# COMPACTNESS AND METRIZABILITY IN THE SPACE OF VECTOR MEASURES IN LOCALLY CONVEX SPACES

Jun Kawabe

Received July 16, 2001

Abstract. The purpose of this paper is to extend Prokhorov-LeCam's criteria for compactness and Varadarajan's criterion for metrizability to vector measures with values in certain locally convex spaces. Our results contain not only Prokhorov-LeCam's and Varadarajan's criteria for real measures but a sequential compactness criterion which was recently given by März and Shortt for vector measures with values in a Banach space.

**1 Introduction.** Compactness and metrizability for the weak topology of measures are important and applicative properties in the space of positive or real measures on topological spaces.

In 1956, Prokhorov [15] gave a compactness criterion for the weak topology of measures in the space of all positive, finite measures on a complete separable metric space. This criterion was extended by LeCam [11] to real Radon measures on an arbitrary completely regular space. These results are called *Prokhorov-LeCam's compactness criteria*, and play an important role in the study of stochastic convergence in probability theory and statistics.

As to the metrizability in the space of measures, it was proved in [15] that the space of all positive, finite measures on a separable metric space is metrizable. This is not the case for real measures, and in fact it was proved in Varadarajan [19] that the set of all real Radon measures on a metric space $S$ is metrizable if and only if $S$ is a finite set. Nevertheless, it was also proved in [19] that not the whole space but a relatively compact subset of real measures on a locally compact separable metric space is metrizable. This criterion is called *Varadarajan's metrizability criterion*.

The purpose of this paper is to extend these criteria for compactness and metrizability to vector measures with values in certain locally convex spaces. Our criteria contain not only Prokhorov-LeCam's compactness criteria and Varadarajan's metrizability criterion for real measures but a sequential compactness criterion which was recently given by März and Shortt [13] for vector measures with values in a Banach space. We should note here that in Theorem 1.5 of [13], they have not treated metrizability in the space of vector measures.

In the following section, we prepare notation and definitions, and recall some necessary results concerning vector measures and an integral of scalar functions with respect to vector measures.

In Section 3, we give a general compactness criterion for a set of vector measures, which are defined on an arbitrary completely regular space and take values in a sequentially complete locally convex space.

In Section 4, using the compactness criterion established in Section 3, we show that any uniformly bounded and uniformly tight subset of vector measures with values in a

2000 *Mathematics Subject Classification.* Primary 28B05, 28C15; Secondary 46G10.
*Key words and phrases.* vector measure, weak convergence of vector measures, uniform tightness, compactness, metrizability, Fréchet space.

Fréchet space is relatively sequentially compact with respect to the weak topology of vector measures. During its proof, we have actually established the result, which is a generalization of Varadarajan's metrizability criterion.

In this paper, all the topological spaces and topological linear spaces are Hausdorff, and the scalar fields of topological linear spaces are taken to be the field $\mathbb{R}$ of real numbers.

**2 Preliminaries.** Let $X$ be a locally convex space. Denote by $X^*$ the topological dual of $X$ and by $\langle x, x^* \rangle$ the natural duality between $X$ and $X^*$. The weak topology of $X$ means the $\sigma(X, X^*)$-topology on $X$. If $x^* \in X^*$ and $p$ is a seminorm on $X$, we write $x^* \leq p$ whenever $|\langle x, x^* \rangle| \leq p(x)$ for all $x \in X$.

Let $\mathcal{S}$ be a $\sigma$-field of subsets of a non-empty set $S$ and $\mu : \mathcal{S} \to X$ be a finitely additive set function. We say that $\mu$ is a *vector measure* if it is countably additive, i.e., for any sequence $\{E_n\}$ of pairwise disjoint subsets of $\mathcal{S}$, we have $\sum_{n=1}^{\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$ in the initial topology of $X$. If $\mu$ is a vector measure, then $x^*\mu := \langle \mu, x^* \rangle$ is a real measure for each $x^* \in X^*$. Conversely, a theorem of Orlicz and Pettis ensures that a finitely additive set function $\mu : \mathcal{S} \to X$ is countably additive if $x^*\mu$ is countably additive for each $x^* \in X^*$ (see, for instance, Corollary 1 of McArthur [14]).

Let $\mu : \mathcal{S} \to X$ be a vector measure and $p$ be a seminorm on $X$. Then the *p-semivariation* of $\mu$ is the set function $\|\mu\|_p : \mathcal{S} \to [0, \infty)$ defined by

$$\|\mu\|_p(E) := \sup_{x^* \leq p} |x^*\mu|(E)$$

for all $E \in \mathcal{S}$, where $|x^*\mu|(\cdot)$ is the total variation of the real measure $x^*\mu$. See Lewis [12] and Kluvánek and Knowles [10] for some properties of $p$-semivariations.

