Scientiae Mathematicae Japonicae Online, **Vol. 5**, (2001), 341–346

HIGH ACCURATE RATIONAL CUBIC CURVE

Zulfiqar Habib and Manabu Sakai

Received June 28, 2001

Abstract. We consider a uniformly parameterized planar rational cubic interpolation. The necessary and sufficient conditions on inflection points and singularities for a convex data have been presented. This paper uses the remaining two degrees of freedom. Shape preserving parameters are automatically generated and the degree of smoothness attained is G^2. Numerical examples with their curvature plots demonstrate high accuracy of our scheme for both C-type and S-type data.

1 Introduction Data interpolation is a useful and powerful tool for curve and surface design. Smooth curve representation, to visualize the data without undesirable infelction points and singularities, is of great significance. In CAGD applications, it is often desirable to find the conditions for high degree rational cubic interpolant such that a curve may or may not have cusps and inflection points. Many efforts and proposals recently have been made for an approximation (interpolation) theory of parametric curves or planar data by geometric splines with polynomial or rational segments from the viewpoints of approximation orders and shape preserving properties; refer to [2], [3] and therein for a unified theory of the general geometric Hermite interpolation. Most authors consider G^2 (2nd order geometric continuity) two-point Hermite interpolation by a planar rational cubic curve $\boldsymbol{z}(t) = (x(t), y(t)), 0 \le t \le 1$ with four Bézier control vertices $\boldsymbol{p}_i, 0 \le i \le 3$ of the form:

$$\text{(1)} \qquad \boldsymbol{z}(t) = \sum_{i=0}^{3} \boldsymbol{p}_i w_i B_i(t) / \sum_{i=0}^{3} w_i B_i(t), \quad B_i(t) = \left(\begin{array}{c} 3 \\ i \end{array} \right) t^i (1-t)^{3-i}$$

where $B_i(t)$ is the cubic Bernstein polynomial.

The following osculatory (G^2 two point Hermite interpolation) problem is well-known: given the four control vertices and a curvature value at each end point, find weights w_1, w_2 subject to $w_0 = w_3 = 1$ or $w_1 = w_2 = 1$ such that the resulting rational cubic of the form (1) assumes the given curvatures at the end points, see [4],[5]. de Boor, et al. ([1]) considered another osculatory cubic polynomial interpolation (i.e., $w_i = 1,\ 0 \le i \le 3$) passing through given two points and satisfying the corresponding tangent directions and consistent curvatures at the end points for a C-shaped control polygon. Then the curvatures $\kappa_i, i = 0, 1$ can be viewed as shape parameters while our results ([5]) enable us to view the weights $w_i, i = 1, 2$ as the shape ones for the given (fixed) curvatures, though the weights can also be used to achieve a high order approximation or a high order of contact at the end points, i.e., use of rational cubic approximations instead of polynomial yields considerable improvement in accuracy.

The main object of our paper is to consider the same G^2 two-point Hermite interpolation problem by use of a more general rational cubic of the form (1) with weights $(w_0, w_1, w_2, w_3) = (v, v+2, w+2, w)$ where $v, w > 0$. Inflection point and singularity tests for both C-type and S-type data are given in section 2. Section 3 gives numerical examples to show the accuracy of our scheme with cutvature plots.

Key words and phrases. Data visualization, Rational cubic curve, Interpolation, Shape preserving, Convex, Inflection points, Singularities.

2 Rational Cubic Interpolant We consider a rational cubic interpolant $\boldsymbol{z}$ through $\boldsymbol{a}$, $\boldsymbol{b}$ with the tangent vectors $\boldsymbol{T}_0, \boldsymbol{T}_1$ and curvatures κ_0, κ_1 at the two points. The angle θ is measured from $\boldsymbol{T}_0$ to $\boldsymbol{ab}$, ψ is measured from $\boldsymbol{ab}$ to $\boldsymbol{T}_1$ and γ is measured from $\boldsymbol{ab}$ to X-axis where assume that $0 < \theta, |\psi| < \pi$ and that $\theta + \psi > 0$. Then, note that with $r_0, r_1 > 0$

(2) $$\boldsymbol{T}_0 = r_0(-\cos(\theta+\gamma), \sin(\theta+\gamma)), \quad \boldsymbol{T}_1 = -r_1(\cos(\psi-\gamma), \sin(\psi-\gamma))$$

