INVARIANCE PRINCIPLES FOR A LINEAR COMBINATION OF U-STATISTICS

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Abstract. Invariance principles or functional limit theorems are well-known for U-statistic and V-statistic. In the case that the kernel is non-degenerate, we show invariance principles for a linear combination of U-statistics which includes V-statistic and LB-statistic.

1 Introduction Let $\theta(F)$ be an estimable parameter or a regular functional of a distribution F and $g(x_1,...,x_k)$ be its kernel of degree k. We assume that the kernel $g(x_1,...,x_k)$ is symmetric and not degenerate. Let $X_1,...,X_n$ be a random sample of size n from the distribution F. U-statistic U_n and V-statistic V_n are well-known as estimators of $\theta(F)$, which are given by the followings.

$$U_n = \binom{n}{k}^{-1} \sum_{1\le j_1<\cdots<j_k\le n} g(X_{j_1},\dots,X_{j_k}), \tag{1.1}$$

where $\sum_{1\le j_1<\cdots<j_k\le n}$ denotes the summation over all integers $j_1,\dots,j_k$ satisfying $1\le j_1<\cdots<j_k\le n$. V-statistic V_n is given by

$$V_n = \frac{1}{n^k}\sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n} g(X_{j_1},\dots,X_{j_k}) \tag{1.2}$$

(see, for example, Lee(1990)).

As an estimator of $\theta(F)$, Toda and Yamato (2001) introduce a linear combination Y_n of U-statistics as follows: Let $w(r_1,\dots,r_j;k)$ be a nonnegative and symmetric function of positive integers $r_1,\dots,r_j$ such that $j=1,\dots,k$ and $r_1+\cdots+r_j=k$, where k is the degree of the kernel g and fixed. We assume that at least one of $w(r_1,\dots,r_j;k)$'s is positive. We put

$$d(k,j) = \sum\nolimits^{+}_{r_1+\cdots+r_j=k} w(r_1,\dots,r_j;k)$$

for $j=1,2,...,k$, where the summation $\sum^{+}_{r_1+\cdots+r_j=k}$ is taken over all positive integers $r_1,...,r_j$ satisfying $r_1+\cdots+r_j=k$ with j and k fixed. For $j=1,\dots,k$, let $g_{(j)}(x_1,...,x_j)$ be the kernel given by

$$g_{(j)}(x_1,\dots,x_j) = \frac{1}{d(k,j)}\sum\nolimits^{+}_{r_1+\cdots+r_j=k} w(r_1,\dots,r_j;k)\, g(\underbrace{x_1,\dots,x_1}_{r_1},\dots,\underbrace{x_j,\dots,x_j}_{r_j}).$$

Let $U_n^{(j)}$ be the U-statistic associated with this kernel $g_{(j)}(x_1,\dots,x_j)$ for $j=1,\dots,k$. The kernel $g_{(j)}(x_1,\dots,x_j)$ is symmetric because of the symmetry of $w(r_1,\dots,r_j;k)$. If $d(k,j)$

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is equal to zero for some j, then the associated $w(r_1,\dots,r_j;k)$'s are equal to zero. In this case, we let the corresponding statistic $U_n^{(j)}$ be zero.
Then the linear combination Y_n of U-statistics is given by

$$Y_n = \frac{1}{D(n,k)} \sum_{j=1}^{k} d(k,j) \binom{n}{j} U_n^{(j)}, \tag{1.3}$$

where $D(n,k) = \sum_{j=1}^{k} d(k,j)\binom{n}{j}$. Since w's are nonnegative and at least one of them is positive, $D(n,k)$ is positive.

If $w(1,1,\dots,1;k) = 1$ and $w(r_1,\dots,r_j;k) = 0$ for positive integers $r_1,\dots,r_j$ such that $j = 1,\dots,k-1$ and $r_1+\cdots+r_j = k$, then $d(k,k) = 1$, $d(k,j) = 0$ $(j = 1,\dots,k-1)$ and $D(n,k) = \binom{n}{k}$. The corresponding statistic Y_n is equal to U-statistic U_n given by (1.1).

If w is the function given by $w(r_1,\dots,r_j;k) = k!/(r_1!\cdots r_j!)$ for positive integers $r_1,\dots,r_j$ such that $j = 1,\dots,k$ and $r_1+\cdots+r_j = k$, then $d(k,j) = j!\mathcal{S}(k,j)$ $(j = 1,\dots,k)$ and $D(n,k) = n^k$ where $\mathcal{S}(k,j)$ are the Stirling number of the second kind. The corresponding statistic Y_n is equal to V-statistic V_n given by (1.2).