In this paper, we need an integral of real valued measurable functions with respect to vector measures with values in locally convex spaces. Let $\mu : \mathcal{S} \to X$ be a vector measure. A real valued $\mathcal{S}$-measurable function $f$ on $S$ is said to be *$\mu$-integrable* if (a) $f$ is $x^*\mu$-integrable for each $x^* \in X^*$, and (b) for each $E \in \mathcal{S}$, there exists an element of $X$, denoted by $\int_E f d\mu$, such that

$$\left\langle \int_E f d\mu, x^* \right\rangle = \int_E f d(x^*\mu)$$

for each $x^* \in X^*$. We note here that if $X$ is a sequentially complete, then every bounded, $\mathcal{S}$-measurable, real valued function $f$ is $\mu$-integrable, and

$$p\left(\int_E f d\mu\right) \leq \sup_{x^* \leq p} \int_E |f| d|x^*\mu| \leq \sup_{s \in E} |f(s)| \cdot \|\mu\|_p(E) \tag{2.1}$$

for every $E \in \mathcal{S}$ and every continuous seminorm $p$ on $X$. See [12] and [10] for some additional properties of this integral.

In what follows, let $S$ be a topological space and $\mathcal{B}(S)$ be the $\sigma$-field of all Borel subsets of $S$. Denote by $\mathcal{M}(S; X)$ the set of all vector measures $\mu : \mathcal{B}(S) \to X$. When $X = \mathbb{R}$, we write $\mathcal{M}(S)$ instead of $\mathcal{M}(S; \mathbb{R})$. Then, $\mathcal{M}(S)$ is a Banach space with the total variation norm $|m| := |m|(S)$.

A vector measure $\mu : \mathcal{B}(S) \to X$ is said to be *Radon* if for each $\varepsilon > 0$, $E \in \mathcal{B}(S)$, and continuous seminorm $p$ on $X$, there exists a compact subset $K$ of $E$ such that $\|\mu\|_p(E - K) < \varepsilon$, and it is said to be *tight* if the condition is satisfied for $E = S$. Then it is known that a vector measure $\mu : \mathcal{B}(S) \to X$ is Radon if and only if $x^*\mu$ is Radon for every $x^* \in X^*$ (see Theorem 1.6 of [12]).

By $\mathcal{M}_t(S;X)$ we denote the set of all Radon vector measures $\mu : \mathcal{B}(S) \to X$. As before, we write $\mathcal{M}_t(S)$ instead of $\mathcal{M}_t(S;\mathbb{R})$. Denote by $C(S)$ the Banach space of all bounded, continuous, real valued functions on $S$ with the norm $\|f\| := \sup_{s\in S} |f(s)|$.

Let $\mu \in \mathcal{M}(S;X)$. If $X$ is sequentially complete, then we can define a continuous linear operator $T_\mu : C(S) \to X$ by

$$T_\mu(f) = \int_S f d\mu, \quad f \in C(S),$$

which is called the operator *determined by* $\mu$. Recall that a linear operator $T : C(S) \to X$ is said to be *weakly compact* if it maps every bounded subset of $C(S)$ into a relatively weakly compact subset of $X$. When $S$ is compact and $X$ is a Banach space, it was first shown in Bartle, Dunford, and Schwartz [1] that every weakly compact operator from $C(S)$ into $X$ is determined by a Radon vector measure whose values in $X$ (see also Theorem VI.5 of Diestel and Uhl [4]).

This type of representation theorem was extended to several other cases. For the case that $S$ is compact and $X$ is an arbitrary locally convex space see Theorem 3.1 of [12].

On the other hand, for the case that $X = \mathbb{R}$, but $S$ is an arbitrary completely regular space see Chapter IV, §5, no. 2, Proposition 5 of Bourbaki [2] and Theorem 2 in §3 of Smolyanov and Fomin [18]. See [7] for the case that $X$ is the weak* dual of a barreled locally convex space.

The following proposition, which may be virtually known, insists that every continuous linear operator, satisfying some tightness condition, from $C(S)$ into $X$ can be determined by a vector measure $\mu \in \mathcal{M}_t(S;X)$ even for the case that $S$ is an arbitrary completely regular space and $X$ is an arbitrary locally convex space. See [8] for its proof.

**Proposition 1.** *Let $S$ be a completely regular space and $X$ a locally convex space. Assume that a weakly compact operator $T : C(S) \to X$ satisfies the following condition* (*): *For each $\varepsilon > 0$ and $x^* \in X^*$, there exists a compact subset $K$ of $S$ such that $|\langle T(f), x^*\rangle| \le \varepsilon\|f\|$ for all $f \in C(S)$ with $f(K) = 0$.*
*Then, there exists a unique vector measure $\mu \in \mathcal{M}_t(S;X)$ such that*

(a) *every bounded, Borel measurable real function is $\mu$-integrable, and*

(b) $T(f) = \int_S f d\mu$ *for all* $f \in C(S)$.