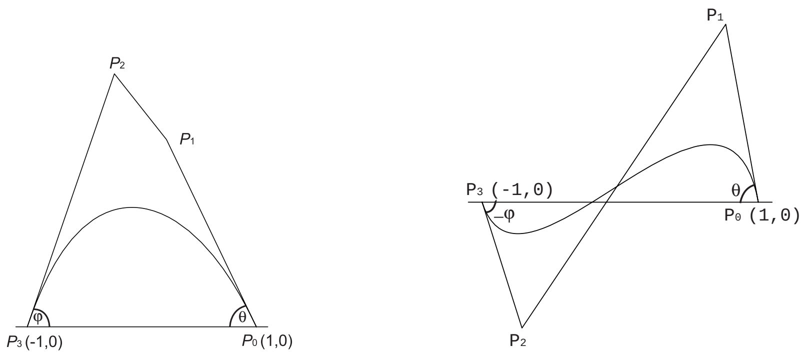

Figure 1: C-shaped $(\theta > 0, \psi > 0)$ and S-shaped $(\theta > 0, \psi < 0)$ polygons.

Figure 1 gives typical C- and S-shaped control polygons with their interpolated curves where $\boldsymbol{a}$=(1,0) and $\boldsymbol{b}$=(-1,0), (γ=0). Assume that the both end points, the end tangent directions and curvature are consistent where the "consistent" curvatures at the end points $\boldsymbol{a}$, $\boldsymbol{b}$ with respect to the corresponding tangent directions mean "sign of $\boldsymbol{z}'(i) \times (\boldsymbol{ab}) = (-1)^i$ sign of $\kappa_i, i = 0, 1$" or $\kappa_0 \sin\theta > 0, \kappa_1 \sin\psi > 0$ ([7]). Then, we consider a rational cubic curve $\boldsymbol{z}(t), 0 \le t \le 1$ of the form with four control vertices $\boldsymbol{p}_0(=\boldsymbol{a}), \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3(=\boldsymbol{b})$:

(3) $$\boldsymbol{z}(t) = \frac{v(1-t)^3\boldsymbol{p}_0 + (v+2)t(1-t)^2\boldsymbol{p}_1 + (w+2)t^2(1-t)\boldsymbol{p}_2 + wt^3\boldsymbol{p}_3}{v(1-t)^3 + (v+2)t(1-t)^2 + (w+2)t^2(1-t) + wt^3}$$

where

$$\begin{aligned}
\boldsymbol{p}_1 &= \boldsymbol{a} + \frac{r_0 v}{v+2}(-\cos(\theta+\gamma), \sin(\theta+\gamma)) \\
\boldsymbol{p}_2 &= \boldsymbol{b} + \frac{r_1 w}{w+2}(\cos(\psi-\gamma), \sin(\psi-\gamma))
\end{aligned}$$

The curvature $\kappa(t)$ of a plane curve $\boldsymbol{z}(t)$ is

(4) $$\kappa(t) = (\boldsymbol{z}'(t) \times \boldsymbol{z}''(t)) / \|\boldsymbol{z}'(t)\|^3$$

Then, it satisfies the required conditions:

$$\boldsymbol{z}(0) = \boldsymbol{a}, \boldsymbol{z}(1) = \boldsymbol{b}; \quad \boldsymbol{z}'(0) = \boldsymbol{T}_0, \boldsymbol{z}'(1) = \boldsymbol{T}_1; \quad \kappa(0) = \kappa_0, \kappa(1) = \kappa_1$$

where $r = \|\boldsymbol{ab}\|, \quad \boldsymbol{b} = \boldsymbol{a} + r(-\cos\gamma, \sin\gamma)$.

We take r_0, r_1 as positive parameters, i.e., we assume that the unit tangent vectors (tangent directions) are given at the two points $\boldsymbol{a}, \boldsymbol{b}$. Then we consider the sufficient condition on the given data $\theta, \psi, \kappa_0, \kappa_1$ and the weights (v, w). ¿From $\kappa(0) = \kappa_0, \kappa(1) = \kappa_1$, we get a system of quadratic equations:

(5) $$\begin{aligned}
&(i) \qquad \kappa_0 r_0^2 v + 2r_1 w \sin(\theta+\psi) = 2r(w+2)\sin\theta \\
&(ii) \qquad \kappa_1 r_1^2 w + 2r_0 v \sin(\theta+\psi) = 2r(v+2)\sin\psi
\end{aligned}$$