If w is the function given by $w(r_1,\dots,r_j;k) = 1$ for positive integers $r_1,\dots,r_j$ such that $j = 1,\dots,k$ and $r_1+\cdots+r_j = k$, then $d(k,j) = \binom{k-1}{j-1}$ $(j = 1,\dots,k)$ and $D(n,k) = \binom{n+k-1}{k}$. The corresponding statistic Y_n is equal to LB-statistic B_n which is given by

$$B_n = \binom{n+k-1}{k}^{-1} \sum_{r_1+\cdots+r_n=k} g(\underbrace{X_1,\dots,X_1}_{r_1},\dots,\underbrace{X_n,\dots,X_n}_{r_n}), \tag{1.4}$$

where $\sum_{r_1+\cdots+r_n=k}$ denote the summation over all non-negative integers $r_1,\dots,r_n$ satisfying $r_1+\cdots+r_n = k$ (see Toda and Yamato (2001)).

In Section 2 we quote the invariance principles for the U-statistic from Miller and Sen (1972), Sen (1974), Denker (1985) and Borovskikh (1996).

Our purpose is to show the invariance principles for the statistic Y_n given by (1.3), using the invariance principles for the U-statistic. These are shown in Section 3. For V-statistic the invariance principles are already shown (see Miller and Sen (1972), Sen (1974), Denker (1985) and Koroljuk and Borovskich (1994)). Our results are obtained for a linear combination of U-statistics including V-statistic and even under stronger conditions than the ones for V-statistic.

2 Invariance principles for U-statistic

For the kernel $g(x_1,\dots,x_k)$, we put

$$\psi_j(x_1,\dots,x_j) = E[g(X_1,\dots,X_k) \mid X_1 = x_1,\dots,X_j = x_j], \quad j = 1,\dots,k$$

and

$$h^{(1)}(x_1) = \psi_1(x_1) - \theta,$$
$$h^{(2)}(x_1,x_2) = \psi_2(x_1,x_2) - h^{(1)}(x_1) - h^{(1)}(x_2) - \theta,$$
$$h^{(c)}(x_1,\dots,x_c) = \psi_c(x_1,\dots,x_c) - \sum_{j=1}^{c-1} \sum_{(c,j)} h^{(j)}(x_{i_1},\dots,x_{i_j}) - \theta$$

for $c = 3, 4, ..., k$, where the sum $\sum_{(c,j)}$ is taken over all integers such that $1 \le i_1 < \cdots < i_j \le c$. Let σ_1^2 be the variance of $h^{(1)}(X_1)$. Since we consider the non-degenerate kernel g in this paper, we have $\sigma_1^2 > 0$.

Let $\{U_n(t) : 0 \le t \le 1\}$ be a random process given by

$$U_n(t) = \begin{cases} 0 & \text{if} \quad t = j/n \quad (\ j = 0, 1, ..., k-1) \\ j(U_j - \theta)/k\sigma_1\sqrt{n} & \text{if} \quad t = j/n \quad (\ j = k, ..., n) \end{cases} \tag{2.1}$$

and by linear interpolation elsewhere. That is,

$$U_n(t) = U_n(\frac{j}{n}) + (nt - j)[U_n(\frac{j+1}{n}) - U_n(\frac{j}{n})]$$

for $j/n < t < (j+1)/n$ $(j = k-1, ..., n-1)$. The following lemma is the weak invariance principle for U-statistic (see, for example, Lee (1990), p.136-137).

Lemma 2.1 (Miller and Sen (1972)) *We assume that* $E \mid g(X_1, \ldots, X_k) \mid^2 < \infty$. *Then* $\{U_n(t) : 0 \le t \le 1\}$ *converges weakly in* $C[0,1]$ *to a standard Brownian motion* W.

The space $C[0,1]$ is the space of all continuous real functions on $[0,1]$ with the norm $\rho(x,y) = \sup_{0 \le t \le 1} \mid x(t) - y(t) \mid$ for $x, y \in C[0,1]$. The σ-field of Borel subsets of $C[0,1]$ is generated by the open subsets of $C[0,1]$.