## 3 A general compactness criterion.

We first introduce the notion of weak convergence of vector measures. Let $S$ be a completely regular space and $X$ be a locally convex space. Let $\{\mu_\alpha\}$ be a net in $\mathcal{M}(S;X)$ and $\mu \in \mathcal{M}(S;X)$. According to the definition by Dekiert [3], we say that $\{\mu_\alpha\}$ *converges weakly to* $\mu$ and write $\mu_\alpha \xrightarrow{w} \mu$ if the corresponding net of real measures $\{x^*\mu_\alpha\}$ converges weakly to $x^*\mu$, i.e., for each $f \in C(S)$, we have

$$\lim_\alpha \int_S f d(x^*\mu) = \int_S f d(x^*\mu).$$

It is obvious that $\mu_\alpha \xrightarrow{w} \mu$ if and only if the net $\{T_{\mu_\alpha}\}$ of operators determined by $\mu_\alpha$ converges to the operator $T_\mu$ determined by $\mu$ in the weak operator topology. In the following, we equip $\mathcal{M}(S;X)$ with the topology determined by this weak convergence and call it the *weak topology of vector measures.* In the case that $X = \mathbb{R}$, we call it the *weak topology of measures.*

A subset $\mathcal{V}$ of $\mathcal{M}(S;X)$ is said to be *uniformly bounded* if $\sup_{\mu\in\mathcal{V}} \|\mu\|_p(S) < \infty$ for every continuous seminorm $p$ on $X$. Since every weakly bounded subset of $X$ is bounded, it is readily seen that $\mathcal{V}$ is uniformly bounded if $x^*(\mathcal{V}) := \{x^*\mu : \mu \in \mathcal{V}\}$ is uniformly bounded for each $x^* \in X^*$, i.e., $\sup_{\mu\in\mathcal{V}} |x^*\mu|(S) < \infty$ for each $x^* \in X^*$. Further, the principle of uniform boundedness (see Corollary to III.4.2 of Schaefer [16]) ensures that if every element of $x^*(\mathcal{V})$ is Radon, then the uniform boundedness of $x^*(\mathcal{V})$ follows from the condition that $\sup_{\mu\in\mathcal{V}} |\int_S f d(x^*\mu)| < \infty$ for each $x^* \in X^*$ and $f \in C(S)$.

We say that $\mathcal{V}$ is *uniformly tight* if for each $\varepsilon > 0$ and continuous seminorm $p$ on $X$, there exists a compact subset $K$ of $S$ such that $\|\mu\|_p(S-K) < \varepsilon$ for all $\mu \in \mathcal{V}$. It is obvious that if $\mathcal{V}$ is uniformly tight, so is $x^*(\mathcal{V})$ for each $x^* \in X^*$. However, the converse statement is not valid in general; see an example of [8]. This concept is very important in the study of compactness for sets of real or vector measures.

The following theorem gives a general weak compactness criterion for vector measures which will be used in Section 4. Theorem 1 has been proved in [9], but we give its proof for the completeness of this paper.

**Theorem 1.** *Let $S$ be a completely regular space and $X$ be a sequentially complete locally convex space. Assume that $\mathcal{V} \subset \mathcal{M}_t(S;X)$ satisfies the following two conditions:*

(a) *For each $x^* \in X^*$, the set $x^*(\mathcal{V})$ is relatively compact in $\mathcal{M}_t(S)$.*

(b) *There exists a relatively weakly compact subset $W$ of $X$ such that*

$$\left\{ \int_S f d\mu : f \in C(S), \|f\| \le 1, \mu \in \mathcal{V} \right\} \subset W.$$

*Then, $\mathcal{V}$ is relatively compact in $\mathcal{M}_t(S;X)$. If $X$ is semi-reflexive,* (b) *follows from* (a).

*Proof.* For each $\mu \in \mathcal{M}(S;X)$, we define a continuous linear operator $T_\mu : C(S) \to X$ by

$$T_\mu(f) = \int_S f d\mu, \quad f \in C(S).$$

Then it follows from (b) that $T_\mu$ is weakly compact for every $\mu \in \mathcal{V}$. Let $X_\sigma$ be the space $X$ with the weak topology $\sigma(X, X^*)$. Denote by $\mathcal{L}(C(S), X_\sigma)$ the space of all continuous linear operators from $C(S)$ into $X_\sigma$, and by $\mathcal{L}_\sigma(C(S), X_\sigma)$ the same space with the topology of simple convergence. We also denote by $X^{C(S)}$ the set of all mappings from $C(S)$ into $X$. Put $\mathcal{H} = \{T_\mu : \mu \in \mathcal{V}\}$ and denote by $\mathcal{H}_1$ the closure of $\mathcal{H}$ in $X^{C(S)}$ for the topology of simple convergence. Then, it follows from (b) and Tychonoff's theorem that $\mathcal{H}_1$ is compact in $X^{C(S)}$ for the topology of simple convergence. To prove that $\mathcal{H}$ is a relatively compact subset of $\mathcal{L}_\sigma(C(S), X_\sigma)$, we have only to show that $\mathcal{H}_1 \subset \mathcal{L}(C(S), X_\sigma)$. Since (b) implies that the set $\{\langle T_\mu(f), x^*\rangle : \mu \in \mathcal{V}\}$ is bounded for each $x^* \in X^*$ and $f \in C(S)$, it follows from Banach-Steinhaus theorem (see, e.g., Theorem III.4.2 of [16]) that $\mathcal{H}$ is an equicontinuous subset of $\mathcal{L}(C(S), X_\sigma)$. Then $\mathcal{H}_1 \subset \mathcal{L}(C(S), X_\sigma)$ by Theorem III.4.3 of [16]. Thus, we have finished the proof of the relative compactness of $\mathcal{H}$, so that for any net $\{\mu_\alpha\}$ of $\mathcal{V}$, we can find a subnet $\{\mu_{\alpha'}\}$ of $\{\mu_\alpha\}$ and an operator $T \in \mathcal{L}(C(S), X_\sigma)$ such that

$$\langle T(f), x^*\rangle = \lim_{\alpha'} \langle T_{\mu_{\alpha'}}(f), x^*\rangle = \lim_{\alpha'} \left\langle \int_S f d\mu_{\alpha'}, x^* \right\rangle \tag{3.1}$$

for all $x^* \in X^*$ and $f \in C(S)$.