Solving (5) for v and w, we get:

$$v = 4r(\kappa_1 r_1^2 \sin\theta + 2r\sin\theta\sin\psi - 2r_1\sin\psi\sin(\theta+\psi))/D$$
$$w = 4r(\kappa_0 r_0^2 \sin\psi + 2r\sin\theta\sin\psi - 2r_0\sin\theta\sin(\theta+\psi))/D$$

where

$$D = \kappa_0\kappa_1 r_0^2 r_1^2 - 4(r\sin\psi - r_0\sin(\theta+\psi))(r\sin\theta - r_1 sin(\theta+\psi))$$

Introduce $R_0, R_1; \rho_0, \rho_1$ (note that consistent tangent directions and curvatures at the end points mean $R_0, R_1 > 0$; refer to [1],[7]) as:

$$R_0 = \frac{\kappa_0 r(v+2)^2\sin^2\psi}{2v(w+2)\sin\theta\sin^2(\theta+\psi)}, \quad R_1 = \frac{\kappa_1 r(w+2)^2\sin^2\theta}{2w(v+2)\sin\psi\sin^2(\theta+\psi)};$$
$$\rho_0 = \frac{vr_0\sin(\theta+\psi)}{r(v+2)\sin\psi}, \quad \rho_1 = \frac{wr_1\sin(\theta+\psi)}{r(w+2)\sin\theta}$$

Then we have a reduced quadratic system of equations in (ρ_0, ρ_1) from (5):

$$\rho_0 = 1 - R_1\rho_1^2, \quad \rho_1 = 1 - R_0\rho_0^2 \tag{6}$$

Depending on the C- and S- shapes of the control polygons as shown in Figure 1, we give the following theorem on N (= the number of solutions of the quadratic system (6)). Cases 1-2 and 3-4 are C- and S-shaped respectively, where $\theta > 0$ and $\psi > 0$ (< 0) for C- (S-) shaped.

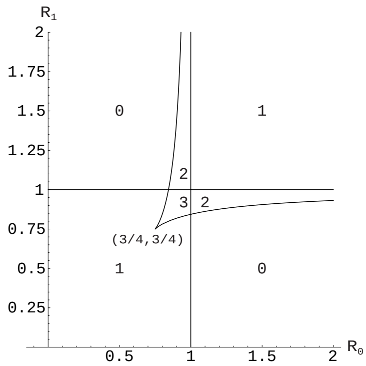

Figure 2: Number of solutions of quadratic system (6)

Theorem 1 ([1], [5]) *(Number of solutions of quadratic system (6)).*
Case 1 *($0 < \theta+\psi < \pi$): N is given in Fig. 2, where the curve separating the regions is represented by $R_0R_1\{256R_0R_1 - 256(R_0+R_1) + 288\} = 27$ having a cusp at (3/4, 3/4).*
Case 2 *($\pi < \theta+\psi < 2\pi$): $N = 1$ for $R_0, R_1 > 0$.*
Case 3 *($-\pi < \theta+\psi < 0$): $N = 1$ for $R_0 > 1, R_1 > 0$ and $N = 0$ for $R_0 \le 1, R_1 > 0$.*
Case 4 *($0 < \theta+\psi < \pi$): $N = 1$ for $R_0 > 0, R_1 > 1$ and $N = 0$ for $R_0 > 0, 0 < R_1 \le 1$.*

We consider the inflections and singularities (loop and cusp) on the curve(s) of the form (3).

Inflection points: The change of variable $t = 1/(1+s)$ converts the interval $(0,1)$ to $(0,\infty)$. For $\boldsymbol{z}(t) = (x(t), y(t))$, *Mathematica* (if necessary) helps us obtain:

$$y''(t)x'(t) - x''(t)y'(t) = \frac{2(1+s)^3}{(2s+s^2v+w)^3}\sum_{i=0}^{3} d_i s^i, \quad t = 1/(1+s) \tag{7}$$

with

$$d_0 = \kappa_1 r_1^3 w^3/2, \quad d_1 = 2rr_1vw^2\sin\psi, \quad d_2 = 3rr_0v^2w\sin\theta, \quad d_3 = \kappa_0 r_0^3 v^3/2$$

Use of Decartes' rule of signs and intermediate value of theorem shows that eq (7) has no positive (just one) zero acoording to C- (S-) shaped control polygons since then $(d_0, d_1, d_2, d_3) = (+,+,+,+)$ $((-,-,+,+))$. Hence, we have

Theorem 2 *(Inflection points) The segment of the form (3) has no (or just one) inflection point for the C-(or S-) shape of the polygon.*

Next, we consider the singularities.