Lemma 2.2 (Borovskikh (1996), p.166.) *We assume that* $E \mid h^{(c)}(X_1, \ldots, X_c) \mid^{\gamma_c} < \infty$, *for each* $c = 1, 2, \ldots, k$ *where* $\gamma_c = 2c/(2c-1)$.
Then $\{U_n(t) : 0 \le t \le 1\}$ *converges weakly in* $C[0,1]$ *to a standard Brownian motion* W.

Borovskikh (1996) states this fact in $D[0,1]$

Next, we quote strong invariance principle. Let $\{\xi(t) : 0 \le t < \infty\}$ be a random process given by

$$\xi(t) = \begin{cases} 0 & t = 0, 1, ..., k-1 \\ n(U_n - \theta) & t = n, n \ge k \end{cases} \tag{2.2}$$

and by linear interpolation elsewhere.
Let $f(t)$ be a positive function satisfying the following conditions:
(i) $f(t)$ is increasing on $[0, \infty)$,
(ii) $t^{-1}f(t)$ is decreasing on $[0, \infty)$,
(iii) $\sum_{n \ge 1}[f(cn)]^{-1} \int_{[h^{(1)}(x)]^2 > f(cn)} [h^{(1)}(x)]^2 dF(x) < \infty$ for $\forall c > 0$.

Since the kernel g is assumed to be non-degenerate, $h^{(1)}(x_1)$ is not equal to a constant almost surely (a.s.). The following lemma is the strong invariance principle for U-statistic (see, for example, Lee (1990), p.139).

Lemma 2.3 (Sen (1974)) *We assume that* $E \mid g(X_1, \ldots, X_k) \mid^2 < \infty$. *Then there exists a standard Brownian motion* $W(t)$ *on* $[0, \infty]$ *such that as* $t \to \infty$,

$$\xi(t) = k\sigma_1 W(t) + O((tf(t))^{\frac{1}{4}} \log t) \quad a.s. \tag{2.3}$$

If we choose $f(t) = t/(\log t)^4$, then $(tf(t))^{1/4} \log t = t^{1/2}$ and we have the following.

Lemma 2.4 (Denker (1985)) *We assume that* $E \mid g(X_1, \ldots, X_k) \mid^2 < \infty$. *Then*

$$\lim_{t\to\infty} (t \log\log t)^{-\frac{1}{2}} \mid \sigma^{-\frac{1}{2}} k^{-1} \xi(t) - W(t) \mid = 0 \quad a.s. \tag{2.4}$$

The following random process is defined in a space different from $C[0,1]$. Let $\{\nu_n(t), t \in [0,1]\}$ be a random process given by

$$\nu_n(t) = \frac{\sqrt{n}}{k\sigma_1}(U_{n(t)} - \theta),$$

$$n(t) = \min\{j \geq 1 : nj^{-1} \leq t\} = -[-\frac{n}{t}], \quad t \in [0,1],$$

where $\nu_n(t)$ belong to the space $D[0,1]$ of all real functions on $[0,1]$ which are right continuous and have left-hand limits. The Skorokhod metric is considered on the space $D[0,1]$.

Lemma 2.5 (Borovskikh (1996), p.169.) *We assume that* $E \mid h^{(c)}(X_1, \ldots, X_c) \mid^{\gamma_c} < \infty$, *for each* $c = 1, 2, \ldots, k$ *where* $\gamma_c = 2c/(2c-1)$.
Then $\{\nu_n(t) : 0 \leq t \leq 1\}$ *converges weakly in* $D[0,1]$ *to a standard Brownian motion* W.

Borovskikh (1996) says this result reversed invariance principle.

3 Invariant principles for Y-statistic Let $\{Y_n(t) : 0 \leq t \leq 1\}$ be a random process given by

$$Y_n(t) = \begin{cases} 0 & \text{if} \quad t = j/n \quad (\ j = 0, 1, \ldots, k-1) \\ j(Y_j - \theta)/k\sigma_1\sqrt{n} & \text{if} \quad t = j/n \quad (\ j = k, \ldots, n) \end{cases} \tag{3.1}$$

and by linear interpolation elsewhere.
Then by (1.3) we have

$$Y_n(\frac{j}{n}) = 0, \quad j = 0, 1, \ldots, k-1$$

$$Y_n(\frac{j}{n}) = \frac{1}{D(j,k)} \sum_{r=1}^{k} d(k,r) \binom{j}{r} \frac{j(U_j^{(r)} - \theta)}{k\sigma_1\sqrt{n}}, \quad j = k, k+1, \ldots, n, \tag{3.2}$$

