to 0}\lim_{R\to\infty} \int_\epsilon^{\frac{\pi}{2}} R^{\nu+2} e^{-aR\sin\theta} e^{-2(a-\lambda)R\frac{\theta}{\pi}} d\theta$$

$$= \lim_{\epsilon\to 0}\lim_{R\to\infty} \frac{\pi R^{\nu+1}}{2(a-\lambda)} \left[e^{-2(a-\lambda)R\epsilon} - e^{-2(a-\lambda)R} \right]$$

$$(2.9) \qquad\qquad\qquad\qquad = 0, \qquad a > \lambda.$$

Therefore, if we let $R \to \infty$ and $\epsilon \to 0$, (2.5) and (2.8) imply that

$$(2.10) \qquad I_1 = \int_0^\infty t^{\nu+1} H_\nu^{(1)}(at) F(t)dt = \int_0^{i\infty} t^{\nu+1} H_\nu^{(1)}(at) F(t)dt.$$

By similar argument one can show that

$$(2.11) \qquad I_2 = \int_0^\infty t^{\nu+1} H_\nu^{(2)}(at) F(t)dt = \int_0^{-i\infty} t^{\nu+1} H_\nu^{(2)}(at) F(t)dt.$$

Now, using the change of variable $t = iy$, equation (2.9) become

$$(2.12) \qquad I_1 = \int_0^\infty t^{\nu+1} H_\nu^{(1)}(at) F(t)dt = -(i)^\nu \int_0^\infty y^{\nu+1} H_\nu^{(1)}(iay) F(iy)dy.$$

On the other hand, using the change of variable $t = -iy$, equation (2.10) becomes

$$(2.13) \qquad I_2 = \int_0^\infty t^{\nu+1} H_\nu^{(2)}(at) F(t)dt = -(-i)^\nu \int_0^\infty y^{\nu+1} H_\nu^{(2)}(-iay) F(-iy)dy.$$

By [3],

$$(-i)^\nu H_\nu^{(2)}(-iay) = -(i)^\nu H_\nu^{(1)}(iay).$$

Thus (2.4), (2.11) and (2.12) give

$$I = -\frac{(i)^\nu}{2} \int_0^\infty y^{\nu+1} H_\nu^{(1)}(iay) F(iy)dy + \frac{(i)^\nu}{2} \int_0^\infty y^{\nu+1} H_\nu^{(1)}(iay) F(-iy)dy$$

$$(2.14) \qquad = \frac{(i)^\nu}{2} \int_0^\infty y^{\nu+1} H_\nu^{(1)}(iay) \left[F(-iy) - F(iy) \right] dy.$$

Hence by combining the condition (ii) in definition 1 and (2.13) we get (2.1).

Following the same procedure, we can show that (2.2) holds.   □

A direct application of this theorem, we get the following integrals which are tabulated in [2] :

I. $\int\limits_{0}^{\infty} t^{\nu+1} J_\nu(at) \cos t \; dt = 0$, $a > 1$.

II. $\int\limits_{0}^{\infty} t^{\nu+1} J_\nu(at) J_\mu(\alpha t) J_{-\mu}(\beta t) dt = 0, \alpha, \beta > 0, \alpha + \beta < a$, and $-1 < Re(\nu) < \frac{1}{2}$.

In proving the above theorem a lot of work is saved in evaluating any integral belongs to the class of the integrals considered above.

In the above theorem, we may allow $F(z)$ to have simple poles in the right half plane, in which case there are also residue contributions. In fact, we prove the following result:

Theorem 2. *Suppose that* $F \in \mathcal{F}_\lambda$ *for some* $\lambda > 0$. *Let* $G(z) = \dfrac{F(z)}{\prod\limits_{j=1}^{m}(z^2 + b_j^2)}$, *where*

$b_1, \cdots, b_m$ *are nonnegative distinct real numbers. For* $1 \le j \le m$ *let* $p_j(t) = \prod\limits_{l \ne j}^{m}(t^2 + b_l^2)$.