Now we shall prove that $T$ is weakly compact and satisfies assumption (*) of Proposition 1. Put $D = \{f \in C(S) : \|f\| \le 1\}$. By (b), there exists a relatively weakly compact subset $W$ of $X$ such that

$$\{T_\mu(f) : f \in D, \mu \in \mathcal{V}\} \subset W,$$

so that we have $\bigcup_{\alpha'} T_{\mu_{\alpha'}}(D) \subset W$. On the other hand, by (3.1) it is easy to see that $T(D)$ is contained in the closure of $\bigcup_{\alpha'} T_{\mu_{\alpha'}}(D)$ for the weak topology $\sigma(X, X^*)$. Thus $T(D)$ is a relatively weakly compact subset of $X$, so that $T$ is weakly compact.

Next we show that $T$ satisfies assumption $(^*)$ of Proposition 1. Fix $\varepsilon > 0$ and $x^* \in X^*$. By (3.1), we have

$$|\langle T(f), x^* \rangle| = \lim_{\alpha'} \left| \left\langle \int_S f d\mu_{\alpha'}, x^* \right\rangle \right| = \lim_{\alpha'} \left| \int_S f d(x^* \mu_{\alpha'}) \right| \tag{3.2}$$

for all $f \in C(S)$. On the other hand, by (a) there exists a subnet $\{m_{\alpha''}\}$ of $\{x^*\mu_{\alpha'}\}$ and a real measure $m \in \mathcal{M}_t(S)$ such that

$$m_{\alpha''} \xrightarrow{w} m. \tag{3.3}$$

Since $m$ is Radon, there exists a compact subset $K$ of $S$ such that

$$|m|(S - K) < \varepsilon. \tag{3.4}$$

Fix $f \in C(S)$ with $f(K) = 0$. Then, it follows from (3.2)–(3.4) that

$$\begin{aligned} |\langle T(f), x^* \rangle| &= \lim_{\alpha''} \left| \int_S f dm_{\alpha''} \right| = \left| \int_S f dm \right| \\ &= \left| \int_{S-K} f dm \right| \le \|f\| \cdot |m|(S - K) < \varepsilon \|f\|, \end{aligned}$$

and this implies that $T$ satisfies assumption $(^*)$ of Proposition 1. Thus, by Proposition 1, we can find a vector measure $\mu \in \mathcal{M}_t(S; X)$ such that

$$T(f) = \int_S f d\mu$$

for all $f \in C(S)$. Hence, by (3.1) we have

$$\lim_{\alpha'} \left\langle \int_S f d\mu_{\alpha'}, x^* \right\rangle = \left\langle \int_S f d\mu, x^* \right\rangle,$$

and this implies the relative weak compactness of $\mathcal{V}$.

Assume that $X$ is semi-reflexive. By (a), $x^*(\mathcal{V})$ is uniformly bounded for each $x^* \in X^*$, so that $\mathcal{V}$ is uniformly bounded as is stated in the beginning of this section. Then, the set

$$\left\{ \int_S f d\mu : f \in C(S), \|f\| \le 1, \mu \in \mathcal{V} \right\}$$

is a bounded subset of $X$, and hence it is weakly relatively compact by IV.5.5 of [16]. $\square$

It is well known as Prokhorov-LeCam's compactness criterion that every uniformly bounded and uniformly tight subset $M$ of $\mathcal{M}_t(S)$ is relatively compact in $\mathcal{M}_t(S)$ (see [18], and also [11] and [19]). The following is a vector measure version of this criterion.

**Corollary.** *Let $S$ be a completely regular space and $X$ be a sequentially complete locally convex space. Assume that $\mathcal{V} \subset \mathcal{M}_t(S; X)$ satisfies the following three conditions*:

(a) *For each $x^* \in X^*$, $x^*(\mathcal{V})$ is uniformly bounded.*

(b) *For each $x^* \in X^*$, $x^*(\mathcal{V})$ is uniformly tight.*

(c) *There exists a relatively weakly compact subset $W$ of $X$ such that*

$$\left\{ \int_S f d\mu : f \in C(S), \|f\| \le 1, \mu \in \mathcal{V} \right\} \subset W.$$

*Then, $\mathcal{V}$ is relatively compact in $\mathcal{M}_t(S;X)$. Further, if $X$ is semi-reflexive,* (c) *follows from* (a).

*Proof.* By (a), (b), and Theorem 2a in §5 of [18], for each $x^* \in X^*$, $x^*(\mathcal{V})$ is relatively compact in $\mathcal{M}_t(S)$. Consequently, $\mathcal{V}$ is relatively compact in $\mathcal{M}_t(S;X)$ by Theorem 1. □

**4 Sequential compactness and metrizability.** In this section, we turn our attention to a sequential compactness criterion and a metrizability criterion in the space of vector measures.