For d and D given in Section 1, we suppose that there exists a positive constant β_1 such that

$$1 - \frac{d(k,k)}{D(n,k)} \binom{n}{k} \leq \frac{\beta_1}{n}. \tag{3.3}$$

We note that the left-hand side is nonnegative from the assumption. The inequality (3.3) is equivalent to

$$\frac{1}{D(n,k)} \sum_{j=1}^{k-1} d(k,j) \binom{n}{j} \leq \frac{\beta_1}{n}. \tag{3.4}$$

For the LB-statistic given by (1.4), $\beta_1 = k(k-1)$ and for the V-statistic given by (1.2), $\beta_1 = k(k-1)/2$ (see Toda and Yamato (2001)). Since we have $\beta_1 = 0$ for the U-statistic, the U-statistic U_n is not included in the following discussion.

Proposition 3.1 *We suppose* (3.3), *and that* $E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty$ *for* $1 \leq i_1 \leq \cdots \leq i_k \leq k$. *Then* $\sup_{0\leq t\leq 1} \mid Y_n(t) - U_n(t) \mid$ *converges to zero in probability as* $n \to \infty$.

Proof: Since $U_n^{(k)} = U_n$, by (3.2) we have

$$\sup_{0\leq t\leq 1} \mid Y_n(t) - U_n(t) \mid \leq I_{1n} + \sum_{r=1}^{k-1} I_{2n}^{(r)}, \tag{3.5}$$

where

$$I_{1n} = \max_{k\leq j\leq n} \mid \frac{d(k,k)}{D(j,k)}\binom{j}{k} - 1 \mid \times \mid U_n(\frac{j}{n}) \mid$$

and

$$I_{2n}^{(r)} = \max_{k\leq j\leq n} \frac{d(k,r)}{D(j,k)}\binom{j}{r}\frac{j \mid U_j^{(r)} - \theta \mid}{k\sigma_1\sqrt{n}}, \quad r = 1, \ldots, k-1.$$

By using (3.3) to I_{1n}, we have

$$\begin{aligned} I_{1n} &= \max_{k\leq j\leq n} \mid \frac{d(k,k)}{D(j,k)}\binom{j}{k} - 1 \mid \frac{j \mid U_j - \theta \mid}{k\sigma_1\sqrt{n}} \\ &\leq \frac{\beta_1}{k\sigma_1} \max_{k\leq j\leq n} \frac{\mid U_j - \theta \mid}{\sqrt{n}}. \end{aligned}$$

We note that $\{U_j, j = k, k+1, \ldots\}$ is a reverse martingale with respect to the σ-fields $\sigma(U_j, U_{j+1}, \ldots)$ and therefore $\{\mid U_j - \theta \mid, j = k, k+1, \ldots\}$ is a reverse submartingale. So by applying the inequality given by Koroljuk and Borovskich (1994), p.78 to $P(\sup_{j\geq k} \mid U_j - \theta \mid /\sqrt{n} > \varepsilon)$, for $\forall\varepsilon > 0$ we have

$$P(\max_{k\leq j\leq n} \frac{\mid U_j - \theta \mid}{\sqrt{n}} > \varepsilon) \leq P(\sup_{j\geq k} \mid U_j - \theta \mid > \varepsilon\sqrt{n}) \leq \frac{E \mid U_k - \theta \mid}{\varepsilon\sqrt{n}}, \tag{3.6}$$

which converges to zero as $n \to \infty$. Thus $\max_{k\leq j\leq n} \mid U_j - \theta \mid /\sqrt{n}$ and therefore I_{1n} converges to zero in probability as $n \to \infty$.

By (3.4) , for $r = 1, \ldots, k-1$ we have

$$I_{2n}^{(r)} \leq \frac{\beta_1}{k\sigma_1}\{\max_{k\leq j\leq n} \frac{\mid U_j^{(r)} - \theta_r \mid}{\sqrt{n}} + \frac{\mid \theta_r - \theta \mid}{\sqrt{n}}\},$$

where $\theta_r = EU_j^{(r)}$. By the same reason as I_{1n}, $\max_{k\leq j\leq n} \mid U_j^{(r)} - \theta_r \mid /\sqrt{n}$ converges to zero in probability as $n \to \infty$. Thus $I_{2n}^{(r)}$ converges to zero in probability as $n \to \infty$ for $r = 1, \ldots, k-1$. Hence by (3.5), $\sup_{0\leq t\leq 1} \mid Y_n(t) - U_n(t) \mid$ converges to zero in probability as $n \to \infty$. □

¿From Lemmas 1.1, 1.2 and Proposition 3.1, we have the following theorems.