Then for $a > \lambda$ *we have*

$$(2.15) \qquad \int\limits_{0}^{\infty} t^{\nu+1} J_\nu(at) G(t) dt = \sum_{j=1}^{m} \frac{(b_j)^\nu F(ib_j)}{p_j(ib_j)} K_\nu(ab_j).$$

Where we use the convention $\prod\limits_{j \in \emptyset} = 1$.

Proof. Let $R > 0$ be large so that $\max\limits_{1 \le j \le m} \{b_j\} < R$. Let

$$D_R = \left\{ z \in \mathcal{C} : |Z| < R \text{ and } 0 \le \arg z \le \frac{\pi}{2} \right\}$$

and let

$$D_\epsilon = \left\{ z \in \mathcal{C} : |Z| < \epsilon \text{ and } 0 \le \arg z \le \frac{\pi}{2} \right\}$$

where ϵ is so small such that $b_j \notin D_\epsilon$ for all $1 \le j \le m$.

For $1 \le j \le m$, let D_j be a small disk around ib_j with boundary C_j. Consider the integral $\dfrac{1}{2\pi i} \int\limits_{C} z^{\nu+1} H_\nu^{(1)}(az) \dfrac{F(z)}{\prod\limits_{j=1}^{m}(z^2 + b_j^2)} dz$ where $C =$ the boundary of $D_R - \left\{ \bigcup\limits_{j=1}^{m} D_j \cup D_\epsilon \right\}$. Then by similar argument as in the proof of Theorem 1, we have

$$\frac{1}{2\pi i} \int\limits_{0}^{\infty} t^{\nu+1} \left[H_\nu^{(1)}(at) - e^{\nu\pi i} H_\nu^{(1)}(ate^{\pi i}) \right] \frac{F(t)}{\prod\limits_{l=1}^{m}(z^2 + b_l^2)} dt$$

$$(2.16) \qquad = \frac{1}{2\pi i} \sum_{j=1}^{m} \int\limits_{C_j} z^{\nu+1} \frac{F(z)}{\prod\limits_{l=1}^{m}(z^2 + b_l^2)} H_\nu^{(1)}(az) \, dz.$$

Now by [3], we have

$$(2.17) \qquad (-i)^\nu H_\nu^{(2)}(-iat) = -(i)^\nu H_\nu^{(1)}(iat).$$

This implies that (2.15) becomes

$$\frac{1}{2\pi i}\int_0^\infty t^{\nu+1}\left[H_\nu^{(1)}(at)+H_\nu^{(2)}(at)\right]\frac{F(t)}{\prod_{l=1}^m(z^2+b_l^2)}dt$$

$$(2.18)\qquad =\frac{1}{2\pi i}\sum_{j=1}^m\int_{C_j}z^{\nu+1}\frac{F(z)}{\prod_{l=1}^m(z^2+b_l^2)}H_\nu^{(1)}(az)\,dz.$$

Using the identity $H_\nu^{(1)}(z)=\dfrac{J_{-\nu}(z)-J_\nu(z)e^{-\nu\pi i}}{i\sin\nu\pi}$ and the Bessel's representation of $J_{\pm\nu}(z)$ we have

(2.19)
$$H_\nu^{(1)}(az)=\frac{1}{i\sin\nu\pi}\left[\sum_{n=0}^\infty\frac{(-1)^n(\frac{az}{2})^{2n-\nu}}{\Gamma(n+1)\Gamma(-\nu+n+1)}-e^{-\nu\pi i}\sum_{n=0}^\infty\frac{(-1)^n(\frac{az}{2})^{2n+\nu}}{\Gamma(n+1)\Gamma(\nu+n+1)}\right].$$