It is well known as Prokhorov-LeCam's sequential compactness criterion that any uniformly bounded and uniformly tight subset of real measures on a completely regular space $S$ is relatively compact and relatively sequentially compact, provided that every compact subset of $S$ is metrizable; see Theorem 6 of [11] and Theorem 2 in §5 of [18]. The following theorem extends this important and applicative result to vector measures with values in a Fréchet space.

**Theorem 2.** *Let $S$ be a completely regular space whose compact subsets are all metrizable, and let $X$ be a Fréchet space. Assume that $\mathcal{V} \subset \mathcal{M}_t(S;X)$ satisfies the following three conditions*:

(a) *$\mathcal{V}$ is uniformly bounded.*

(b) *$\mathcal{V}$ is uniformly tight.*

(c) *There exists a relatively weakly compact subset $W$ of $X$ such that*

$$\left\{ \int_S f d\mu : f \in C(S), \|f\| \le 1, \mu \in \mathcal{V} \right\} \subset W.$$

*Then, $\mathcal{V}$ is relatively compact and relatively sequentially compact in $\mathcal{M}_t(S;X)$. Further, if $X$ is reflexive,* (c) *follows from* (a).

To prove Theorem 2, we need the following

**Lemma 1.** *Let $S$ be a space as in Theorem 2 above and $X$ a sequentially complete locally convex space. Let $\mathcal{V} \subset \mathcal{M}_t(S;X)$ be uniformly tight. Let $p$ be a continuous seminorm on $X$. Then, there exists a countable dense subset $I$ of $C(S)$ which satisfies the following condition*: *For any $\varepsilon > 0$ and $f \in C(S)$, we can find a $g \in I$ satisfying*

$$p\left( \int_S (f-g) d\mu \right) \le \varepsilon \left( \|\mu\|_p(S) + 2\|f\| + \varepsilon \right)$$

*for all $\mu \in \mathcal{V}$.*

*Proof.* By uniform tightness of $\mathcal{V}$, there exists a sequence $\{K_n\}$ of compact subsets of $S$ such that

$$\|\mu\|_p(S - K_n) < \frac{1}{n} \tag{4.1}$$

for all $\mu \in \mathcal{V}$ and $n \ge 1$. Since each $K_n$ is metrizable, $C(K_n)$ is separable. Fix $n \ge 1$ for a moment, and let $\{g_{i,n}\}_{i=1}^{\infty}$ be a countable dense subset of $C(K_n)$. Then, each $g_{i,n}$ has an extension $\tilde{g}_{i,n} \in C(S)$ such that

$$\|\tilde{g}_{i,n}\| = \|g_{i,n}\|_{K_n} := \sup_{s \in K_n} |g_{i,n}(s)|. \tag{4.2}$$

Put $I = \{\tilde{g}_{i,n}\}_{i,n=1}^{\infty}$. Fix $f \in C(S)$ and $\varepsilon > 0$, and choose $n_0$ such that $1/n_0 < \varepsilon$. We set $f_{n_0} = f\lceil_{K_{n_0}}$ (the restriction of $f$ onto $K_{n_0}$) $\in C(K_{n_0})$, then there exists a $g_{i_0,n_0} \in C(K_{n_0})$ such that

$$\|f_{n_0} - g_{i_0,n_0}\|_{K_{n_0}} < \frac{1}{n_0} \tag{4.3}$$

since $\{g_{i,n_0}\}_{i=1}^{\infty}$ is a dense subset of $C(K_{n_0})$. On the other hand, by (4.2) and (4.3), we have

$$\begin{aligned}
\|f - \tilde{g}_{i_0,n_0}\| &\leq \|f\| + \|\tilde{g}_{i_0,n_0}\| = \|f\| + \|g_{i_0,n_0}\|_{K_{n_0}} \\
&\leq \|f\| + \left(\frac{1}{n_0} + \|f_{n_0}\|_{K_{n_0}}\right) \\
&\leq 2\|f\| + \frac{1}{n_0}.
\end{aligned}$$

By (4.1), (4.3) and the inequality above, together with (2.1) and the fact that $\|\mu\|_p(\cdot)$ is increasing on $\mathcal{B}(S)$, for each $\mu \in \mathcal{V}$, we have

$$\begin{aligned}
p\left(\int_S (f - \tilde{g}_{i_0,n_0}) d\mu\right) &\leq p\left(\int_{K_{n_0}} (f_{n_0} - g_{i_0,n_0}) d\mu\right) + p\left(\int_{S-K_{n_0}} (f - \tilde{g}_{i_0,n_0}) d\mu\right) \\
&\leq \|\mu\|_p(K_{n_0}) \cdot \|f_{n_0} - g_{i_0,n_0}\|_{K_{n_0}} \\
&\qquad + \|\mu\|_p(S - K_{n_0}) \cdot \|f - \tilde{g}_{i_0,n_0}\| \\
&\leq \frac{1}{n_0}\|\mu\|_p(S) + \frac{1}{n_0}\|f - \tilde{g}_{i_0,n_0}\| \\
&\leq \frac{1}{n_0}\|\mu\|_p(S) + \frac{1}{n_0}\left(2\|f\| + \frac{1}{n_0}\right) \\
&\leq \varepsilon\left(\|\mu\|_p(S) + 2\|f\| + \varepsilon\right).
\end{aligned}$$

Hence, the proof of Lemma 1 is complete if we put $g = \tilde{g}_{i_0,n_0} \in I$. □

*Proof of Theorem 2.* Denote by $\sigma_{\mathcal{M}}$ the weak topology of vector measures on $\mathcal{M}_t(S;X)$, and by $\overline{A}$ the closure of a subset $A$ of $\mathcal{M}_t(S;X)$ with respect to $\sigma_{\mathcal{M}}$.