Theorem 3.2 *We assume* (3.3), $E \mid g(X_1, \ldots, X_k) \mid^2 < \infty$, *and that* $E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty$ *for* $1 \leq i_1 \leq \cdots \leq i_k \leq k$. *Then* $\{Y_n(t) : 0 \leq t \leq 1\}$ *converges weakly in* $C[0,1]$ *to a standard Brownian motion* W.

Theorem 3.3 *We assume that* $E \mid h^{(c)}(X_1, \ldots, X_c) \mid^{\gamma_c} < \infty$, *for each* $c = 1, 2, \ldots, k$, *where* $\gamma_c = 2c/(2c-1)$ *and* $E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty$ *for* $1 \leq i_1 \leq \cdots \leq i_k \leq k$. *We also suppose* (3.3).
Then $\{Y_n(t) : 0 \leq t \leq 1\}$ *converges weakly in* $C[0,1]$ *to a standard Brownian motion* W.

Now we consider the strong invariance principle for the statistic Y_n.

Lemma 3.4 *We suppose* (3.3), *and that* $E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty$ *for* $1 \leq i_1 \leq \cdots \leq i_k \leq k$. *Then* $(n/\log\log n) \mid Y_n - U_n \mid$ *converges to zero almost surely as* $n \to \infty$.

Proof: By (1.3) we have

$$Y_n - U_n = (U_n - \theta)[\frac{d(k,k)}{D(n,k)}\binom{n}{k} - 1] + \sum_{j=1}^{k-1} \frac{d(k,j)}{D(n,j)}\binom{n}{j}(U_n^{(j)} - \theta).$$

Using (3.3) and (3.4) to the right-hand side of the above, we have

$$\frac{n}{\log\log n} \mid Y_n - U_n \mid \leq \frac{\beta_1}{\log\log n}[\mid U_n - \theta \mid + \sum_{j=1}^{k-1} \mid U_n^{(j)} - \theta \mid].$$

Under the assumption for $j = 1, \ldots, k$, $U_n^{(j)} \to \theta_j$ a.s. as $n \to \infty$ and therefore the right-hand side converges to zero a.s. as $n \to \infty$. Hence $(n/\log\log n) \mid Y_n - U_n \mid$ converges to zero a.s. as $n \to \infty$. □

Let $\{\eta(t) : 0 \leq t < \infty\}$ be a random process given by

$$\eta(t) = \begin{cases} 0 & t = 0, 1, \ldots, k-1 \\ n(Y_n - \theta) & t = n,\ n \geq k \end{cases}$$

and by linear interpolation elsewhere.
Then for all $n \geq k$, we have

$$\eta(n) - \xi(n) = n(Y_n - U_n),$$

which converges to zero a.s. as $n \to \infty$ by Lemma 3.4. So we have

$$\mid \eta(n) - \xi(n) \mid = o((nf(n))^{\frac{1}{4}} \log n).$$

Thus by this result and Lemma 2.3 we have the following.

Theorem 3.5 *We assume that* $E \mid g(X_1, \ldots, X_k) \mid^2 < \infty$ *and that* $E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty$ *for* $1 \leq i_1 \leq \cdots \leq i_k \leq k$. *We also suppose* (3.3). *Then there exists a standard Brownian motion* $W(t)$ *on* $[0, \infty]$ *such that as* $t \to \infty$,

$$\eta(t) = k\sigma_1 W(t) + O((tf(t))^{\frac{1}{4}} \log t) \quad a.s.$$

Theorem 3.6 *We assume that* $E \mid g(X_1, \ldots, X_k) \mid^2 < \infty$ *and that* $E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty$ *for* $1 \leq i_1 \leq \cdots \leq i_k \leq k$. *We also suppose* (3.3). *Then*

$$\lim_{t \to \infty} (t \log\log t)^{-\frac{1}{2}} \mid \sigma^{-\frac{1}{2}} k^{-1} \eta(t) - W(t) \mid = 0 \quad a.s.$$

The following is the reversed invariance principle for Y_n. Let $\{\zeta_n(t), t \in [0,1]\}$ be a random process given by

$$\zeta_n(t) = \frac{\sqrt{n}}{k\sigma_1}(Y_{n(t)} - \theta), \quad t \in [0,1].$$

The random process $\zeta_n(t)$ belong to $D[0,1]$.