Singularities (loops): As in the inflection points, convert the interval (0,1) to (0,∞) by a change of variable; $t = 1/(1+p)$. Assume the existance of a loop, i.e., for $p, q > 0$:

$$x(1/(1+p)) - x(1/(1+q)) = 0 \quad and \quad y(1/(1+p)) - y(1/(1+q)) = 0 \tag{8}$$

Here, letting $p + q = m$ and $pq = n$, then

$$p = (m + \sqrt{m^2 - 4n})/2 \quad and \quad q = (m - \sqrt{m^2-4n})/2, \quad m^2 > 4n.$$

Mathematica helps us reduce (8) to a system of linear equations in r_0, r_1:

$$\begin{aligned} \{\mu\cos(\gamma+\theta)\}r_0 + \{\lambda\cos(\gamma-\psi)\}r_1 &= \eta\cos\gamma \\ \{\mu\sin(\gamma+\theta)\}r_0 + \{\lambda\sin(\gamma-\psi)\}r_1 &= \eta\sin\gamma \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\mu = w(mw + n(w+2) - n^2v), \quad \lambda = w(nmv + n(v+2) - w) \\ &\eta = r(m^2vw + 2nv + mw(v+2) + (2n+mnv)(w+2))(>0) \end{aligned}$$

Solving (9) for r_0 and r_1, we get:

$$(i) \quad r_0 = \frac{\eta\sin\psi}{\mu\sin(\theta+\psi)}(>0), \quad (ii) \quad r_1 = \frac{\eta\sin\theta}{\lambda\sin(\theta+\psi)}(>0) \tag{10}$$

from which we derive:

$$\begin{aligned} (i) \quad & r(v+2)\sin\psi - r_0 v\sin(\theta+\psi)(= \kappa_1 r_1^2 w/2) \\ &= -\frac{rv^2(2mn + 2n^2 + n^2v + w(m^2-n) + mnw)\sin\psi}{\mu} \\ (ii) \quad & r(w+2)\sin\theta - r_1 w\sin(\theta+\psi)(= \kappa_0 r_0^2 v/2) \\ &= -\frac{rw^2(2 + 2m + mv + v(m^2-n) + w)\sin\theta}{\lambda} \end{aligned} \tag{11}$$

Now depending on the polygon, we consider the following four cases:

Case 1 ($0 < \theta + \psi < \pi$): Note 10(i) to obtain $\mu > 0$ and so, 11(i) can not be valid since $k_1 > 0$.
Case 2 ($\pi < \theta + \psi < 2\pi$): Note 10(i)-(ii) to obtain $\lambda, \mu < 0$ which can not be valid at the same time since:

$$\mu + \lambda m = wn(w+2) + 2mn + nv(m^2 - n + m) > 0$$

Case 3 ($-\pi < \theta + \psi < 0$): Note 10(i) to obtain $\mu > 0$ and so, 11(i) can not be valid since $k_1 < 0$ and $\sin\psi < 0$.
Case 4 ($0 < \theta + \psi < \pi$): Note 10(ii) to obtain $\lambda > 0$ and so, 11(ii) can not be valid since $k_0 > 0$.

The above consideration for all cases shows that the linear system (9) can not be valid, i.e., the segment has no loop.

Singularities (cusps): Assume that the rational cubic $\boldsymbol{z}(t)(= (x(t), y(t)))$ of the form (3) would have a cusp at $t = m, 0 < m < 1$ for which the necessary condition is $x'(m) = y'(m) = 0$ (the curvature becomes unbounded). The condition gives with $m = 1/(1+s), 0 < s < \infty$ the same two equations (9) with $m = 2s, n = s^2 (\Leftrightarrow m^2 = 4n)$. Since then (10) and (11) can not be valid, i.e., the system (9) has no positive solutions r_0, r_1. Hence the curve $\boldsymbol{z}$ has no cusp. Therefore we obtain:

Theorem 3 *(Singularities): The segment of the form (3) has no singularity regardless of the shape of the control polygon.*