Assume that $\mathcal{V} \subset \mathcal{M}_t(S;X)$ satisfies (a)–(c) of Theorem 2. Then, $\mathcal{V}$ is relatively compact by Theorem 1, so that $\overline{\mathcal{V}}$ is compact. Take a sequence $\{\mu_n\} \subset \mathcal{V}$, and put $M = \overline{\{\mu_n\}}$. Then, $M$ is compact since $M$ is a closed subset of $\overline{\mathcal{V}}$. Consequently, to prove Theorem 2, we have only to show that the weak topology $\sigma_{\mathcal{M}}$ is metrizable on $M$, since compactness and sequential compactness coincide on metrizable spaces. For this end, we need the following

**Lemma 2.** *Let $p$ be a continuous seminorm on $X$. Then, there exists a semi-metric $d_p$ on $M$ which satisfies the following two conditions*:

(a) *The weak topology $\sigma_{\mathcal{M}}$ on $M$ is finer than the topology generated by $d_p$.*

(b) *Let* $\mu, \nu \in M$. *Then,* $d_p(\mu, \nu) = 0$ *implies that*

$$p\left(\int_S f d\mu - \int_S f d\nu\right) = 0$$

*for all* $f \in C(S)$.

*Proof.* Let $I$ be a countable dense subset of $C(S)$ in Lemma 1, and put $I = \{g_m\}_{m=1}^{\infty}$ for simplicity. Put

$$Q = \left\{\int_S g_m d\mu_n : m \geq 1, n \geq 1\right\},$$

and denote by $X_0$ the closed linear subspace spanned by $Q$. Since $X_0$ is a separable metrizable locally convex space, $X_0^*$ is separable for $\sigma(X_0^*, X_0)$ by IV.1.7 of [16], so that there exists a countable subset $\{u_l^*\}_{l=1}^{\infty}$ of $X_0^*$ which is dense in $X_0^*$ for $\sigma(X_0^*, X_0)$. By Hahn-Banach Theorem, each $u_l^*$ has an extension $x_l^* \in X^*$. Put $H = \{x_l^*\}_{l=1}^{\infty}$. Since $\{u_l^*\}_{l=1}^{\infty}$ is dense in $X_0^*$ for $\sigma(X_0^*, X_0)$, $H$ separates points of $X_0$, i.e., $x_0 \in X_0$ and $\langle x_0, x^* \rangle = 0$ for all $x^* \in H$, then $x_0 = 0$.

Define a semi-metric $d_p$ on $M$ by

$$d_p(\mu, \nu) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2^l} \cdot \frac{1}{2^m} \cdot \frac{\left|\left\langle \int_S g_m d\mu - \int_S g_m d\nu, x_l^* \right\rangle\right|}{1 + \left|\left\langle \int_S g_m d\mu - \int_S g_m d\nu, x_l^* \right\rangle\right|}$$

for all $\mu, \nu \in M$. Then, we shall show that $d_p$ satisfies (a) and (b). It is easy to prove (a), so that we shall prove (b). Let $\mu, \nu \in M$, and $d_p(\mu, \nu) = 0$. Then, it is readily shown that $\int_S g_m d\mu, \int_S g_m d\nu \in X_0$ for all $m \geq 1$, since $M = \overline{\{\mu_n\}}$ by the definition and $X_0$ is the (weakly) closed linear subspace spanned by $Q = \{\int_S g_m d\mu_n : m \geq 1, n \geq 1\}$. Hence, by the definition of $d_p$ and the fact that $H = \{x_l^*\}$ separates points of $X_0$, we have

$$\int_S g_m d\mu = \int_S g_m d\nu \tag{4.4}$$

for all $m \geq 1$. Fix $f \in C(S)$ and $\varepsilon > 0$. By Lemma 1, there exists a $g_{m_0} \in I$ such that

$$p\left(\int_S (f - g_{m_0}) d\lambda\right) \leq \varepsilon \left(\|\lambda\|_p(S) + 2\|f\| + \varepsilon\right) \tag{4.5}$$

for all $\lambda \in \mathcal{V}$. Thus, by (4.4) and (4.5), we have

$$\begin{aligned} p\left(\int_S f d\mu - \int_S f d\nu\right) &\leq p\left(\int_S (f - g_{m_0}) d\mu\right) + p\left(\int_S g_{m_0} d\mu - \int_S g_{m_0} d\nu\right) \\ &\quad + p\left(\int_S (g_{m_0} - f) d\nu\right) \\ &\leq \varepsilon \left(\|\mu\|_p(S) + 2\|f\| + \varepsilon\right) + \varepsilon \left(\|\nu\|_p(S) + 2\|f\| + \varepsilon\right). \end{aligned}$$