Theorem 3.7 *We assume that* $E \mid h^{(c)}(X_1, \ldots, X_c) \mid^{\gamma_c} < \infty$, *for each* $c = 1, 2, \ldots, k$ *where* $\gamma_c = 2c/(2c-1)$ *and that* $E \mid g(X_{i_1}, \ldots, X_{i_k}) \mid < \infty$ *for* $1 \le i_1 \le \cdots \le i_k \le k$. *We also suppose* (3.3).
Then $\{\zeta_n(t) : 0 \le t \le 1\}$ *converges weakly in* $D[0,1]$ *to a standard Brownian motion* W.

Proof. By the definition of $\nu_n(t)$, $\zeta_n(t)$, we have

$$\sup_{0 \le t \le 1} \mid \zeta_n(t) - \nu_n(t) \mid = \frac{\sqrt{n}}{k\sigma_1} \sup_{0 \le t \le 1} \mid Y_{n(t)} - U_{n(t)} \mid = \frac{\sqrt{n}}{k\sigma_1} \sup_{j \ge n} \mid Y_j - U_j \mid .$$

By (1.3), (3.3) and (3.4), for $j \ge n$ we get

$$\mid Y_j - U_j \mid \le \frac{\beta_1}{n}[\mid U_j - \theta \mid + \sum_{r=1}^{k-1}(U_j^{(r)} - \theta)].$$

Thus

$$\sup_{0 \le t \le 1} \mid \zeta_n(t) - \nu_n(t) \mid \le \frac{\beta_1}{k\sigma_1}\{\frac{1}{\sqrt{n}} \sup_{j \ge n} \mid U_j - \theta \mid + \sum_{r=1}^{k-1} \frac{1}{\sqrt{n}} \sup_{j \ge n} \mid U_j^{(r)} - \theta \mid\}. \tag{3.7}$$

By the same reason stated with respect to (3.6), for $\forall \varepsilon > 0$ we have

$$P(\frac{1}{\sqrt{n}} \sup_{j \ge n} \mid U_j - \theta \mid > \varepsilon) \le \frac{1}{\varepsilon\sqrt{n}} E \mid U_n - \theta \mid,$$

which converges to zero as $n \to \infty$. Thus $\sup_{j \ge n} \mid U_j - \theta \mid / \sqrt{n}$ converges to zero in probability as $n \to \infty$. Similarly for $r = 1, \ldots, k-1$, $\sup_{j \ge n} \mid U_j^{(r)} - \theta \mid / \sqrt{n}$ converges to zero in probability as $n \to \infty$ by the assumption. Hence by (3.7), $\sup_{0 \le t \le 1} \mid \zeta_n(t) - \nu_n(t) \mid$ converges to zero in probability as $n \to \infty$. This fact and Lemma 2.5 give the weak convergence of $\{\zeta_n(t) : 0 \le t \le 1\}$. □

References

[1] Borovskikh, Yu.V. (1996). *U-statistics in Banach spaces.* VSP, Utrecht.

[2] Denker, M. (1985). *Asymptotic Distribution Theory in Nonparametric Statistics.* Friedr. Vieweg & Sohn, Wiesbaden.

[3] Koroljuk, V.S. and Borovskich, Yu.V. (1994). *Theory of U-statistics.* Kluwer Academic Publishers, Dordrecht.

[4] Lee, A. J. (1990). *U-statistics.* Marcel Dekker, New York.

[5] Miller, R.G. and Sen, P.K. (1972). Weak convergence of U-statistics and von Mises differentiable statistical functions. *Ann. Math. Statist.* **43**, 31–41.

[6] Sen, P.K. (1974). Almost sure behaviour of U-statistics and von Mises differentiable statistical functions. *Ann. Statist.* **2**, 387–395.

[7] Toda, K. and Yamato, H. (2001). Berry-Esseen bounds for a class of statistics including LB-statistic and V-statistic. *J. Japan Statis.Soc.* (To appear.)

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