Now

$$\frac{1}{\pi i}\int_0^\infty J_\nu(at)\frac{t^{\nu+1}F(t)}{\prod_{j=1}^m(t^2+b_j^2)}dt$$

$$(2.20)\qquad =\frac{-1}{2\pi\sin\nu\pi}\sum_{n=0}^\infty\frac{(-1)^n}{\Gamma(n+1)}\left[\frac{A_n}{\Gamma(-\nu+n+1)}-\frac{B_n}{\Gamma(\nu+n+1)}\right]$$

where

$$A_n=(\frac{a}{2})^{2n-\nu}\sum_{j=1}^m\int_{C_j}\frac{z^{\nu+1+2n-\nu}}{\prod_{l=1}^m(z^2+b_l^2)}F(z)dz\qquad\text{and}$$

$$(2.21)\qquad B_n=-e^{-\nu\pi i}(\frac{a}{2})^{2n+\nu}\sum_{j=1}^m\int_{C_j}\frac{z^{\nu+1+2n+\nu}}{\prod_{l=1}^m(z^2+b_l^2)}F(z)dz.$$

Therefore, one can show that

(2.22)
$$A_n=\pi i\sum_{j=1}^m\frac{(-1)^n(b_j)^\nu(\frac{ab_j}{2})^{2n-\nu}F(ib_j)}{p_j(ib_j)}\text{ and }B_n=\pi i\sum_{j=1}^m\frac{(-1)^n(b_j)^\nu(\frac{ab_j}{2})^{2n+\nu}F(ib_j)}{p_j(ib_j)}.$$

Thus

$$\frac{1}{\pi i}\int_0^\infty t^{\nu+1}J_\nu(at)\frac{F(t)}{\prod_{l=1}^m(z^2+b_l^2)}dt$$

(2.23)
$$=\frac{1}{2i\sin\nu\pi}\sum_{j=1}^m\frac{(b_j)^\nu F(ib_j)}{p_j(ib_j)}\left[\sum_{n=0}^\infty\frac{(\frac{ab_j}{2})^{2n-\nu}}{\Gamma(n+1)\Gamma(-\nu+n+1)}-\sum_{n=0}^\infty\frac{(\frac{ab_j}{2})^{2n+\nu}}{\Gamma(n+1)\Gamma(\nu+n+1)}\right].$$

Now using the series representation of the Bessel functions of the third kind $I_{\pm\nu}$ we have

$$(2.24) \quad \frac{1}{\pi i} \int_0^\infty t^{\nu+1} J_\nu(at) \frac{F(t)}{\prod\limits_{l=1}^m (z^2 + b_l^2)} dt = \frac{1}{2i \sin \nu\pi} \sum_{j=1}^m \frac{(b_j)^\nu F(ib_j)}{p_j(ib_j)} \left(I_{-\nu}(ab_j) - I_\nu(ab_j) \right).$$

The last equation when combined with the identity $K_\nu(z) = \frac{\pi}{2 \sin \nu\pi} \left(I_{-\nu}(z) - I_\nu(z) \right)$, we obtain

$$(2.25) \qquad \int_0^\infty t^{\nu+1} J_\nu(at) \frac{F(t)}{\prod\limits_{l=1}^m (z^2 + b_l^2)} dt = \sum_{j=1}^m \frac{(b_j)^\nu F(ib_j)}{p_j(ib_j)} K_\nu(ab_j)$$

The result in Theorem 2, generalized the integral $\int_0^\infty t^{\nu+1} J_\nu(at) \frac{\cos t}{t^2 + b^2} dt = b^\gamma \cosh b \, K_\nu(ab)$ which is tabulated in [2]. Finally, a direct application of Theorem 2 implies the following integral which is tabulated in [2]:

$$\int_0^\infty t^\nu J_\nu(\gamma t) \frac{\sin \alpha t}{t^2 + \beta^2} dt = \beta^{\nu-1} \sinh \alpha\beta K_\nu(\beta\gamma)$$

for $0 < \alpha \leq \gamma, Re(\beta) > 0, -1 < Re(\nu) < \frac{3}{2}$.

References

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