Since $\varepsilon$ is arbitrary, we have

$$p\left(\int_S f d\mu - \int_S f d\nu\right) = 0,$$

and the proof of Lemma 2 is complete. □

Let us return to the proof of Theorem 2. Let $\{p_n\}$ be a countable set of continuous seminorms which generates the topology of $X$. Put $d_n = d_{p_n}$ for simplicity and define a semi-metric on $M$ by

$$d(\mu,\nu) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(\mu,\nu)}{1+d_n(\mu,\nu)}$$

for all $\mu,\nu \in M$. To prove that $d$ is actually a metric, we assume that $d(\mu,\nu)=0$, $\mu,\nu \in M$. Then, $d_n(\mu,\nu)=0$ for all $n \geq 1$, so that, by (b) of Lemma 2, we have

$$p_n\left(\int_S f\,d\mu - \int_S f\,d\nu\right) = 0 \tag{4.6}$$

for all $n \geq 1$ and all $f \in C(S)$. Since $\{p_n\}$ generates the topology of $X$, it follows from (4.6) that

$$\int_S f\,d\mu = \int_S f\,d\nu$$

for all $f \in C(S)$, which implies $\mu = \nu$, since $\mu$ and $\nu$ are Radon.

By (a) of Lemma 2, it is easy to see that the weak topology $\sigma_{\mathcal{M}}$ on $M$ is finer than the metric topology generated by $d$. ¿From this and the compactness of $M$ it follows that both topologies are equivalent (see, e.g., I.5.8 of Dunford and Schwartz [6]). Hence, the weak topology $\sigma_{\mathcal{M}}$ on $M$ is metrizable, and the proof of the relative weak sequential compactness of Theorem 2 is complete.

Finally, assume that $X$ is reflexive. Then, it follows from (a) that the set

$$\left\{\int_S f\,d\mu : f \in C(S), \|f\| \leq 1, \mu \in \mathcal{V}\right\}$$

is a bounded subset of $X$, so that it is weakly relatively compact by Theorem IV.5.6 of [16]. □

In the proof of Theorem 26 in Part II of [19], it was proved that any relatively compact subset of real measures on a locally compact separable metric space is *metrizable* with respect to the weak topology of measures, so that it is relatively sequentially compact (Varadarajan's metrizability criterion). Since every locally compact separable metric space is a Polish space (see Theorem 6 in Chapter II of Schwartz [17]), by Theorem 30 in Part II of [19], relative compactness coincides with the combination of uniform boundedness and uniform tightness for subsets of $\mathcal{M}_t(S)$. Therefore, the following Theorem 3 extends Varadarajan's metrizability criterion to vector measures with values in a separable Fréchet space.

**Theorem 3.** *Let $S$ be a completely regular space whose compact subsets are all metrizable. Let $X$ be a Fréchet space whose topological dual $X^*$ has a countable set which separates points of $X$ (this is equivalent to $X^*$ being separable for the weak topology $\sigma(X^*,X)$). Assume that $\mathcal{V} \subset \mathcal{M}_t(S;X)$ satisfies the following three conditions:*

(a) *$\mathcal{V}$ is uniformly bounded.*

(b) *$\mathcal{V}$ is uniformly tight.*

(c) *There exists a relatively weakly compact subset $W$ of $X$ such that*

$$\left\{\int_S f\,d\mu : f \in C(S), \|f\| \leq 1, \mu \in \mathcal{V}\right\} \subset W.$$

*Then, $\mathcal{V}$ is metrizable and relatively compact in $\mathcal{M}_t(S;X)$, so that it is relatively sequentially compact in $\mathcal{M}_t(S;X)$.*

To prove Theorem 3, we need the following

**Lemma 3.** *Let $S$ be a completely regular space and $X$ a locally convex space. Assume that $\mathcal{V} \subset \mathcal{M}_t(S;X)$ is uniformly bounded and uniformly tight. Then the closure $\overline{\mathcal{V}}$ of $\mathcal{V}$ in $\mathcal{M}_t(S;X)$ is also uniformly bounded and uniformly tight.*

*Proof.* Fix $\varepsilon > 0$ and a continuous seminorm $p$ on $X$. Since $\mathcal{V}$ is uniformly tight, there exists a compact subset $K$ of $S$ such that

$$\|\mu\|_p(S-K) < \varepsilon \tag{4.7}$$

for all $\mu \in \mathcal{V}$.