3 Demonstration Let us consider numerical examples of C-type and S-type data:

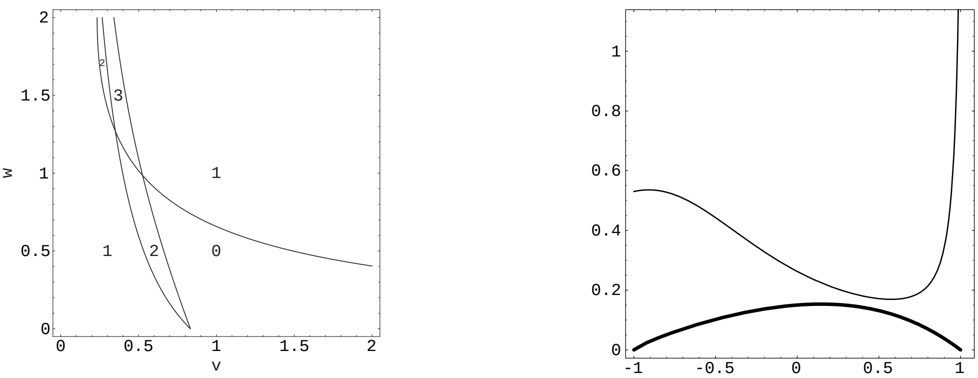

Figure 3: The number of curves and curve together with the curvature plot for C-shaped polygon

C**-shaped polygon:**

$$\boldsymbol{a} = (1,0), \boldsymbol{b} = (-1,0), (\theta, \psi, \kappa_0, \kappa_1, r_0, r_1) = (\pi/8, \pi/6, 0.2, 0.6, 1, 1)$$

S**-shaped polygon:**

$$\boldsymbol{a} = (1,0), \boldsymbol{b} = (-1,0), (\theta, \psi, \kappa_0, \kappa_1, r_0, r_1) = (\pi/4, -\pi/8, 0.5, -0.2, 1, 1)$$

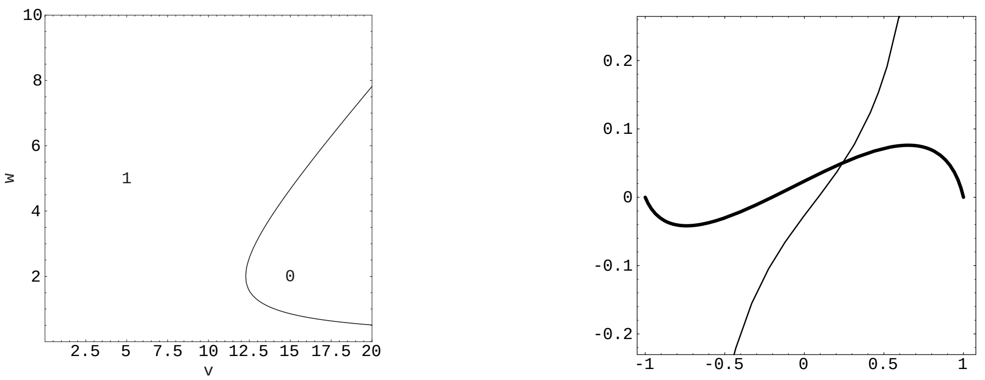

Figure 4: The number of curves and curve together with the curvature plot for S-shaped polygon

The left graphs in Figures 3 and 4 give $N(=$ the number of the curves with respect to $(v, w))$. The right ones in Figures 3 and 4 give the curves (represented in bold) and the curvature plots along the curves as offset distances proportional to the curvatures. For the C-shaped polygon, $(v, w) = (5.75583, 0.16943)(\Leftrightarrow (R_0, R_1) = (1, 1))$. Its curvature plot in Figure 3 shows neither inflection point nor singularity. For the S-shaped polygon, $(v, w) = (1, 1)$ gives $R_1 > 1$ from which the curve has just one inflection point. Its curvature plot in Figure 4 shows the existence of the inflection point where it crosses the curve.

Acknowledgements

We acknowledge the financial support of Monbusho (Ministry of education, Japan).

References

[1] C. de Boor, K. Hollig, and M. Sabin. High accuracy geometric hermite interpolation. *CAGD*, 4:269–278, 1987.
[2] W. Degen. High accurate rational approximation of parametric curves. *Computer Aided Geometric Design*, 10:293–313, 1993.
[3] W. Degen. *Mathematical Methods for Curves and Surfaces.* Vanderbilt University Press, New York, 1995.
[4] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design - Practical Guide.* Academic Press, New York, 1995.
[5] M. Sakai. Osculatory interpolation. *Computer Aided Geometric Design (accepted for publication)*, 2001.
[6] N. Sapidis. *Progress in Computer Graphics.* Ablex Publishing, Norwood, NJ, 1992.
[7] C. Seymour and K. Unsworth. Interactive shape preserving interpolation by curvature continuous rational cubic splines. *J. Comp. and Maths*, 102:87–117, 1999.

Department of Mathematics and Computer Science, Graduate School of Science and Engineering, Kagoshima University, Kagoshima 890-0065, Japan.

$E - mail : habib@po.minc.ne.jp; \quad msakai@sci.kagoshima - u.ac.jp$