Take $\mu \in \overline{\mathcal{V}}$ arbitrarily. Then there exists a net $\{\mu_\alpha\} \subset \mathcal{V}$ such that $\mu_\alpha \xrightarrow{w} \mu$. By (4.7) and Chapter IX, §5, no. 3, Proposition 6 of [2], for any $x^* \in X^*$ with $x^* \leq p$, we have

$$\begin{aligned} |x^*\mu|(S-K) &\leq \liminf_\alpha |x^*\mu_\alpha|(S-K) \\ &\leq \sup_{\nu\in\mathcal{V}} |x^*\nu|(S-K) \leq \sup_{\nu\in\mathcal{V}} \|\nu\|_p(S-K) \leq \varepsilon, \end{aligned}$$

so that $\|\mu\|_p(S-K) \leq \varepsilon$. Since $\mu \in \overline{\mathcal{V}}$ is arbitrary, we have

$$\sup_{\mu\in\overline{\mathcal{V}}} \|\mu\|_p(S-K) \leq \varepsilon,$$

and this implies that $\overline{\mathcal{V}}$ is uniformly tight. Uniform boundedness of $\overline{\mathcal{V}}$ can be proved similarly. □

*Proof of Theorem 3.* In the proof of Theorem 2, put $M = \overline{\mathcal{V}}$ instead of $M = \overline{\{\mu_n\}}$. Then, by Lemma 3, $M$ is uniformly bounded and uniformly tight. Therefore, Lemma 1 is valid for $M$. Consequently, Lemma 2 can be proved for the set $M$, since $X^*$ has a countable set which separates points of $X$. The rest of the proof is the same as the proof of Theorem 2. □

**Remark.** When $X$ is a Banach space, condition (c) of Theorems 1 and 2 follows from uniform tightness of $\mathcal{V}$ and the following condition (d) which was assumed in Theorem 1.5 of [13]:

(d) For each compact subset $D$ of $S$, there exists a weakly compact subset $W_D$ of $X$ such that

$$\left\{ \int_D f d\mu : f \in C(S), \|f\| \leq 1, \mu \in \mathcal{V} \right\} \subset W_D.$$

To prove this, we fix $\varepsilon > 0$. By uniform tightness of $\mathcal{V}$, there exists a compact subset $D$ of $S$ such that $\|\mu\|(S-D) < \varepsilon$ for all $\mu \in \mathcal{V}$, where $\|\mu\|$ denotes the semivariation of $\mu$ with respect to the norm on $X$. Then we have

$$\left\| \int_S f d\mu - \int_D f d\mu \right\| = \left\| \int_{S-D} f d\mu \right\| \leq \|f\| \cdot \|\mu\|(S-D) \leq \varepsilon$$

for all $\mu \in \mathcal{V}$ and all $f \in C(S)$ with $\|f\| \leq 1$. Let $W_D$ be the weakly compact subset of $X$ in condition (d). Then

$$W := \left\{ \int_S f d\mu : f \in C(S), \|f\| \leq 1, \mu \in \mathcal{V} \right\} \subset W_D + \varepsilon B_X,$$

where $B_X$ denotes the closed unit ball of $X$. Thus, $W$ is relatively weakly compact by Grothendieck's lemma (see Lemma XIII.2 of Diestel [5]).

## References

[1] R. G. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. 7 (1955), 289–305.

[2] N. Bourbaki, *Integration*, Ch. IX, Intégration sur les espaces topologiques separés, Hermann, Paris, 1969.

[3] M. Dekiert, *Kompaktheit, Fortsetzbarkeit und Konvergenz von Vectormassen*, Dissertation, University of Essen, 1991.

[4] J. Diestel and J. J. Uhl, *Vector Measures*, Amer. Math. Soc. Surveys No. 15, Providence, 1977.

[5] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, 1984.

[6] N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, John Wiley & Sons, 1957.

[7] J. Kawabe, *A type of Strassen's theorem for positive vector measures with values in dual spaces*, Proc. Amer. Math. Soc. 128 (2000), 3291–3300.

[8] J. Kawabe, *Sequential compactness for the weak topology of vector measures in certain nuclear spaces*, Georgian Math. J. 8 (2001), 283–295.

[9] J. Kawabe, *Compactness criteria for the weak compactness of vector measures in locally convex spaces*, to appear in Publ. Math. Debrecen.

[10] I. Kluvánek and G. Knowles, *Vector Measures and Control Systems*, North-Holland, 1976.

[11] L. LeCam, *Convergence in distribution of stochastic processes*, Univ. California Publ. Statist. 2 (1957), 207–236.

[12] D. R. Lewis, *Integration with respect to vector measures*, Pacific J. Math. 33 (1970), 157–165.

[13] M. März and R. M. Shortt, *Weak convergence of vector measures*, Publ. Math. Debrecen 45 (1994), 71–92.

[14] C. W. McArthur, *On a theorem of Orlicz and Pettis*, Pacific J. Math. 22 (1967), 297–302.

[15] Yu. V. Prokhorov, *Convergence of random processes and limittheorems in probability theory*, Theory Probab. Appl. 1 (1956), 157–214.

[16] H. H. Schaefer, *Topological Vector Spaces*, Springer, New York, 1971.

[17] L. Schwartz, *Radon Measures on Arbitrary Topological Spacesand Cylindrical Measures*, Tata Institute of Fundamental Research, Oxford University Press, 1973.

[18] O. G. Smolyanov and S. V. Fomin, *Measures on linear topological spaces*, Russian Math. Surveys 31 (1976), 1–53.

[19] V. S. Varadarajan, *Measures on topological spaces*, Amer. Math. Soc. Transl. Ser. II 48 (1965), 161–228.

DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING
SHINSHU UNIVERSITY
4-17-1 WAKASATO, NAGANO 380-8553, JAPAN
E-mail: jkawabe@gipwc.shinshu-u.ac.jp